ON AN INVARIANT RELATED TO A LINEAR INEQUALITY

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Abstract. Let $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m_{>0}$. Let $\underline{\alpha}_{i,j}$ be the vector obtained from $\underline{\alpha}$ on deleting the entries α_i and α_j. We investigate some invariants and near invariants related to the solutions $\underline{\epsilon} \in \{\pm 1\}^{m-2}$ of the linear inequality $|\alpha_i - \alpha_j| < \langle \underline{\epsilon}, \underline{\alpha}_{i,j} \rangle < \alpha_i + \alpha_j$, where $\langle , \rangle$ denotes the usual inner product. One of our methods relates, by the use of Rademacher functions, integrals involving products of trigonometric functions to these quantities.

1. Introduction

The purpose of this note is to construct a certain invariant related to a linear inequality. To give an example, consider the numbers $4, 6, 7, 9$ and 11. Pick a pair out of them, say 4 and 6. Then consider the linear combinations of the form $\pm 7 \pm 9 \pm 11$ that are in the open interval $(|4-6|, 4+6)$. There are two of them: $1\cdot 7 + (-1)\cdot 9 + 1\cdot 11$, which we give weight $1\cdot(-1)\cdot 1 = -1$ and $1\cdot 7 + 1\cdot 9 + (-1)\cdot 11$, which we give weight $1\cdot 1\cdot(-1) = -1$. Adding yields -2. If we now pick any two other numbers and repeat the same construction, we also get -2. This is no coincidence; one obtains the same invariance result for any sequence $\alpha_1, \cdots, \alpha_m$ of positive reals with m is odd, provided that $\pm\alpha_1 \pm \alpha_2 \pm \cdots \pm \alpha_m \neq 0$. The value of the invariant depends on the numbers chosen at the outset. If the collection of numbers has even size, we can only obtain an invariance result modulo 2.

A word about the structure of this note: We first give the results and a direct proof that the quantity computed is independent of the choice of the two numbers. Based on our method of proof we can then supply a closed formula for this quantity. At the same time, having found the closed formula, we can give a much easier alternative proof of the theorem. We also give some explanation regarding our choice of weights. Finally, in Section 3 we rederive some of the results using Rademacher functions and obtain some new ones.

2. Results

Theorem 2.1. *Let $m \geq 3$. Let $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m_{>0}$ and suppose that there is no $\underline{\epsilon} \in \{\pm 1\}^m$ satisfying $\langle \underline{\epsilon}, \underline{\alpha} \rangle = 0$. Let $1 \leq i < j \leq m$. Let $\underline{\alpha}_{i,j} \in \mathbb{R}^{m-2}_{>0}$ be the vector obtained from $\underline{\alpha}$ on deleting α_i and α_j. Let*

$$S_{i,j}(\underline{\alpha}) := \{\underline{\epsilon} \in \{\pm 1\}^{m-2} : |\alpha_i - \alpha_j| < \quad \langle \underline{\epsilon}, \underline{\alpha}_{i,j} \rangle \quad < \alpha_i + \alpha_j\}.$$

a) *The reduction of $\# S_{i,j}(\underline{\alpha})$ mod 2 only depends on $\underline{\alpha}$.*

b) *Define $N_{i,j}(\underline{\alpha}) = \sum_{\underline{\epsilon} \in S_{i,j}} \prod_{k=1}^{m-2} \epsilon_k$. Suppose that m is odd. Then $N_{i,j}(\underline{\alpha})$ only depends on $\underline{\alpha}$.*

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Corollary 2.2. *Suppose $m = 4$. Then $|N_{i,j}(\underline{\alpha})|$ only depends on $\underline{\alpha}$.*

Proof. In this case obviously $\#S_{i,j}(\underline{\alpha}) \in \{0, 1, 2\}$. If $\#S_{i,j}(\underline{\alpha}) \in \{0, 2\}$, then $|N_{i,j}(\underline{\alpha})|$ is easily seen to equal zero. If $\#S_{i,j}(\underline{\alpha}) = 1$, then $|N_{i,j}(\underline{\alpha})| = 1$. By property a) it then follows that $|N_{i,j}(\underline{\alpha})|$ is an invariant. □

Remark 2.3. An alternative notation for $N_{i,j}(\underline{\alpha})$ and $S_{i,j}(\underline{\alpha})$ we will use that turns out to be more convenient on occasion is $N_{\alpha_i,\alpha_j}(\underline{\alpha})$, respectively $S_{\alpha_i,\alpha_j}(\underline{\alpha})$.

