Transformée de Laplace Exercices Simples

1) Laplace

Calculer les transformées de Laplace suivantes :

a) $\mathscr{L}\left[\left(t^2+t-e^{-3t}\right)\mathscr{U}(t)\right]$

b) $\mathscr{L}\left[\left(t+2\right)\mathscr{U}(t)+\left(t+3\right)\mathscr{U}(t-2)\right]$

c) $\mathscr{L}\left[\left(t^2+t+1\right)e^{-2t}\mathscr{U}(t)\right]$

2) Laplace inverse

Calculer les originaux suivants :

a) $\mathscr{L}^{-1}\left[\dfrac{p+2}{(p+3)(p+4)}\right]$

b) $\mathscr{L}^{-1}\left[\dfrac{3}{(p+5)^2}\right]$

c) $\mathscr{L}^{-1}\left[\dfrac{p-1}{(p^2+2p+5)}\right]$

3) Équations différentielles

Utiliser la transformée de Laplace pour déterminer la solution particulière de chacune des équations différentielles suivantes :

a) $x'(t)+x(t)=t\,\mathscr{U}(t)-t\,\mathscr{U}(t-1)$ condition initiale : $x(0)=0$

b) $x''(t)+x'(t)=\mathscr{U}(t)$ conditions initiales : $\begin{cases} x(0) = 0 \\ x'(0) = 0 \end{cases}$

c) $x''(t)+4x(t)=2\,\mathscr{U}(t)$ conditions initiales : $\begin{cases} x(0) = 0 \\ x'(0) = 1 \end{cases}$

d) $x''(t)+5x'(t)+4x(t)=e^{-2t}\,\mathscr{U}(t)$ conditions initiales : $\begin{cases} x(0) = 1 \\ x'(0) = 0 \end{cases}$

e) $x''(t)+2x'(t)+2x(t)=0$ conditions initiales : $\begin{cases} x(0) = 1 \\ x'(0) = 1 \end{cases}$

Transformée de Laplace Exercices d'entraînement

1) Calculer les transformées de Laplace suivantes :

a) $\mathscr{L}\left[\cos(t)e^{-t}\,\mathscr{U}(t)\right]$ **b)** $\mathscr{L}\left[(5t)^2e^{-5t}\,\mathscr{U}(t)\right]$

c) $\mathscr{L}\left[(\cos(2t)-\sin(t))\,e^{-3t}\,\mathscr{U}(t)\right]$ **d)** $\mathscr{L}\left[(t^2+t+1)e^{-2t}\,\mathscr{U}(t)\right]$

2) Calculer les originaux suivants :

a) $\mathscr{L}^{-1}\left[\dfrac{3}{p+2}-\dfrac{1}{p^3}\right]$

b) $\mathscr{L}^{-1}\left[\dfrac{-2}{(p+3)^2}\right]$

c) $\mathscr{L}^{-1}\left[\dfrac{5}{(p+3)(p^2+3p+5)}\right]$

d) $\mathscr{L}^{-1}\left[\dfrac{p}{p^2+4p+6}\right]$

e) $\mathscr{L}^{-1}\left[\dfrac{p}{(p+1)^2}\right]$

f) $\mathscr{L}^{-1}\left[\dfrac{2p+3}{2p^2+4p+5}\right]$

3) Équations différentielles

Utiliser la transformée de Laplace pour résoudre les équations suivantes :

a) $x''(t)+3x'(t)+2x(t)=0$ avec : $x(0)=1$ et $x'(0)=0$

b) $x''(t)+6x'(t)+9x(t)=e^{-2t}\,\mathscr{U}(t)$ avec : $x(0)=0$ et $x'(0)=0$

c) $x''(t)-x(t)=(3e^{-2t}+t^2+1)\,\mathscr{U}(t)$ avec : $x(0)=0$ et $x'(0)=0$

d) $x''(t)-4x(t)=(3e^{-t}-t^2)\,\mathscr{U}(t)$ avec : $x(0)=0$ et $x'(0)=1$

e) $x''(t)+x(t)=e^t\cos(t)\,\mathscr{U}(t)$ avec : $x(0)=0$ et $x'(0)=0$

f) $x''(t)+x(t)=\mathscr{U}(t)-\mathscr{U}(t-1)$ avec : $x(0)=2$ et $x'(0)=0$

Exercices Simples (Solutions)

