A survey of the cell-growth problem and some its variations [1]

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Abstract

A very brief survey of the main results concerning the cell–growth problem and its variations is given. The name stems from an analogy with an animal which, starting from a single cell of some specified basic polygonal shape, grows step by step in the plane by adding at each step a cell of the same shape to its periphery. The fundamental combinatorial problem concerning these animals is "How many animals with n cells are there?" This problem was included in the list of unsolved problems in the enumeration of graphs by Frank Harary in 1960. Despite serious efforts over the last 40 years, this problem is completely open. However, a few asymptotic results are known. For example, let $p(n)$ denote the number of polyominoes (square animals) having n cells. It was proved that $(p(n))^{1/n}$ tends to a limit Θ, which satisfies the following inequality: $3.87 < \Theta < 4.65$. The situation could hardly be worse, since the first digit of Θ is not even known...

The difficulty of the classical cell–growth problem has led to the study of various restricted classes of polyominoes. Some variations of this problem are considered. Unsolved problems are stated. Chemical applications of this problem are mentioned too.

1. Classical cell–growth problem

Combinatorial problem known as *cell-growth problem* is stated as follows [1–7]. The name stems from an analogy with an *animal* which, starting from a single *cell* of some specified basic polygonal shape, grows step by step in the plane by adding at each step a cell of the same shape to its periphery. Thus if the basic shape is a square, the animals are the *polyominoes* (Fig.1a). If the basic shape is an equilateral triangle or a regular hexagon, we obtain triangular and hexagonal animals looking like those in Fig.1b and Fig.1c. Animals are defined as *simply–connected* ones if they have no *holes* and as *multiply–connected* ones otherwise. All animals presented in Fig.1 are simply–connected ones. The smallest multiply–connected polyomino is shown in Fig.2.

The fundamental combinatorial problem concerning these animals is *"How many animals with n cells are there?"* This problem was included in the list of unsolved problems in the enumeration of graphs by Harary in 1960 [8]. Polyominoes have the most long history, going to the start of the 20th century, but

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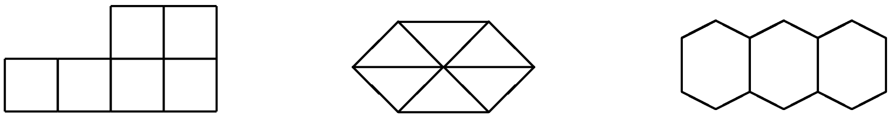


a *b* *c*

Figure 1. Simply-connected square (a) , triangular (b) and hexagonal (c) animals

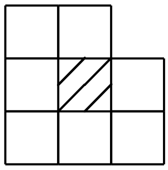


Figure 2. The smallest multiply-connected polyomino

they were popularized in the present era by Golomb [9–11] and by Gardner [12, 13] in his *Scientific American* columns "Mathematical Games". Another notable book on the subject is written by Martin [14]. There are a great many articles and problems concerning polyominoes to be found in the magazine *Recreational Mathematics* [15–20].

The answer on the main question "how many animals are there?" depends on how we distinguish animals. There are some distinguishing rules commonly used, and for each set there is a name for the animals.

Free animals are considered distinct if they have different shapes. Their orientation and location in the plane is no importance. For example, the two animals:

are the same free square animal since they differ only in orientation. We use *free(n)* to denote the number of free animals with n cells.

Fixed animals are considered distinct if they have different shapes or orientations. Thus two animals above are different fixed animals. We use *fixed(n)* to denote the number of fixed animals with n cells.

Originally the cell–growth problem was considered for the polyominoes. The most general discussion of polyominoes was done by Golomb [10], however the number of polyominoes was only briefly discussed. In 1962 Read [21] derived several theoretical results about the number of polyominoes. He presented a method for deriving generating functions to calculate the number of simply–connected and multiply–connected polyominoes, but these become intractable very quickly. He calculated *free(n)* only for n up to 10 and his value for $n = 10$ was incorrect.

Klarner [22, 23] found bounds for *free(n)* and *fixed(n)* polyominoes. The values seem to be growing exponentially, and indeed they have exponential bonds. It is easy to see that for each n,

$$\frac{fixed(n)}{8} \leq free(n) \leq fixed(n)$$

Eden [24] seems to have been the first person to give upper and lower bounds for *fixed(n)*. His bounds are

$$(3.14)^n < fixed(n) < 4^n,$$

for sufficiently large n. The proof of his upper found was questionable. Later these bounds were improved by Klarner and Rivest [25]. Using automata theory and building on earlier works of Eden, Klarner and Read they have shown

Theorem 1 [25]

$$\lim_{n\to\infty} (fixed(n))^{\frac{1}{n}} = \Theta \quad \textit{exists}, \quad \textit{and} \quad 3.87 < \Theta < 4.65. \tag{1}$$

Considerable effort has been expended to find a formula for the number of fixed polyominoes, with no success. Lunnon [26] has made the most successful previous enumeration. He computed the numbers of free, fixed and symmetric polyominoes up to 18 cells. Later Lunnon [27] computed the numbers of free and fixed triangular and hexagonal animals up to $n = 16$ and $n = 12$ respectively. The results are given in Table 1 and Table 2.