Proof of Theorem 2.1. One checks the result easily in the case $m = 3$, so we assume that $m \geq 4$. Suppose we are given two pairs of indices, (i_1, j_1) and (i_2, j_2), with one index repeated. We can choose an index l different from all of the above indices. We will show that as function of α_l we have $N_{i_1,j_1}(\underline{\alpha}) = N_{i_2,j_2}(\underline{\alpha})$ when the other components of $\underline{\alpha}$ are kept fixed, and that the same holds with $N_{*,*}$ replaced by $\#S_{*,*}$ modulo 2. If this holds for all possible pairs with the above restriction, then also $N_{i_1,j_1}(\underline{\alpha}) = N_{i_2,j_2}(\underline{\alpha})$ holds in the case the indices are all different (and the same with $N_{*,*}$ replaced by $\#S_{*,*}$ modulo 2).

Notice that if α_l is sufficiently large, then $S_{*,*}(\underline{\alpha}) = \{\emptyset\}$. Thus in this case we get an equality with $\#S_{*,*}(\underline{\alpha}) = 0$ and $N_{*,*}(\underline{\alpha}) = 0$. As α_l decreases, a change in some $S_{i,j}(\underline{\alpha})$ will only occur if α_l crosses the point where $|\alpha_j - \alpha_i| = \langle \underline{\epsilon}, \underline{\alpha}_{i,j} \rangle$ or $\alpha_j + \alpha_i = \langle \underline{\epsilon}, \underline{\alpha}_{i,j} \rangle$, that is at most at those α_l such that $\langle \underline{\epsilon}, \underline{\alpha} \rangle = 0$ for some $\underline{\epsilon} \in \{\pm 1\}^m$. We will actually see that on moving across such $\underline{\alpha}$ a change always occurs. In order to prove Theorem 2.1 it is enough to prove that both $N_{i,j}(\underline{\alpha})$ and $\#S_{i,j}(\underline{\alpha})$ change by the same amount (respectively the same amount mod 2), independent of i, j, when we go from $(\alpha_1, \ldots, \alpha_{l-1}, \alpha_l + \delta, \ldots, \alpha_m)$ to $(\alpha_1, \ldots, \alpha_{l-1}, \alpha_l - \delta, \ldots, \alpha_m)$, where $\langle \underline{\epsilon}, \underline{\alpha} \rangle = 0$ and δ is sufficiently small, but positive. Without loss of generality we may assume that $\alpha_j \geq \alpha_i$. Put $\alpha_l^+ = \alpha_l + \delta, \alpha_l^- = \alpha_l - \delta$ and, for $k \neq l$, $\alpha_k^+ = \alpha_k^- = \alpha_k$. Let $N_{i,j}^{\pm}(\underline{\alpha})$ and $S_{i,j}^{\pm}(\underline{\alpha})$ have the obvious definitions. Note that $\langle \underline{\epsilon}, \underline{\alpha} \rangle = 0$ implies that

$$-\epsilon_j \sum_{k \neq i,j} \epsilon_k \alpha_k^+ = \alpha_j^+ + \epsilon_i \epsilon_j \alpha_i^+ - \epsilon_j \epsilon_l \delta$$

and also that

$$-\epsilon_j \sum_{k \neq i,j} \epsilon_k \alpha_k^- = \alpha_j^- + \epsilon_i \epsilon_j \alpha_i^- + \epsilon_j \epsilon_l \delta.$$

If $\epsilon_i \epsilon_j = 1$ and $\epsilon_j \epsilon_l = -1$, the passage from α_l^+ to α_l^- leads to $-\epsilon_j \cdot \underline{\epsilon}_{i,j}$ to be added to $S_{i,j}(\underline{\alpha})$. Similarly looking at the other possible values for $\epsilon_i \epsilon_j$ and $\epsilon_j \epsilon_l$, we see that for all sign possibilities a solution is added or deleted according to whether the sign of $(\epsilon_i \epsilon_j) \cdot (\epsilon_j \epsilon_l) = \epsilon_i \epsilon_l$ is negative, respectively positive and therefore we conclude that $\#S_{i,j}^-(\underline{\alpha}) = \#S_{i,j}^+(\underline{\alpha}) - \epsilon_i \epsilon_l$. In general, let $\underline{\epsilon}(1), \ldots, \underline{\epsilon}(s)$ denote all the different solutions to $\langle \underline{\epsilon}, \underline{\alpha} \rangle = 0$, where $\underline{\epsilon}$ and $-\underline{\epsilon}$ are considered the same solution. Each of them leads to a contribution of $\epsilon_i(r)\epsilon_l(r)$ to $\#S_{i,j}^-(\underline{\alpha}) - \#S_{i,j}^+(\underline{\alpha})$ that is not yet accounted for, where $1 \leq r \leq s$. If there would be further changes in the passage from $+$ to $-$ they would lead to additional solutions of $\langle \underline{\epsilon}, \underline{\alpha} \rangle = 0$. We deduce that $\#S_{i,j}^-(\underline{\alpha}) = \#S_{i,j}^+(\underline{\alpha}) - \sum_{r=1}^s \epsilon_i(r)\epsilon_l(r)$. In particular mod 2 the latter sum is independent of the choice of i and j. This proves part a. Note that $N_{i,j}(\underline{\alpha})$ changes by $\sum_{r=1}^s \epsilon_i(r)\epsilon_l(r) \prod_{k \neq i,j}(-\epsilon_j(r)\epsilon_k(r))$. If m is odd this is equal to