1) Laplace

Calculer les transformées de Laplace suivantes :

a) $\mathscr{L}\left[\left(t^2+t-e^{-3t}\right)\mathscr{U}(t)\right]$

$$f(t)=\left(t^2+t-e^{-3t}\right)\mathscr{U}(t) \;=\; t^2\,\mathscr{U}(t)+t\,\mathscr{U}(t)-e^{-3t}\,\mathscr{U}(t)$$

$$F(p)=\boxed{\frac{2}{p^3}+\frac{1}{p^2}-\frac{1}{p+3}}$$

b) $\mathscr{L}\left[\left(t+2\right)\mathscr{U}(t)+\left(t+3\right)\mathscr{U}(t-2)\right]$

$$f(t)=\left(t+2\right)\mathscr{U}(t)+\left(t+3\right)\mathscr{U}(t-2)=\left(t+2\right)\mathscr{U}(t)+\left((t-2)+5\right)\mathscr{U}(t-2)$$

$$\mathscr{L}\left[\left(t+5\right)\mathscr{U}(t)\right]=\frac{1}{p^2}+\frac{5}{p}$$

$$\mathscr{L}\left[\left((t-2)+5\right)\mathscr{U}(t-2)\right]=\left(\frac{1}{p^2}+\frac{5}{p}\right)e^{-2p}$$

$$\mathscr{L}\left[\left(t+2\right)\mathscr{U}(t)\right]=\frac{1}{p^2}+\frac{2}{p}=\frac{2p+1}{p^2}$$

$$F(p)=\boxed{\frac{2p+1}{p^2}+\left(\frac{1}{p^2}+\frac{5}{p}\right)e^{-2p}}$$

c) $\mathscr{L}\left[\left(t^2+t+1\right)e^{-2t}\,\mathscr{U}(t)\right]$

$$f(t)=\left(t^2+t+1\right)e^{-2t}\,\mathscr{U}(t)$$

$$\mathscr{L}\left[\left(t^2+t+1\right)\mathscr{U}(t)\right]=\frac{2}{p^3}+\frac{1}{p^2}+\frac{1}{p}\;=\;\frac{p^2+p+2}{p^3}$$

$$\mathscr{L}\left[\left(t^2+t+1\right)e^{-2t}\,\mathscr{U}(t)\right]=\frac{(p+2)^2+(p+2)+2}{(p+2)^3}$$

$$F(t)=\boxed{\frac{p^2+5p+8}{(p+2)^3}}$$

2) Laplace inverse

Calculer les originaux suivants :

a) $\mathscr{L}^{-1}\left[\frac{p+2}{(p+3)(p+4)}\right]$

$$F(p)=\frac{p+2}{(p+3)(p+4)}\;=\;\frac{2}{p+4}+\frac{-1}{p+3}$$

$$f(t)=\boxed{\left(2\,e^{-4t}-e^{-3t}\right)\mathscr{U}(t)}$$

b) $\mathscr{L}^{-1}\left[\dfrac{3}{(p+5)^2}\right]$

$$F(p) = \frac{3}{(p+5)^2}$$
$$\mathscr{L}^{-1}\left[\frac{3}{p^2}\right] = 3\,t\,\mathscr{U}(t)$$
$$f(t) = \boxed{3\,t\,e^{-5t}\,\mathscr{U}(t)}$$

c) $\mathscr{L}^{-1}\left[\dfrac{p-1}{(p^2+2p+5)}\right]$

$$F(p) = \frac{p-1}{(p^2+2p+5)} = \frac{p+1}{(p+1)^2+2^2} - \frac{2}{(p+1)^2+2^2}$$
$$\mathscr{L}^{-1}\left[\frac{p}{p^2+2^2} - \frac{2}{p^2+2^2}\right] = \Big(\cos(2t) - \sin(2t)\Big)\,\mathscr{U}(t)$$
$$f(t) = \boxed{\Big(\cos(2t) - \sin(2t)\Big)\,e^{-t}\,\mathscr{U}(t)}$$

