Bertrand's postulate

Bertrand's postulate states that for each integer $n \geq 2$ there is a prime number p with $n < p < 2n$. The following proof is due to Erdős. This account is based on my reading of Hardy and Wright, *Introduction to the Theory of Numbers* and Rose, *A Course in Number Theory* (both Oxford University Press).

We need to prove a bound due to Chebyshev on the theta function. This is defined by

$$\vartheta(n) = \sum_{p \leq n} \log p,$$

where p runs over primes. Chebyshev's bound is $\vartheta(n) \leq n \log 4$ for all integers n. We use induction on n. For $n = 1$ and $n = 2$ the claim is obvious. If $n \geq 4$ is even then the case of n is immediate from that of $n - 1$. So let $n = 2m + 1$ be odd with $m \geq 1$. The binomial coefficient $\binom{2m+1}{m}$ occurs twice in the expansion of $(1 + 1)^{2m+1}$, and so $\binom{2m+1}{m} \leq 4^m$. But each prime p with $m + 1 < p \leq 2m + 1$ divides $\binom{2m+1}{m}$ and so

$$\vartheta(2m + 1) - \vartheta(m + 1) \leq \log \binom{2m+1}{m} \leq \log(4^m) = m \log 4.$$

Inductively $\vartheta(m + 1) \leq (m + 1) \log 4$, and so $\vartheta(2m + 1) \leq (2m + 1) \log 4$, establishing Chebyshev's bound.

Now to the main proof. Suppose that $n \geq 2$ and there is no prime p with $n < p < 2n$. Suppose first that $n \geq 2^{11} = 2048$. As $\binom{2n}{n}$ is the largest of the $2n + 1$ terms in the expansion of $(1 + 1)^{2n}$ then $\binom{2n}{n} \geq 4^n/(2n + 1)$. For a prime p we shall denote the highest power of p dividing $\binom{2n}{n}$ by $p^{r(p,n)}$. But

$$r(p, n) = \sum_{j=1}^{\lfloor \log 2n/ \log p \rfloor} \left(\lfloor 2n/p^j \rfloor - 2\lfloor n/p^j \rfloor \right).$$

Each of these terms is 0 or 1, and so $r(p, n) \leq \lfloor \log 2n/ \log p \rfloor$. Consequently $p^{r(p,n)} \leq 2n$. For $p > \sqrt{2n}$ we have $\lfloor \log 2n/ \log p \rfloor \leq 1$. Hence for $p > \sqrt{2n}$ we have $r(p, n) = \lfloor 2n/p \rfloor - 2\lfloor n/p \rfloor$. By assumption there are no primes p with $n < p < 2n$. If $2n/3 < p \leq n$, then $p > \sqrt{2n}$ and $r(p, n) = \lfloor 2n/p \rfloor - 2\lfloor n/p \rfloor = 2 - 2 = 0$. Thus each prime factor of $\binom{2n}{n}$ is at most $2n/3$. Hence

$$\binom{2n}{n} = \prod_{p \leq 2n} p^{r(p,n)} \leq \prod_{p \leq \sqrt{2n}} 2n \cdot \prod_{p \leq 2n/3} p \leq (2n)^{\sqrt{2n}} \exp(\vartheta(2n/3)).$$

Chebyshev's inequality gives

$$\frac{4^n}{2n + 1} \leq \binom{2n}{n} \leq (2n)^{\sqrt{2n}} 4^{2n/3}.$$

For our values of n, $2n+1 < (2n)^2$ and so

$$4^{n/3} \leq (2n)^{2+\sqrt{2n}}.$$

Also $2 \leq \sqrt{2n}/3$ and so

$$4^{n/3} \leq (2n)^{4\sqrt{2n}/3}.$$

Taking logarithms gives

$$\sqrt{2n}\log 2 \leq 4\log 2n.$$

Write $2n = 4^t$ so that $t \geq 6$. We then get

$$2^t/t \leq 8.$$

The function $x \mapsto 2^x/x$ is increasing for $x > 1/\log 2$. So $2^t/t \geq 2^6/6 > 10$, a contradiction.

Now assume $2 \leq n < 2^{11}$. Then one of the prime numbers 3, 5, 7, 13, 23, 43, 83, 163, 317, 631, 1259, 2503 satisfies $n < p < 2n$ as each is less than twice its predecessor. This completes the proof.