$-\sum_{r=1}^{s}\epsilon_l(r)\prod_{k=1}^{m}\epsilon_k(r)$. This is independent of i and j, thus proving part b. □

The more interesting part of Theorem 2.1, that is part b, raises the question of giving an alternative description of $N_{i,j}(\underline{\alpha})$ that does not involve i and j. The next theorem will give such a description. Let $\operatorname{sgn}(\beta)$ denote the function that equals 1 if $\beta > 0$, 0 if $\beta = 0$ and -1 if $\beta < 0$. Let $m \geq 3$ be odd. Let δ_0 be the delta distribution at 0. This is a generalised function which is the derivative of any step function of jump 1 at 0, e.g. $(1/2)\operatorname{sgn}$. When we checked how $N_{i,j}(\underline{\alpha})$ varied when we changed α_l, we were in fact computing the derivative of $N_{i,j}(\underline{\alpha})$ with respect to α_l. The computation we made in the proof of Theorem 2.1 clearly gives the following formula:

$$\frac{\partial N_{i,j}(\underline{\alpha})}{\partial \alpha_l} = -\frac{1}{2}\sum_{\underline{\epsilon}\in\{\pm 1\}^m}\epsilon_l\delta_0(\langle\underline{\epsilon},\underline{\alpha}\rangle)\prod_{k=1}^{m}\epsilon_k.$$

The factor of 1/2 comes from the fact that we count each solution together with its negative. This almost proved the next result:

Theorem 2.4. *For $m \geq 3$ odd and $\underline{\alpha}$ as in Theorem 2.1, we have*

$$N_{i,j}(\underline{\alpha}) = -\frac{1}{4}\sum_{\underline{\epsilon}\in\{\pm 1\}^m}\operatorname{sgn}(\langle\underline{\epsilon},\underline{\alpha}\rangle)\prod_{k=1}^{m}\epsilon_k. \tag{1}$$

We give two proofs in this section and another proof, based on Rademacher functions, in Section 3.

Proof 1. It follows from the computation above that both sides of (1) have the same partial derivatives. It is also easy to see that

$$\lim_{\alpha_l\to\infty}\text{R.H.S.} = -\frac{1}{4}\sum_{\underline{\epsilon}\in\{\pm 1\}^m}\epsilon_l\prod_{k=1}^{m}\epsilon_k = 0,$$

and the theorem follows. □

We wish, however, to give a second, more direct proof, which does not use the computation done while proving Theorem 2.1.

Proof 2. Denote the right hand side in (1) by $g(\underline{\alpha})$ and the summand by $h(\underline{\alpha},\underline{\epsilon})$. Since m is odd by assumption, we have $h(\underline{\alpha},-\underline{\epsilon}) = h(\underline{\alpha},\underline{\epsilon})$. Thus, we can write

$$g(\underline{\alpha}) = -\frac{1}{2}\sum_{\langle\underline{\epsilon}_{i,j},\underline{\alpha}_{i,j}\rangle>0}h(\underline{\alpha},\underline{\epsilon}),$$

where we sum over all $\underline{\epsilon}\in\{\pm 1\}^m$ satisfying the condition. (Note that

$$\sum_{\langle\underline{\epsilon}_{i,j},\underline{\alpha}_{i,j}\rangle=0}h(\underline{\alpha},\underline{\epsilon}) = 0.)$$

From this

$$g(\underline{\alpha}) = -\frac{1}{2}\sum_{\langle\underline{\epsilon}_{i,j},\underline{\alpha}_{i,j}\rangle>0}\left[\sum_{\epsilon_i,\epsilon_j}\epsilon_i\epsilon_j\operatorname{sgn}\Big(\langle\underline{\epsilon}_{i,j},\underline{\alpha}_{i,j}\rangle+\epsilon_i\alpha_i+\epsilon_j\alpha_j\Big)\right]\prod_{k\neq i,j}\epsilon_k.$$