Table 1. The numbers of fixed and free triangular animals [27]

n	*fixed(n)*	*free(n)*
1	2	1
2	3	1
3	6	1
4	14	3
5	36	4
6	94	12
7	250	24
8	675	66
9	1838	160
10	5053	448
11	14.016	1186
12	39.169	3334
13	110.194	9235
14	311.751	26.166
15	886.160	73.983
16	2.529.260	211.297

Table 2. The numbers of fixed and free hexagonal animals [27]

n	*fixed(n)*	*free(n)*
1	1	1
2	3	1
3	11	3
4	44	7
5	186	22
6	814	82
7	3652	333
8	16.689	1448
9	77.359	6572
10	362.671	30.490
11	1.716.033	143.552
12	8.182.213	683.101

Table 3. The numbers of fixed and free polyominoes [28]

n	*fixed(n)*	*free(n)*
1	1	1
2	2	1
3	6	2
4	19	5
5	63	12
6	216	35
7	760	108
8	2725	369
9	9910	1285
10	36.446	4655
11	135.268	17.073
12	505.861	63.600
13	1.903.890	238.591
14	7.204.874	901.971
15	27.394.666	3.426.576
16	104.592.937	13.079.255
17	400.795.844	50.107.909
18	1.540.820.542	192.622.052
19	5.940.738.676	742.624.232
20	22.964.779.660	2.870.671.950
21	88.983.512.783	11.123.060.678
22	345.532.572.678	43.191.857.688
23	1.344.372.335.524	168.047.007.728
24	5.239.988.770.268	654.999.700.403

Redelmeier [28] enumerated all free and fixed polyominoes up to 24 cells. His algorithm, which produced the entries in Table 3 (and took over ten months of computer time to run), generates the fixed polyominoes one by one and counts them. The running time is (necessarily) exponential. At present, the computation of fixed(n) for $n > 30$ seems intractable.

Klarner [29] presented some unsolved problems arising in the cell–growth problem for polyominoes.

Problem 1. *Can the number of fixed animals with n cells be computed by a polynomial–time algorithm?*

A related problem concerns the constant Θ defined above.

Problem 2. *Is there a polynomial algorithm to find, for each n, an approximation Θ_n of Θ satisfying*

$$10^{-n} < |\Theta_n - \Theta| < 10^{-n+1}?$$

The lower–bound method of Klarner and Satterfield [30] gives an algorithm for approximating Θ from below that has exponential complexity; no such method is known for approximating Θ from above.

Problem 3. *Define some decreasing sequence $\beta = (\beta_1, \beta_2, ...)$ that tends to Θ, and give an algorithm to compute β_n for every n.*

It is known that $(fixed(n)^{1/n}) \leq \Theta$ for all n, and it seems that the ratios $\tau(n) = fixed(n+1)/fixed(n)$ increase for all n. If the latter is true, $\tau(n)$ would approach Θ from below. This gives two more unsolved problems:

Problem 4. *Show that* $(fixed(n))^{1/n} \leq (fixed(n+1))^{1/(n+1)}$ *for all* n

Problem 5. *Show that* $\tau(n) \leq \tau(n+1)$ *for all* n

Problem 6. *Is the generating function $T(z) = \sum_{n=1}^{\infty} fixed(n) z^n$ rational function?*
Is $T(z)$ even algebraic?

One can consider all of these problems for triangular and hexagonal animals.

So we gave the answer for the following question *"How many free and fixed animals with n cells are there?"*

Actually one can say about another distinguishing rule among animals. We can ask *"How many simply–connected and multiply–connected animals with n cells are there?"*

Read [21] calculated the numbers of simply–connected and multiply–connected square animals up to $n = 10$. Later Trinajstić, etc., [31, 32] computed the numbers of simply–connected animals up to 10 cells and the numbers of multiply–connected animals with the only hole up to 10 cells.