3) Équations différentielles

a) $x'(t) + x(t) = t\,\mathscr{U}(t) - t\,\mathscr{U}(t-1)$ condition initiale : $x(0) = 0$

$$x'(t) + x(t) = t\,\mathscr{U}(t) - \big((t-1)+1\big)\,\mathscr{U}(t-1)$$
$$(p\,X(p) - 0) + X(p) = \frac{1}{p^2} - \left(\frac{1}{p^2} + \frac{1}{p}\right)e^{-p}$$
$$(p+1)X(p) = \frac{1}{p^2} - \frac{p+1}{p^2}\,e^{-p}$$
$$X(p) = \frac{1}{p^2(p+1)} - \frac{1}{p^2}\,e^{-p}$$
$$X(p) = \frac{1}{p^2} - \frac{1}{p} + \frac{1}{p+1} - \frac{1}{p^2}\,e^{-p}$$
$$\boxed{x(t) = \big(t - 1 + e^{-t}\big)\,\mathscr{U}(t) - \big(t-1\big)\,\mathscr{U}(t-1)}$$

b) $x''(t) + x'(t) = \mathscr{U}(t)$ conditions initiales : $\begin{cases} x(0) = 0 \\ x'(0) = 0 \end{cases}$

$$(p^2\,X(p) - 0 - 0) + (p\,X(p) - 0) = \frac{1}{p}$$
$$(p^2 + p)X(p) = \frac{1}{p}$$
$$X(p) = \frac{1}{p(p^2+p)} = \frac{1}{p^2} - \frac{1}{p} + \frac{1}{p+1}$$
$$\boxed{x(t) = \Big(t - 1 + e^{-t}\Big)\,\mathscr{U}(t)}$$

c) $x''(t) + 4x(t) = 2\,\mathscr{U}(t)$ conditions initiales : $\left\{\begin{array}{rcl} x(0) & = & 0 \\ x'(0) & = & 1 \end{array}\right.$

$$\begin{aligned}
(p^2\,X(p) - 0 - 1) + 4X(p) &= \frac{2}{p} \\
(p^2+4)X(p) &= \frac{2}{p} + 1 \\
X(p) &= \frac{p+2}{p(p^2+4)} \;=\; \frac{\frac{1}{2}}{p} + \frac{-\frac{1}{2}p+1}{p^2+4} \\
&= \frac{1}{2}\left(\frac{1}{p} - \frac{p}{p^2+4} + \frac{2}{p^2+4}\right)
\end{aligned}$$

$$\boxed{x(t) = \frac{1}{2}\Big(1 - \cos(2t) + \sin(2t)\Big)\,\mathscr{U}(t)}$$

d) $x''(t) + 5x'(t) + 4x(t) = e^{-2t}\,\mathscr{U}(t)$ conditions initiales : $\left\{\begin{array}{rcl} x(0) & = & 1 \\ x'(0) & = & 0 \end{array}\right.$

$$\begin{aligned}
(p^2\,X(p) - p - 0) + 5(p\,X(p) - 1) + 4X(p) &= \frac{1}{p+2} \\
(p^2+5p+4)X(p) &= \frac{1}{p+2} + p + 5 \\
X(p) &= \frac{p^2+7p+11}{(p+2)(p^2+5p+4)} \;=\; \frac{p^2+7p+11}{(p+2)(p+1)(p+4)} \\
&= \frac{5/3}{p+1} + \frac{-1/2}{p+2} + \frac{-1/6}{p+4}
\end{aligned}$$

$$\boxed{x(t) = \left(\frac{5\,e^{-t}}{3} - \frac{e^{-2t}}{2} - \frac{e^{-4t}}{6}\right)\,\mathscr{U}(t)}$$

e) $x''(t) + 2x'(t) + 2x(t) = 0$ conditions initiales : $\left\{\begin{array}{rcl} x(0) & = & 1 \\ x'(0) & = & 1 \end{array}\right.$

$$\begin{aligned}
(p^2\,X(p) - p - 1) + 2(p\,X(p) - 1) + 2X(p) &= 0 \\
(p^2+2p+2)X(p) &= p + 3 \\
X(p) &= \frac{p+3}{p^2+2p+2} \\
&= \frac{p+1}{(p+1)^2+1^2} + \frac{2}{(p+1)^2+1^2}
\end{aligned}$$

$$\boxed{x(t) = \Big(\cos(t) + 2\sin(t)\Big)e^{-t}\,\mathscr{U}(t)}$$

Exercices d'entraînement (Solutions)