One checks that the sum in the square brackets is 0 if $\underline{\epsilon}_{i,j}$ satisfies either $\left\langle \underline{\epsilon}_{i,j}, \underline{\alpha}_{i,j} \right\rangle > \alpha_i + \alpha_j$ or $\left\langle \underline{\epsilon}_{i,j}, \underline{\alpha}_{i,j} \right\rangle < |\alpha_j - \alpha_i|$, and is -2 if $|\alpha_j - \alpha_i| < \left\langle \underline{\epsilon}_{i,j}, \underline{\alpha}_{i,j} \right\rangle < \alpha_i + \alpha_j$. Thus

$$g(\underline{\alpha}) = \sum_{|\alpha_j - \alpha_i| < \left\langle \underline{\epsilon}_{i,j}, \underline{\alpha}_{i,j} \right\rangle < \alpha_i + \alpha_j} \prod_{k \neq i,j} \epsilon_k = N_{i,j}(\underline{\alpha}). \tag{2}$$

(The case where $\left\langle \underline{\epsilon}_{i,j}, \underline{\alpha}_{i,j} \right\rangle = |\alpha_i - \alpha_j|$ or $\left\langle \underline{\epsilon}_{i,j}, \underline{\alpha}_{i,j} \right\rangle = \alpha_i + \alpha_j$ does not occur, since by assumption $\langle \underline{\epsilon}, \underline{\alpha} \rangle \neq 0$.) □

Remark 2.5. Notice that $g(\underline{\alpha}) = 0$ when m is even.

Modulo 2 we have, by (2), that $g(\underline{\alpha}) = \sum_{|\alpha_j - \alpha_i| < \left\langle \underline{\epsilon}_{i,j}, \underline{\alpha}_{i,j} \right\rangle < \alpha_i + \alpha_j} 1 = N_{i,j}(\underline{\alpha})$ in case $m \geq 3$ is odd. Thus Theorem 2.1 a) follows in case m is odd. From Theorem 2.4 the validity of Theorem 2.1 b) immediately follows.

Is there a result similar to Theorem 2.1 b) when m is even? Clearly the exact same statement is false. It is however conceivable that there exists an assignment of weights to the elements of $S_{i,j}(\underline{\alpha})$ that would lead to a similar result. We now show that under certain assumptions this is impossible, while also suggesting how one might have guessed the correct form of the weights in Theorem 2.1 in the first place. We consider a weight function $f : \{\pm 1\}^{m-2} \to \mathbb{R}$. We would like f to be such that $N'_{i,j}(\underline{\alpha})$, where

$$N'_{i,j}(\underline{\alpha}) := \sum_{\underline{\epsilon} \in S_{i,j}(\underline{\alpha})} f(\underline{\epsilon}),$$

is independent of i and j. A look at the proof of Theorem 2.1 shows that f will have this property if and only if the following condition is satisfied:

$(*)$: For any $\underline{\epsilon} \in \{\pm 1\}^m$ the quantity $\epsilon_i f(-\epsilon_j \cdot \underline{\epsilon}_{i,j})$ is independent of i and j.

Indeed, since $N'_{i,j}(\underline{\alpha})$ is independent of whether $\alpha_j \geq \alpha_i$ or $\alpha_i < \alpha_j$, we can assume w.l.o.g. that $\alpha_j \geq \alpha_i$, the proof of Theorem 2.1 then shows that when we pass over a solution of $\langle \underline{\epsilon}, \underline{\alpha} \rangle = 0$, we gain or lose a solution according to the sign of $\epsilon_l \epsilon_i$, with l the coordinate that we vary, and that this solution is $-\epsilon_j \cdot \underline{\epsilon}_{i,j}$. $N_{i,j}$ therefore changes by $\epsilon_l \epsilon_i f(-\epsilon_j \cdot \underline{\epsilon}_{i,j})$ and we may neglect ϵ_l, since it is fixed in the argument. We now have the easy

Proposition 2.6. *If $m \geq 4$ is even, there is no function f satisfying the condition $(*)$. If $m \geq 3$ is odd, every f satisfying $(*)$ is of the form $f(\underline{\epsilon}) = C \cdot \prod_{k=1}^{m-2} \epsilon_k$, with C a constant.*

Proof. Let $m \geq 3$. Consider $(\epsilon_1, \ldots, \epsilon_r, \ldots, \epsilon_{m-2}) \in \{\pm 1\}^{m-2}$. If we apply condition $(*)$ to the vector

$$(\epsilon_1, \ldots, \epsilon_{r-1}, \epsilon_r, -\epsilon_r, \epsilon_{r+1}, \ldots, \epsilon_{m-2}, -1) \in \{\pm 1\}^m,$$

with $(i,j) = (r,m)$ and $(i,j) = (r+1,m)$, we see immediately that f has to satisfy

$$f(\epsilon_1, \ldots, \epsilon_r, \ldots, \epsilon_{m-2}) = -f(\epsilon_1, \ldots, -\epsilon_r, \ldots, \epsilon_{m-2}).$$