The hexagonal animals which are also sometimes called *polyhexes* correspond to the structural formulas of planar polycyclic aromatic hydrocarbons [33–35]. That is the reason why polyhexes have found a big interest among chemists [36–51]. Moreover one more distinguishing rule among polyhexes was considered. For simply–connected hexagonal animals it was done the answer on the following question *"How many animals with n cells and i internal vertices are there?"* This classification is important for

cata-condensed hydrocarbon
a

peri-condensed hydrocarbon
b

Figure 3. Hexagonal animals depicting dibenzo[a,c]anthracene (a) and benzo[e]pyrene (b)

chemists because the hexagonal simply–connected animals without internal vertices correspond to the cata–condensed benzenoid hydrocarbons and the hexagonal simply–connected animals with internal vertices correspond to the peri–condensed benzenoid hydrocarbons (see Fig.3). The obtained results are given in Table 4.

The same classification was used for square and triangular animals by Konstantinova [52, 53]. The numbers of simply–connected square and triangular animals without and with internal vertices up to 11 and 13 cells correspondingly are given in Table 5 and Table 6.

Table 4. The numbers of simply–connected hexagonal animals with i internal vertices [31]

$n\backslash i$	0	1	2	3	4	5	6	7	8	9	10	*total*
1	1											1
2	1											1
3	2	1										3
4	5	1	1									7
5	12	6	3	1								22
6	36	24	14	4	3							81
7	118	106	68	25	10	3	1					331
8	411	453	329	144	67	21	9	1				1435
9	1489	1966	1601	825	396	154	55	15	4			6505
10	5572	8395	7652	4518	2340	1018	416	123	42	9	1	30086

Table 5. The numbers of simply–connected square animals with i internal vertices [52]

$n\backslash i$	0	1	2	3	4	5	*total*
1	1						1
2	1						1
3	2						2
4	4	1					5
5	11	1					12
6	27	7	1				35
7	82	21	4				107
8	250	90	21	2			363
9	815	334	89	9	1		1248
10	2685	1311	391	67	6		4460
11	9072	4978	1674	324	45	1	16094

Table 6. The numbers of simply-connected triangular animals with i internal vertices [53]

$n\backslash i$	0	1	2	3	*total*
1	1				1
2	1				1
3	1				1
4	3				3
5	4				4
6	11	1			12
7	23	1			24
8	62	4			66
9	148	11			159
10	405	38	1		444
11	1041	118	2		1161
12	2825	386	15		3226
13	7541	1189	54	1	8785

She also presented the numbers of multiply–connected square [52], triangular [53] and hexagonal [54] animals with respect to the type of holes.

All multiply–connected square animals with the fixed internal boundaries presented in Fig.4 were generated and enumerated. The obtained data for $n = 9, n = 10, n = 11$ are given in Table 7, Table 8 and Table 9 correspondingly. In these tables t is the type of an internal boundary (see Fig.4) and i is the number of internal vertices. The total numbers of multiply–connected square animals with $7 \le n \le 11$ are given in Table 10. The total numbers of simply– and multiply–connected square animals for $n \le 11$ are given in Table 11. These data correspond to the numbers of free square animals [26, 28].

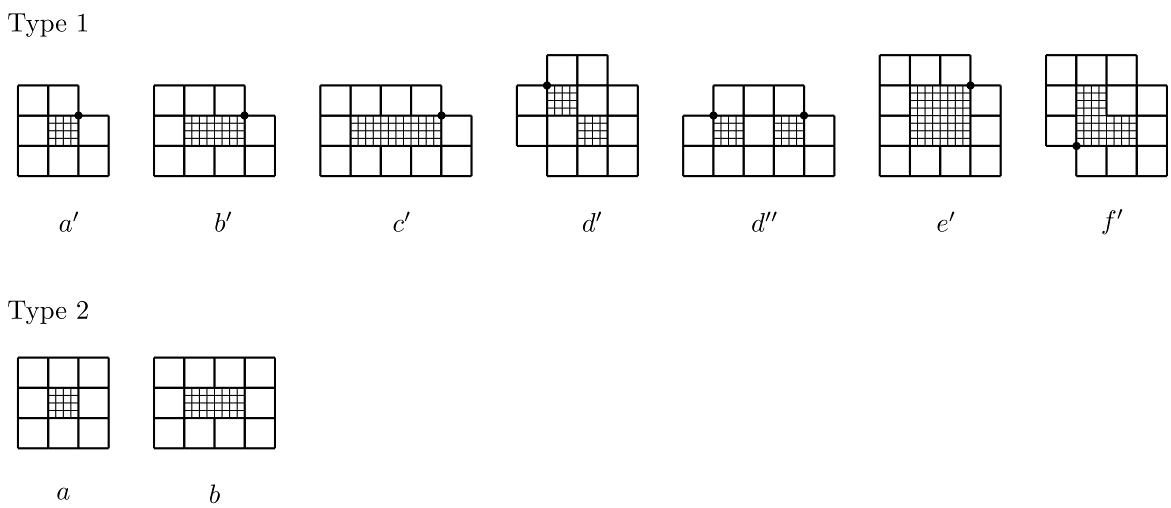


Figure 4. The different types of internal boundaries for multiply–connected square animals with up to 11 cells