1) Calculer les transformées de Laplace suivantes :

a) $\mathscr{L}\left[\cos(t)e^{-t}\,\mathscr{U}(t)\right]$

$$\mathscr{L}\left[\cos(t)\,\mathscr{U}(t)\right] = \frac{p}{p^2+1^2}$$
$$\mathscr{L}\left[\cos(t)e^{-t}\,\mathscr{U}(t)\right] = \frac{(p+1)}{(p+1)^2+1^2}$$

$$\boxed{F(p) = \frac{p+1}{p^2+2p+2}}$$

b) $\mathscr{L}\left[(5t)^2e^{-5t}\,\mathscr{U}(t)\right]$

$$\mathscr{L}\left[25\,t^2\,\mathscr{U}(t)\right] = 25\frac{2!}{p^3}$$
$$\mathscr{L}\left[(5t)^2e^{-5t}\,\mathscr{U}(t)\right] = 25\frac{2}{(p+5)^3}$$

$$\boxed{F(p) = \frac{50}{(p+5)^3}}$$

c) $\mathscr{L}\left[(\cos(2t)-\sin(t))\,e^{-3t}\,\mathscr{U}(t)\right]$

$$\mathscr{L}\left[(\cos(2t)-\sin(t))\;\mathscr{U}(t)\right] = \frac{p}{p^2+2^2} - \frac{1}{p^2+1^2}$$
$$\mathscr{L}\left[(\cos(2t)-\sin(t))\,e^{-3t}\,\mathscr{U}(t)\right] = \frac{(p+3)}{(p+3)^2+2^2} - \frac{1}{(p+3)^2+1^2}$$

$$\boxed{F(p) = \frac{p+3}{p^2+6p+13} - \frac{1}{p^2+6p+10}}$$

d) $\mathscr{L}\left[(t^2+t+1)e^{-2t}\,\mathscr{U}(t)\right]$

$$\mathscr{L}\left[(t^2+t+1)\;\mathscr{U}(t)\right] = \frac{2}{p^3} + \frac{1}{p^2} + \frac{1}{p}$$
$$\mathscr{L}\left[(t^2+t+1)e^{-2t}\,\mathscr{U}(t)\right] = \frac{2}{(p+2)^3} + \frac{1}{(p+2)^2} + \frac{1}{p+2}$$

$$\boxed{F(p) = \frac{2}{(p+2)^3} + \frac{1}{(p+2)^2} + \frac{1}{p+2}}$$

2) Calculer les originaux suivants :

a) $\mathscr{L}^{-1}\left[\dfrac{3}{p+2}-\dfrac{1}{p^3}\right]$ $\boxed{f(t)=\left(3e^{-2t}-\dfrac{t^2}{2}\right)\,\mathscr{U}(t)}$

b) $\mathscr{L}^{-1}\left[\dfrac{-2}{(p+3)^2}\right]$ $\boxed{f(t)=-2t\,e^{-3t}\,\mathscr{U}(t)}$

c) $\mathscr{L}^{-1}\left[\dfrac{5}{(p+3)(p^2+3p+5)}\right]$

$$\begin{aligned}
F(p)=\frac{5}{(p+3)(p^2+3p+5)} &= \frac{1}{p+3}-\frac{p}{p^2+3p+5}\\
&= \frac{1}{p+3}-\frac{p}{(p+\frac{3}{2})^2+(\frac{\sqrt{11}}{2})^2}\\
&= \frac{1}{p+3}-\left(\frac{p+\frac{3}{2}}{(p+\frac{3}{2})^2+(\frac{\sqrt{11}}{2})^2}-\frac{\frac{3}{2}}{(p+\frac{3}{2})^2+(\frac{\sqrt{11}}{2})^2}\right)\\
&= \frac{1}{p+3}-\left(\frac{p+\frac{3}{2}}{(p+\frac{3}{2})^2+(\frac{\sqrt{11}}{2})^2}-\frac{3}{2}\,\frac{2}{\sqrt{11}}\,\frac{\frac{\sqrt{11}}{2}}{(p+\frac{3}{2})^2+(\frac{\sqrt{11}}{2})^2}\right)
\end{aligned}$$