Therefore, $f(\underline{\epsilon}) = C \cdot \prod_{k=1}^{m-2} \epsilon_k$ with $C = f(1, 1, \ldots, 1)$. Now it is immediately checked that this function satisfies $(*)$ only if m is odd. □

2.1. **Shortening vectors.** The quantities above can be related to quantities of the same nature, but for shortened vectors. Let $\underline{\alpha}$ be a vector of the type allowed in Theorem 2.1. For $j \neq k$, let $\underline{\gamma}^{\pm}_{j,k}$ be the vector of length $m-1$ obtained from $\underline{\alpha}$ on replacing α_j by $|\alpha_j \pm \alpha_k|$ and deleting α_k. It can be deduced, for example, that if $m \geq 4$ and $\alpha_k \leq \alpha_j - \alpha_i$ with i, j and k distinct, then

$$\#S_{\alpha_i,\alpha_j}(\underline{\alpha}) = \#S_{\alpha_i,|\alpha_j-\alpha_k|}(\underline{\gamma}^{-}_{j,k}) + \#S_{\alpha_i,\alpha_j+\alpha_k}(\underline{\gamma}^{+}_{j,k}). \tag{3}$$

To see this, note that if $\alpha_k \leq \alpha_j - \alpha_i$ the number of $\underline{\epsilon} \in \{\pm 1\}^{m-2}$ with $\langle \underline{\epsilon}, \underline{\alpha}_{i,j} \rangle = \cdots + \alpha_k + \cdots$ and the number of $\underline{\epsilon}$ with $\langle \underline{\epsilon}, \underline{\alpha}_{i,j} \rangle = \cdots - \alpha_k + \cdots$, equals $\#S_{\alpha_i,|\alpha_j-\alpha_k|}(\underline{\gamma}^{-}_{j,k})$, respectively $\#S_{\alpha_i,\alpha_j+\alpha_k}(\underline{\gamma}^{+}_{j,k})$. We defer further discussion of shortening until Section 3.1, where a more powerful approach in uncovering and proving this type of identities is employed.

3. Results obtained by using Rademacher functions

Let $0 \leq t \leq 1$ be a real number. We define $\epsilon_i(t)$, $i = 1, 2, \cdots$ recursively. Suppose $\epsilon_1(t), \cdots, \epsilon_{i-1}(t)$ are already defined. Then we define $\epsilon_i(t)$ to be 1 if

$$\frac{1}{2^i} \leq t - \sum_{j=0}^{i-1} \frac{\epsilon_j(t)}{2^j}$$

and to be zero otherwise. Note that $0.\epsilon_1(t)\epsilon_2(t)\cdots$ gives a binary representation for t. We define $r_i(t) = 1 - 2\epsilon_i(t)$ for $i = 1, 2, \cdots$. The functions $r_i(t)$ are called Rademacher functions.

Using the Rademacher functions we will prove the following result, which shows again that $N_{*,*}(\underline{\alpha})$ only depends on $\underline{\alpha}$ in case m is odd. Using (4) it is then easy to give yet another proof of Theorem 2.4.

Theorem 3.1. *Let $m \geq 3$ and $\underline{\alpha}$ as be as in Theorem 1. Let $\beta_1, \cdots, \beta_m$ be positive integers and q a real number such that, for $k = 1, \cdots, m$,*

$$|\frac{\beta_k}{q} - \alpha_k| < \frac{\min_{\underline{\epsilon} \in \{\pm 1\}^m} |\langle \underline{\epsilon}, \underline{\alpha} \rangle|}{m},$$

and with the β's satisfying the same ordering and equalities as do the α's (that is if $\alpha_i \leq \alpha_j$, then $\beta_i \leq \beta_j$, where $1 \leq i, j \leq m$ with $i \neq j$). Suppose m is odd. Then

$$N_{i,j}(\underline{\alpha}) = (-1)^{\frac{m+1}{2}} \frac{2^{m-2}}{2\pi} \int_0^{2\pi} \cot(\frac{x}{2}) \sin(\beta_1 x) \cdots \sin(\beta_m x) dx. \tag{4}$$