Table 7. The numbers of multiply–connected square animals with $n = 9$

$i \backslash t$	a′	a	b′	*total*
0	31	2	1	34
1	3			3
total	34	2	1	37

Table 8. The numbers of multiply–connected square animals with $n = 10$

$i \backslash t$	a′	a	b′	b	*total*
0	132	14	12	1	159
1	34	1			35
2	1				1
total	167	15	12	1	195

Table 9. The numbers of multiply–connected square animals with $n = 11$

$i \backslash t$	a'	a	b'	b	c'	d'	d''	e'	f'	*total*
0	575	52	78	4	1	2	2	1	3	718
1	213	13	8							234
2	26	1								27
total	814	66	86	4	1	2	2	1	3	979

Table 10. The total numbers M of multiply–connected square animals

n	7	8	9	10	11
M	1	6	37	195	979

Table 11. The total numbers of simply– and multiply–connected square animals

n	1	2	3	4	5	6	7	8	9	10	11
data	1	1	2	5	12	35	108	369	1285	4655	17073

Moreover the diagrams of all multiply–connected square animals up to 11 cells are given in [52] and diagrams of all multiply–connected triangular animals up to 13 cells are given in [53].

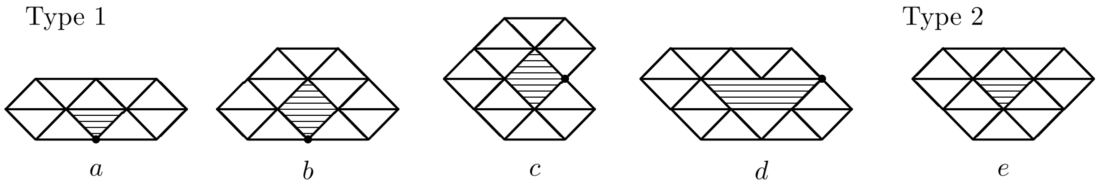


Figure 5. The different types of internal boundaries for multiply–connected triangular animals with up to 13 cells

All multiply–connected triangular animals with the fixed internal boundaries presented in Fig.5 were generated and enumerated. The obtained data are given in Table 12. In this table t is the type of an internal boundary, i is the number of internal vertices and n is the number of cells. The total numbers of simply– and multiply–connected triangular animals for $n \leq 13$ are given in Table 13. These data correspond to the numbers of free triangular animals [27].

Table 12. The number of multiply–connected triangular animals

i	0					1	
$n \backslash t$	a	b	c	d	e	a	total
9	1						1
10	4						4
11	24	1					25
12	100	5	1		1	1	108
13	405	29	5	1	2	8	450

Table 13. The total number of simply– and multiply–connected triangular animals

n	1	2	3	4	5	6	7	8	9	10	11	12	13
data	1	1	1	3	4	12	24	66	160	448	1186	3334	9235

All multiply–connected hexagonal animals with the fixed internal boundaries presented in Fig.6 were generated and enumerated. The obtained data are given in Table 14. In this table t is the type of an internal boundary, i is the number of internal vertices and n is the number of cells. The total numbers of simply– and multiply–connected hexagonal animals for $n \leq 9$ are given in Table 15. These data correspond to the numbers of free hexagonal animals [27].

Table 14. The number of multiply–connected hexagonal animals

i	0			1		2	3	4	
$n \backslash t$	a	b	c	a	b	a	a	a	total
6	1								1
7	1			1					2
8	5	1		3		4			13
9	17	2	1	17	2	17	10	1	67

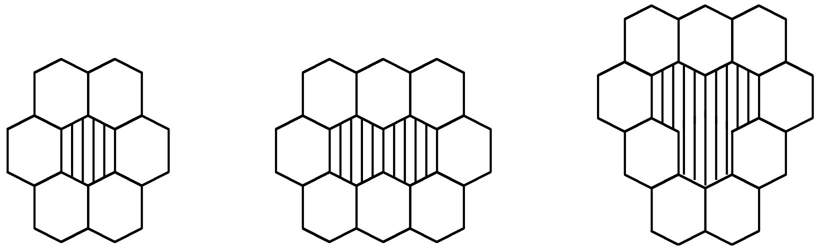


Figure 6. The different types of internal boundaries for multiply–connected hexagonal animals with up to 9 cells

Table 15. The total number of simply– and multiply–connected hexagonal animals

n	1	2	3	4	5	6	7	8	9
data	1	1	3	7	22	82	333	1448	6572

Sometimes more careful classification is used for simply–connected hexagonal animals [55]. *Unbranched tree–like polyhexes* have only two terminal cells (see Fig.1c). *Branched tree-like polyhexes* have more than two terminal cells (see Fig.3a). Unbranched tree–like polyhexes are the graph representations of unbranched cata–condensed benzenoid molecules, including helicenic species (non-embedded to the plane) and play a distinguished role in the theoretical chemistry of benzenoid hydrocarbons [35].