$$\boxed{f(t)=\left(e^{-3t}-\left(\cos(\tfrac{\sqrt{11}}{2}t)-\tfrac{3}{\sqrt{11}}\sin(\tfrac{\sqrt{11}}{2}t)\right)e^{-\frac{3}{2}t}\right)\,\mathscr{U}(t)}$$

d) $\mathscr{L}^{-1}\left[\dfrac{p}{p^2+4p+6}\right]$

$$\begin{aligned}
F(p)=\frac{p}{p^2+4p+6} &= \frac{p}{(p+2)^2+2}\\
&= \frac{p+2}{(p+2)^2+(\sqrt{2})^2}-\frac{2}{(p+2)^2+(\sqrt{2})^2}\\
&= \frac{p+2}{(p+2)^2+(\sqrt{2})^2}-\frac{\sqrt{2}\times\sqrt{2}}{(p+2)^2+(\sqrt{2})^2}
\end{aligned}$$

$$\boxed{f(t)=\left(\cos(\sqrt{2}t)-\sqrt{2}\sin(\sqrt{2}t)\right)e^{-2t}\,\mathscr{U}(t)}$$

e) $\mathscr{L}^{-1}\left[\dfrac{p}{(p+1)^2}\right]=\mathscr{L}^{-1}\left[\dfrac{-1}{(p+1)^2}+\dfrac{1}{p+1}\right]$

$$\boxed{f(t)=(-t+1)e^{-t}\,\mathscr{U}(t)}$$

f) $\mathscr{L}^{-1}\left[\dfrac{2p+3}{2p^2+4p+5}\right]$

$$\begin{aligned}F(p) &= \frac{2p+3}{2p^2+4p+5} = \frac{p+\frac{3}{2}}{p^2+2p+\frac{5}{2}} = \frac{p+\frac{3}{2}}{(p+1)^2+\frac{3}{2}}\\ &= \frac{p+1}{(p+1)^2+\left(\frac{\sqrt{3}}{\sqrt{2}}\right)^2} + \frac{\frac{1}{2}}{(p+1)^2+\left(\frac{\sqrt{3}}{\sqrt{2}}\right)^2}\\ &= \frac{p+1}{(p+1)^2+\left(\frac{\sqrt{6}}{2}\right)^2} + \frac{1}{2}\,\frac{2}{\sqrt{6}}\,\frac{\frac{\sqrt{6}}{2}}{(p+1)^2+\left(\frac{\sqrt{6}}{2}\right)^2}\\ &= \frac{p+1}{(p+1)^2+\left(\frac{\sqrt{6}}{2}\right)^2} + \frac{\sqrt{6}}{6}\,\frac{\frac{\sqrt{6}}{2}}{(p+1)^2+\left(\frac{\sqrt{6}}{2}\right)^2}\end{aligned}$$

$$\boxed{f(t) = \left(\cos(\tfrac{\sqrt{6}}{2}t) + \tfrac{\sqrt{6}}{6}\sin(\tfrac{\sqrt{6}}{2}t)\right) e^{-t}\, \mathscr{U}(t)}$$

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3) Équations différentielles

Utiliser la transformée de Laplace pour résoudre les équations suivantes :

a) $x''(t) + 3x'(t) + 2x(t) = 0$ avec : $x(0) = 1$ et $x'(0) = 0$

$$\begin{aligned}(p^2\,X(p) - p - 0) + 3(p\,X(p) - 1) + 2X(p) &= 0\\ (p^2+3p+2)X(p) &= p+3\\ X(p) &= \frac{p+3}{p^2+3p+2} = \frac{p+3}{(p+1)(p+2)}\\ &= \frac{2}{p+1} + \frac{-1}{p+2}\end{aligned}$$

$$\boxed{x(t) = \left(2e^{-t} - e^{-2t}\right) \mathscr{U}(t)}$$

b) $x''(t) + 6x'(t) + 9x(t) = e^{-2t}\,\mathscr{U}(t)$ avec : $x(0) = 0$ et $x'(0) = 0$

$$\begin{aligned}(p^2\,X(p) - 0 - 0) + 6(p\,X(p) - 0) + 9X(p) &= \frac{1}{p+2}\\ (p^2+6p+9)X(p) &= \frac{1}{p+2}\\ X(p) &= \frac{1}{(p+2)(p^2+6p+9)} = \frac{1}{(p+2)(p+3)^2}\\ &= \frac{1}{p+2} - \frac{1}{(p+3)^2} - \frac{1}{p+3}\end{aligned}$$