Suppose m is even and $\alpha_i \leq \alpha_j$. Then

$$N_{i,j}(\underline{\alpha}) = (-1)^{\frac{m}{2}-1} \frac{2^{m-2}}{2\pi} \int_0^{2\pi} \cot(\frac{x}{2}) \cot(\beta_j x) \sin(\beta_1 x) \cdots \sin(\beta_m x) dx. \tag{5}$$

Let $m \geq 3$ be arbitrary and $\alpha_i \leq \alpha_j$. Then

$$\#S_{i,j}(\underline{\alpha}) = \frac{2^{m-2}}{2\pi} \int_0^{2\pi} \cot(\frac{x}{2}) \tan(\beta_i x) \cos(\beta_1 x) \cdots \cos(\beta_m x) dx. \tag{6}$$

Corollary 3.2.
a) Theorem 2.1 b) *holds true.*
b) If $m \geq 4$ is even, then $N_{*,*}(\underline{\alpha})$ depends only on the largest component omitted from $\underline{\alpha}$. This implies that $N_{*,*}(\underline{\alpha})$ assumes at most $m-1$ values.
c) The value of $\#S_{*,*}(\underline{\alpha})$ depends only on the smallest component omitted from $\underline{\alpha}$.

Proof of Theorem 3.1. The existence of q and $\underline{\beta} = (\beta_1, \cdots, \beta_m)$ is obvious. Clearly the equality $N_{i,j}(\underline{\alpha}) = N_{i,j}(q^{-1}\underline{\beta}) = N_{i,j}(\underline{\beta})$ holds and similarly we have $S_{i,j}(\underline{\alpha}) = S_{i,j}(\underline{\beta})$. We only prove the identity (4), the proofs of the other two being very similar.

Let s be an integer. Note that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{isx}dx = \begin{cases} 1 & \text{if s=0;} \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Let us consider $N_{m-1,m}(\underline{\alpha}) = N_{m-1,m}(\underline{\beta})$. Put $m_1 = m-2$. We have

$$N_{m-1,m}(\underline{\alpha}) = \frac{2^{m_1}}{2\pi} \sum_{s=|\beta_{m-1}-\beta_m|+1}^{\beta_{m-1}+\beta_m-1} \int_0^1 r_1(t)\cdots r_{m_1}(t) \int_0^{2\pi} e^{ix(\beta_1 r_1(t)+\cdots+\beta_{m_1} r_{m_1}(t)-s)}dx\ dt.$$

Every sequence of length m_1 of $+1$'s and -1's corresponds to one and only on interval $(j/2^{m_1}, (j+1)/2^{m_1})$, with $0 \leq j \leq 2^{m_1}-1$. Thus

$$\int_0^1 r_1(t)r_2(t)\cdots r_{m_1}(t)e^{ix\sum_{k=1}^{m_1}\beta_k r_k(t)}dt = 2^{-m_1}\sum_{\underline{\epsilon}\in\{\pm 1\}^{m_1}} (\prod_{k=1}^{m_1}\epsilon_k)e^{ix\sum_{k=1}^{m_1}\epsilon_k\beta_k}$$

$$= i^{m_1}\prod_{k=1}^{m_1}\frac{e^{i\beta_k x} - e^{-i\beta_k x}}{2i} = i^{m_1}\prod_{k=1}^{m_1}\sin(\beta_k x).$$

Our expression for $N_{m-1,m}(\underline{\alpha})$ can thus be rewritten as

$$N_{m-1,m}(\underline{\alpha}) = \frac{(2i)^{m_1}}{2\pi}\sum_{s=|\beta_{m-1}-\beta_m|+1}^{\beta_{m-1}+\beta_m-1}\int_0^{2\pi} e^{-ixs}\sin(\beta_1 x)\cdots\sin(\beta_{m_1}x)dx.$$

Let us consider the case where $\beta_{m-1} > \beta_m$. Note that

$$\sum_{s=\beta_{m-1}-\beta_m+1}^{\beta_{m-1}+\beta_m-1} e^{-ixs} = e^{-ix\beta_{m-1}}\frac{\sin((\beta_m-\frac{1}{2})x)}{\sin(\frac{x}{2})}.$$

Since by assumption m is odd, m_1 is odd and hence $(2i)^{m_1}$ is purely imaginary. Since, a priori, $N_{m-1,m}(\underline{\alpha})$ is real, we see that we only have to retain the purely imaginary part of $e^{-ix\beta_{m-1}}$, that is $-i\sin(\beta_{m-1}x)$. We thus find that

$$N_{m-1,m}(\underline{\alpha}) = -i\frac{(2i)^{m_1}}{2\pi}\int_0^{2\pi}\frac{\sin(\beta_1 x)\cdots\sin(\beta_{m-1}x)\sin((\beta_m-1/2)x)}{\sin(x/2)}dx. \tag{8}$$