The unbranched tree–like polyhexes U_n were counted by Balaban and Harary [36]:

$$U_n = \begin{cases} \frac{1}{4}\,(3^{(n-2)/2}+1)^2, & \text{if } n \text{ is an even} \\ \frac{1}{4}\,(3^{n-2}+3^{(n-1)/2}+3^{(n-3)/2})^2, & \text{if } n \text{ is an odd} \end{cases} \tag{2}$$

Later Dobrynin [56] computed and generated all these polyhexes up to 16 cells. Some variations of this problem were considered by Cyvin, etc., [57–60] for unbranched tree–like systems of congruent polygons.

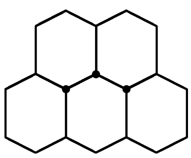
Cyvin [60] formulated the following problem in mathematical chemistry. Let P is the polyhex, i is the number of internal vertices in P and n is the number of hexagons in P. Then a very useful relation for polyhexes holds:

$$i \leq 2n - \lceil (12n-3)^{1/2} \rceil, \tag{3}$$

where $\lceil x \rceil$ is the smallest integer not smaller than x. The upper bound is realized in extremal animals [61] .

For example, for $n = 5$ extremal polyhex P looks like this one:

P



Using the above formula we exactly have the following upper bound

$$i \leq 2 \cdot 5 - \lceil (12 \cdot 5 - 3)^{1/2} \rceil = 10 - \lceil \sqrt{57} \rceil = 10 - 7 = 3, \tag{4}$$

which is realized in polyhex P.

Let define the mono–q–polyhex as the planar graph embedded to the mono–q–hexagonal lattice which is similar to the hexagonal lattice; it consists of exactly one q–gon and otherwise hexagons.

Many hydrocarbons correspond to the mono–q–polyhex graphs, e.g., the (q)circulenes. (5)circulene, (6)circulene, (7)circulene have been synthesized and a synthesis of (8)circulene has been attempted.

Let h be the number of hexagons outside the unique q–gon. Then the following conjecture is proposed for mono–q–polyhexes:

Problem 7 [60]. *Show that*

$$i \leq 2h - \lceil (1/2)(8qh + q^2)^{1/2} - (q/2) \rceil$$

The upper bound is supposed to be realized in the appropriate extremal systems.

One more problem immediately arise here.

Problem 8. *To enumerate all mono–q–polyhex with i internal vertices and h hexagons*

So the main results concerning the cell–growth problem and some unsolved problems arising there are considered above. Actually there is a lot of variations for this combinatorial problem. We will consider some of them.

2. Cell–growth problem for non–embedding animals

The classical cell–growth problem was formulated for the animals embedded to the plane. The animals non–embedded to the plane were investigated in the several papers [3, 22, 23, 36, 41, 56, 57, 62, 63].

As was mentioned above unbranched tree–like polyhexes embedded and non–embedded to the plane were considered by Balaban, Harary and Read [3, 36]. Moreover they have obtained the formula (2) for the number of unbranched tree–like polyhexes embedded and non–embedded to the plane. Actually in [3] it was shown how, by making a fairly drastic change in the definition of hexagonal animals, it is possible to arrive at a combinatorial problem for which an explicit solution exists.

In [62] a variation of the cell–growth problem for so–called $n-$*clusters* was considered. Here is a more formal recursive definition. The graph which is a polygon of order n ($n-$gon) is an $n-$*cluster*, and if G is an $n-$cluster of order p then the graph of order $p + (n-2)$ obtained by identifying an edge of a new $n-$gon with an edge of G lying in exactly one $n-$gon is again an $n-$cluster. The example of 6–cluster is given in Figure 7. Thus three–like polyhexes enumerated in [3] form a subset of hexagonal clusters. The generating function for $n-$clusters was obtained and the results for $3 \leq n \leq 6$ are given in Table 16.