$$\boxed{x(t) = \left(e^{-2t} - (t+1)e^{-3t}\right) \mathscr{U}(t)}$$

c) $x''(t) - x(t) = (3e^{-2t} + t^2 + 1)\,\mathscr{U}(t)$ avec : $x(0) = 0$ et $x'(0) = 0$

$$\begin{aligned}
(p^2\,X(p) - 0 - 0) - X(p) &= \frac{3}{p+2} + \frac{2}{p^3} + \frac{1}{p} \\
(p^2-1)X(p) &= \frac{4p^3+2p^2+2p+4}{(p+2)p^3} \\
X(p) &= \frac{4p^2+4p+4}{(p^2-1)(p+2)p^3} \\
&= \frac{1}{p+2} + \frac{2}{p-1} - \frac{2}{p^3} - \frac{3}{p}
\end{aligned}$$

$$\boxed{x(t) = \left(e^{-2t} + 2e^t - t^2 - 3\right)\,\mathscr{U}(t)}$$

d) $x''(t) - 4x(t) = (3e^{-t} - t^2)\,\mathscr{U}(t)$ avec : $x(0) = 0$ et $x'(0) = 1$

$$\begin{aligned}
(p^2\,X(p) - 0 - 1) - 4X(p) &= \frac{3}{p+1} - \frac{2}{p^3} \\
(p^2-4)X(p) &= \frac{3}{p+1} - \frac{2}{p^3} + 1 \\
X(p) &= \frac{p^4+4p^3-2p-2}{(p+1)(p+2)(p-2)p^3} \\
&= \frac{7/16}{p+2} + \frac{7/16}{p-2} - \frac{1}{p+1} + \frac{1/2}{p^3} + \frac{1/8}{p}
\end{aligned}$$

$$\boxed{x(t) = \left(\frac{7e^{2t}}{16} + \frac{7e^{-2t}}{16} - e^{-t} + \frac{t^2}{4} + \frac{1}{8}\right)\,\mathscr{U}(t)}$$

e) $x''(t) + x(t) = e^t\cos(t)\,\mathscr{U}(t)$ avec : $x(0) = 0$ et $x'(0) = 0$

$$\begin{aligned}
(p^2\,X(p) - 0 - 0) + X(p) &= \frac{(p-1)}{(p-1)^2+1} \\
(p^2+1)X(p) &= \frac{p-1}{p^2-2p+2} \\
X(p) &= \frac{p-1}{(p^2-2p+2)(p^2+1)} \\
&= \frac{\frac{1}{5}(p+1)}{(p-1)^2+1} - \frac{\frac{1}{5}(p+3)}{p^2+1} \\
&= \frac{1}{5}\left(\frac{p-1}{(p-1)^2+1} + \frac{2}{(p-1)^2+1} - \frac{p}{p^2+1} - \frac{3}{p^2+1}\right)
\end{aligned}$$

$$\boxed{x(t) = \frac{1}{5}\Big(\cos(t)e^t + 2\sin(t)e^t - \cos(t) - 3\sin(t)\Big)\,\mathscr{U}(t)}$$

f) $x''(t) + x(t) = \mathscr{U}(t) - \mathscr{U}(t-1)$ avec : $x(0) = 2$ et $x'(0) = 0$

$$\begin{aligned}
(p^2\,X(p) - 2p - 0) + X(p) &= \frac{1}{p} - \frac{e^{-p}}{p} \\
(p^2+1)X(p) &= \frac{1}{p} + 2p - \frac{e^{-p}}{p} \\
X(p) &= \frac{2p^2+1}{p(p^2+1)} - \frac{e^{-p}}{p(p^2+1)} \\
&= \left(\frac{1}{p} + \frac{p}{p^2+1}\right) - \left(\frac{1}{p} - \frac{p}{p^2+1}\right) e^{-p}
\end{aligned}$$

$$\boxed{x(t) = \Big(1 + \cos(t)\Big)\,\mathscr{U}(t) - \Big(1 - \cos(t-1)\Big)\,\mathscr{U}(t-1)}$$