Our assumption on q and $\beta_1, \cdots, \beta_m$ implies that $\langle\underline{\epsilon}, \underline{\beta}\rangle \neq 0$ for every $\underline{\epsilon} \in \{\pm 1\}^m$. Thus instead of summing from $s = \beta_{m-1} - \beta_m + 1$ to $s = \beta_{m-1} + \beta_m - 1$, we might as well sum from $s = \beta_{m-1} - \beta_m$ to $s = \beta_{m-1} + \beta_m$, this then yields

$$N_{m-1,m}(\underline{\alpha}) = -i\frac{(2i)^{m_1}}{2\pi}\int_0^{2\pi} \frac{\sin(\beta_1 x)\cdots\sin(\beta_{m-1}x)\sin((\beta_m + 1/2)x)}{\sin(x/2)}dx. \tag{9}$$

On noting that $\sin(\gamma-\delta)+\sin(\gamma+\delta) = 2\sin\gamma\cos\delta$, we finally obtain (4) for $i = m-1$ and $j = m$ on adding (8) to (9) and averaging. By a completely similar reasoning one deals with the case where $\beta_{m-1} \le \beta_m$ and one also finds (4). Obviously one also arrives at (4) if one considers $N_{i,j}(\underline{\alpha})$ for arbitrary $1 \le i < j \le m$. □

Remark 3.3.
a) The condition on the ordering of the β's is not needed in the derivation of (4). It is in (5) and (6) to infer that the assumption $\alpha_i \le \alpha_j$ implies that $\beta_i \le \beta_j$.
b) Since (7) only holds valid for integral s, we are forced to work with the approximation vector $\underline{\beta}$, rather than $\underline{\alpha}$ itself.
c) Using that the argument in (4) has period 2π and is an odd function if $m \ge 2$ and even, one deduces that the integral in (4) equals zero in this case.

3.1. **The shortening of vectors reconsidered.** The various formulae in Theorem 3.1 can be related to each other by invoking very elementary trigonometric identies such as $2\sin\alpha\sin\beta = \cos(\alpha+\beta)+\cos(\alpha-\beta)$. This then yields shortening formulae. In proving them, which is left to reader, one has to convince oneself that one can choose an 'approximation vector' $\underline{\beta}$ for $\underline{\alpha}$ that will also yield an approximation vector of the shortened vector(s) involved. An alternative method of proof is indicated in Section 3.2. Recall that $\underline{\gamma}^{\pm}_{j,k}$ is defined in Section 2.1.

Theorem 3.4. *Let $\underline{\alpha}$ and $\underline{\alpha}_{i,j}$ be defined as in Theorem 2.1.*
a) Suppose $m \ge 4$ and even and $\alpha_i \le \alpha_j$. Then

$$N_{i,j}(\underline{\alpha}) = \operatorname{sgn}(\alpha_j - \alpha_k)N_{*,*}(\underline{\gamma}^-_{j,k}) - N_{*,*}(\underline{\gamma}^+_{j,k}).$$

b) Let $m \ge 4$ and $\alpha_i \le \alpha_j$. Suppose furthermore there exist $r, s \ne i$ such that $\alpha_r + \alpha_s \ge \alpha_i$ and $|\alpha_r - \alpha_s| \ge \alpha_i$. Then

$$\#S_{\alpha_i,\alpha_j}(\underline{\alpha}) = \#S_{\alpha_i,|\alpha_r-\alpha_s|}(\underline{\gamma}^-_{r,s}) + \#S_{\alpha_i,\alpha_r+\alpha_s}(\underline{\gamma}^+_{r,s}).$$

c) Let $m \ge 5$ be odd. Suppose we have $\alpha_k \le |\alpha_i - \alpha_j|$ for some i, j, k with $k \ne i, j$. Then

$$N_{*,*}(\underline{\alpha}) = N_{\alpha_k,\alpha_i+\alpha_j}(\underline{\gamma}^-_{i,j}) - N_{\alpha_k,|\alpha_i-\alpha_j|}(\underline{\gamma}^+_{i,j}).$$

In case $\alpha_i = \alpha_j$ for some $i \ne j$ and $\alpha_k \le 2\alpha_i$ for some $k \ne i, j$, then

$$N_{*,*}(\underline{\alpha}) = -N_{\alpha_k,\alpha_i+\alpha_j}(\underline{\gamma}^+_{i,j}) - 2N_{*,*}(\underline{\alpha}_{i,j}).$$

3.2. **Theorem 2.4 reconsidered and some analoga.** In this subsection we present a third proof of Theorem 2.4 and present another theorem that can be proved using the same method of proof.