It was also mentioned that the enumeration of $n-$clusters can be viewed as the counting of dissections of a polygon. One can draw a cluster so that its perimeter (the set of outer edges) appears as a regular

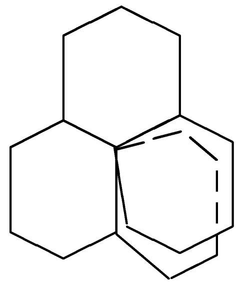


Figure 7. Hexagonal cluster

Table 16. The numbers of $n-$clusters, $3 \leq n \leq 6$, with h cells [62]

h	$n = 3$	$n = 4$	$n = 5$	$n = 6$
1	1	1	1	1
2	1	1	1	1
3	1	2	2	3
4	6	5	8	12
5	4	16	33	68
6	12	60	194	483
7	27	261	1196	3946
8	82	1243	8196	34.485
9	228	6257	58.140	315.810
10	733	32.721	427.975	2.984.570
11	2282	175.760	3.223.610	28.907.970
12	7528	963.900	24.780.752	285.601.251
13	24.834	5.374.400	193.610.550	2.868.869.733
14	83.898	30.385.256	1.534.060.440	29.227.904.840
15	285.357	173.837.631	12.302.123.640	301.430.074.416
16	983.244	1.004.867.079	99.699.690.472	3.141.985.563.575
17	3.412.420	5.861.610.475	815.521.503.060	33.059.739.636.198
18	11.944.614	34.469.014.515	6.725.991.120.004	
19	42.080.170	204.161.960.310	55.882.668.179.880	
20	149.197.152	1.217.145.238.485		
21	531.883.768	7.299.007.647.552		
22	1.905.930.975	44.005.602.441.840		
23	6.861.221.666			
24	24.806.004.996			
25	90.036.148.954			
26	327.989.004.892			
27	1.198.854.697.588			
28	4.395.801.203.290			
29	16.165.198.379.984			
30	59.609.171.366.325			

polygon. Then the cluster gives a dissection of the regular polygon into regions, each of which is an $n-$gon. The simplest of these dissection problems is the one for which $n = 3$, and concerns triangulations of the polygon. For some special cases this problem has been solved by Guy [64] and Motzkin [65]. In this connection see also the catalogue of sequences by Sloane [66], which corrects an error in Guy's list,

Table 17. The numbers of non–embedding polyhex NEP up to 10 hexagon [41]

n	1	2	3	4	5	6	7	8	9	10
NEP						1	8	71	542	3857

and also one in Motzkin's. Note also that the problem of counting clusters rooted at an exterior edge is equivalent to that of counting dissections of a fixed polygon, and has been considered in some detail by Motzkin. The case $n = 3$ for a fixed polygon is particularly well–known, having a history that extends all the way back to Euler, and gives rise to the ubiquitous Catalan number.

Some chemical enumerations of non-embedding animals take place too. Trinajstić, etc., [41] enumerated all simply–connected polyhexes non–embedded to the plane up to 10 hexagon. These polyhexes are the graph representations of cata–helicene and peri–helicene benzenoid hydrocarbons [55]. The data are shown in Table 17.

The following unsolved problems concerning non–embedding animals with n cells are here.

Problem 9. *To enumerate all simply–connected and multiply–connected non–embedding animals*

Problem 10. *To enumerate all non–embedding simply–connected animals with i internal vertices*

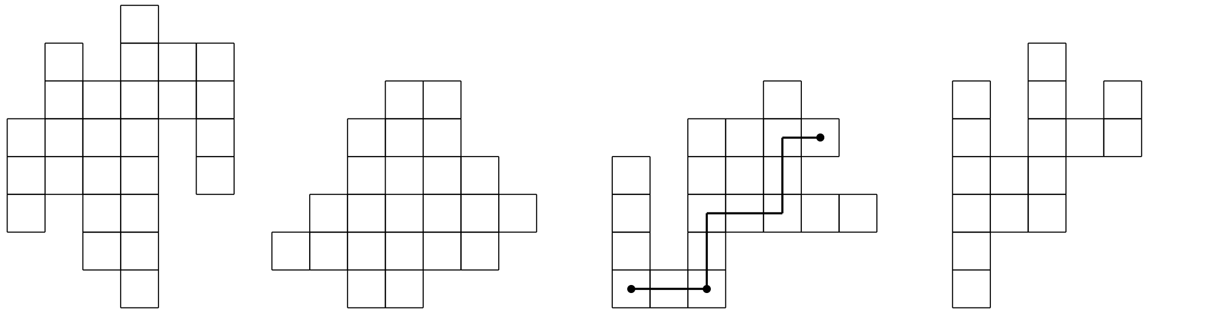
All previous considerations were dealing with the 2-dimensional case. Actually one can consider 3-dimensional case of the cell-growth problem and formulate the cell–growth problem for this case using, for example, cubes instead of squares:

Problem 11. *How many cubical animals are there?*

3. Variation of cell–growth problem: convex polyominoes

The difficulty of the classical cell–growth problem has led to the study of various restricted classes of polyominoes. Most of them can be defined by combining two notions: a geometric notion of *convexity*, and a notion of *directed growth*, which comes from statistical physics. Dhar [67, 68] presented the important example of the correspondence between the enumeration of *directed* polyominoes on a regular lattice in dimension D and the resolution of a gas–model in dimension $D - 1$.

A polyomino is said to be *vertically convex* (or *column–convex*) when its intersection with any vertical line is convex (see Fig.8). We can define similarly a notion of *horizontal* (or *row-*) convexity. A polyomino is *convex* if it is both vertically and horizontally convex. The *area* of a polyomino is the number of cells, and the *perimeter* is the length of the border. A polyomino is said to be *directed* when every it's cell can be reached from a distinguished cell, called a *root*, by a path that is contained in polyomino and has only North and East steps (see Fig.8).



Column–convex polyomino   Convex polyomino   Directed polyomino   Directed column–convex polyomino

Figure 8. Four main subclasses of polyominoes

Combining the two notions described above, one can already define four types of polyominoes, depending on whether they are only column–convex, or also row–convex, directed or not. Namely, here are the four main subclasses of polyominoes: column–convex polyominoes, convex polyominoes, directed and column–convex polyominoes, directed and convex polyominoes.

Usually the enumeration of these objects according to their perimeter and area is considered. Roughly, one can say that two kinds of generating functions occur, depending on the convexity properties of the class of polyominoes that is being enumerated. More precisely:

- the perimeter generating function for any usual convex polyominoes is an algebraic series, whereas the area generating function involves q-series; moreover, taking into account the perimeter (or the width and the height) when one already knows the area generating function is usually a rather easy task;

- the situation is different for families of column–convex polyominoes: the perimeter generating function and the area generating function are both algebraic; but the difficulty consists in taking into account simultaneously the two parameters.

Column–convex polyominoes apparently first appeared in Pólya's diary notes [69] and were independently introduced by Temperley [70]. The area generating function of these polyominoes was found on the spot [69, 70]. Klarner [22] has obtained the area generating function of row–convex polyominoes. He used the following method.

Let a composition of n with k parts is an ordered k-tuple $(a_1, \ldots, a_k)$ of positive integer with $a_1 + \ldots + a_k = n$ and let us assign to each composition a polyomino with n cells and with a horizontal strip of a_i cells in row i. Thus can be done in many ways, and the results are all row–column polyominoes. The examples of 6 row–convex polyominoes with 6 cells corresponding to the composition (3,1,2) of 6 are shown in Figure 9.

Since there are $(m+n-1)$ ways to form polyominoes with $(m+n)$ cells by placing a strip of n cells atop a strip of m cells, it follows that for each composition there are

$$(a_1 + a_2 - 1)(a_2 + a_3 - 1) \cdots (a_{k-1} + a_k - 1)$$

polyominoes with n cells having a strip of a_i cells in the ith row for each i.

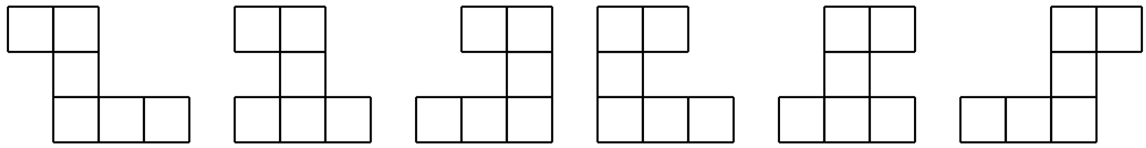


Figure 9. The 6 row–convex polyominoes with 6 cells corresponding to the composition (3,1,2) of 6

It follows that if $b(n)$ is the number of row–convex polyominoes with n cells, then

$$b(n) = \sum (a_1 + a_2 - 1)(a_2 + a_3 - 1) \cdots (a_{k-1} + a_k - 1),$$

where the sum extends over all compositions $(a_1, \ldots, a_k)$ of n into k parts, for all k. $b(n)$, and the area generating function $B(z) = \sum_{n=1}^{\infty} b(n) z^n$, are given by

Theorem 2 [22]

$$b(n+3) = 5b(n+2) - 7b(n+1) + 4b(n), \quad \text{and} \quad B(z) = \frac{z(1-z)^3}{1 - 5z + 7z^2 - 4z^3} \tag{5}$$

for $n = 2, 3, \ldots$, where $b(1) = 1, b(2) = 2, b(3) = 6, \ldots$.

Corollary 1.
$\lim_{n\to\infty} (b(n))^{\frac{1}{n}} = \beta$ where β is the largest real root of $z^3 - 5z^2 + 7z - 4 = 0$; $3.20 < \beta < 3.21$

More general case was considered by Bousquet–Mélou [71] in 1996 for directed column–convex polyominoes. Using 'Temperley methodology' [70] and building on her earlier works [72–79] she has obtained the generating function $V(x, y, n)$ in which the variables x, y, n mark horizontal and vertical edges of a perimeter and the number of cells

Theorem 3 [71]

$$V(x, y, n) = y^2 \frac{\sum_{i=1}^{\infty} \frac{x^{2i}(y^2-1)^{i-1} n^{i(i+1)/2}}{(n)_{i-1}(y^2 n)_{i-1}(y^2 n)_i}}{1 - \sum_{i=1}^{\infty} \frac{x^{2i}(y^2-1)^{i-1} n^{i(i+1)/2}}{(n)_i (y^2 n)_{i-1}(y^2 n)_i}} \tag{6}$$

The method which produced by formula (6) is markedly versatile. Besides the directed column–convex polyominoes, that method can be handle e.g. directed convex, convex, column–convex poliominoes and also some special classes such that parallelogram polyominoes [80]. Some common results for directed polyominoes on the triangular and hexagonal lattice was obtained in [81].

On the contrary, the perimeter generating function $G(x, y)$ of column–convex polyominoes remained unknown for many years after Pólya's and Temperley's works. At last Delest [82, 83] applied the DSV–methodology [84–88] and the computer algebra program MACSYMA to obtain a formula for $G(x, x)$. Subsequently, Brak, etc., [89] rederived the function $G(x, x)$ using the Temperley methodology and *Mathematica*. Thus it turned out that the formula given in [82] can be written in a simpler form. The result of Brak was generalized to the case $x \neq y$ by Lin [90] and confirmed by Feretić [91]. The following remarkably simple formula for $G(x, y)$ takes place:

$$G(x,y) = (1-y^2)\left[1 - \frac{2\sqrt{2}}{3\sqrt{2} - \sqrt{1 + x^2 + \sqrt{(1-x^2)^2 - 16x^2y^2/(1-y^2)^2}}}\right] \tag{7}$$

The area and perimeter generating functions of column–convex polyominoes and directed column–convex polyominoes were obtained also by Brak, etc., [92], Delest and Dulucq [93], Feretić [94].

4. Some related topics

Some another results concerning the discussed topic can be found in [95–99] results related to the classical cell-growth problem can be found in [100–116]. Let us mention some of them.

The polyominoes' problem defined by two vectors has been proposed Navit in 1992 in the course of the seminar held at the Dipartimento di Sistemi e Informatica di Fireze, on September 1992, on the subject *Tiling the plane with a horizontal bar* h_m *and a vertical bar* v_n. It is the problem of establishing the existence of a polyomino with a given number of cells in every column and every row. The problem is solved by Lungo [109] for the following classes of polyominoes: directed column-convex, directed convex, and parallelogram. The problem is also solved in the class of convex polyominoes in a particular case. Also, for each of these classes an algorithm is defined which controls the existence of a polyomino for given vectors.

The following problem concerns polyominoes radically different from convex ones.

Problem 12. **[29]** *Find the smallest natural number* n *such that there exists a polyomino with* n *cells and with no row or column consisting of just a single strip of cells*

An example of a polyomino with 21 cells with this property is shown in Figure 10.

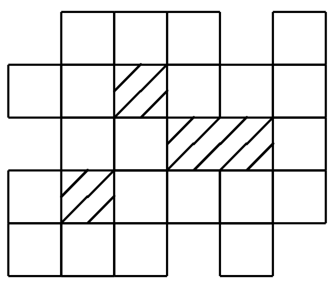


Figure 10. A polyomino with 21 cells and with no row or column a single strip of cells

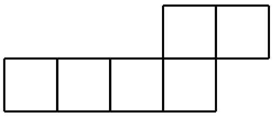


Figure 11. Snaky polyomino

Achievement games for polyominoes are frequently discussed in the literature [18–20,102,111,112,116]. For a given polyomino P two players A and B alternately mark the cells of the tessellation as game board. The player who first completes a copy of P with his marks wins the game. A polyomino P is called a winner if the first player A can win regardless of the moves made by B. Otherwise, P is called a loser.

For the triangular tessellation there are three winners and all other polyominoes are losers (see [20]). For the square tessellation 11 polyominoes are known to be winners. All others except one undecided polyomino, called Snaky (see Fig.11), are losers [111]. For the hexagonal tessellation all but five polyominoes with at most five cells are determined as winners or losers [116]. It may be remarked that for the five platonic solids as game boards all winners and losers are determined in [112].

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