Third proof of Theorem 2.4. On inverting some of the last steps in the proof of Theorem 3.1, one easily checks that for integer β we have

$$\frac{1}{2\pi}\int_0^{2\pi} \cot(\frac{x}{2})\sin(\beta x)dx = \operatorname{sgn}(\beta). \tag{10}$$

On writing $\sin(\beta_j x)$ as $(e^{i\beta_j x} - e^{-i\beta_j x})/2i$ for $j = 1, \cdots, m$ in (4) and multiplying all these factors out, one gets a sum of terms of the form $e^{i\langle \underline{\epsilon}, \underline{\beta} \rangle x} \cos(x/2)/\sin(x/2)$, where the term $e^{i\langle -\underline{\epsilon}, \underline{\beta} \rangle x} \cos(x/2)/\sin(x/2)$ appears with opposite sign, due to the fact that m is odd. This allows one to rewrite (4) in the form

$$N_{i,j}(\underline{\alpha}) = -\frac{1}{4} \sum_{\underline{\epsilon} \in \{\pm 1\}^m} (\prod_{k=1}^{m} \epsilon_k) \frac{1}{2\pi} \int_0^{2\pi} \cot(\frac{x}{2}) \sin(\langle \underline{\epsilon}, \underline{\beta} \rangle x) dx.$$

By (10) we then find the expression for $N_{i,j}(\underline{\alpha})$ as given in Theorem 2.4 with $\langle \underline{\epsilon}, \underline{\alpha} \rangle$ replaced by $\langle \underline{\epsilon}, \underline{\beta} \rangle$. The proof is now completed on noting that $\underline{\beta}$ has the property that $\mathrm{sgn}(\langle \underline{\epsilon}, \underline{\beta} \rangle) = \mathrm{sgn}(\langle \underline{\epsilon}, \underline{\alpha} \rangle)$. □

The latter method of proof can also be applied to equalities (5) and (6) and then yields Theorem 3.5. Theorem 3.5 is also easily derived on employing the method of proof in Proof 2 of Theorem 2.4.

Theorem 3.5. *Let $m \geq 3$ and $\underline{\alpha}$ be as in* Theorem 2.1. *Suppose $\alpha_i \leq \alpha_j$. Then*

$$\#S_{i,j}(\underline{\alpha}) = \frac{1}{2} \sum_{\substack{\underline{\epsilon} \in \{\pm 1\}^m \\ \epsilon_i = 1}} \mathrm{sgn}(\langle \underline{\epsilon}, \underline{\alpha} \rangle).$$

If m is even, we have, moreover,

$$N_{i,j}(\underline{\alpha}) = \frac{1}{4} \sum_{\underline{\epsilon} \in \{\pm 1\}^m} \epsilon_j \mathrm{sgn}(\langle \underline{\epsilon}, \underline{\alpha} \rangle) \prod_{k=1}^{m} \epsilon_k.$$

Applying the method of proof of Theorem 2.4 on the right hand side of the latter identity one finds the following invariant in case m is even. Note that since i, j are required to be distinct from some prescribed number, the result is consistent with Proposition 2.6.

Theorem 3.6. *Let $\underline{\alpha}$ and $\underline{\alpha}_{i,j}$ be as in Theorem 2.1. Let $h \neq i, j$. Let h_1 be an index such that the h_1th component of $\underline{\alpha}_{i,j}$ equlas α_h. Define*

$$N_{i,j}^{(h)}(\underline{\alpha}) = \sum_{\underline{\epsilon} \in S_{i,j}} \epsilon_{h_1} \prod_{k=1}^{m-2} \epsilon_k.$$

Then, for $m \geq 4$ with m even, $N_{i,j}^{(h)}(\underline{\alpha})$ does not depend on i and j. If $\alpha_{h_2} \leq \alpha_h$ for some $h_2 \neq h$, then $N_{i,j}^{(h)}(\underline{\alpha}) = N_{h_2,h}(\underline{\alpha})$.

On invoking Theorems 2.4 and 3.5 it is not difficult to reprove Theorem 3.4. This is left to the reader.

3.3. **An example.** Let $p_1, p_2, \cdots$ denote the consecutive primes. Since for the natural integers we have unique factorisation (up to order of factors), we have that $\pm \log p_1 \pm \log p_2 \pm \cdots \pm \log p_n \neq 0$ and hence we can apply Theorem 2.1 with $\underline{\alpha}^{(n)} = (\log 2, \cdots, \log p_n)$. It is not difficult to show that, for $1 \leq i < j \leq n$,

$$\#N_{i,j}(\underline{\alpha}^{(n)}) = (-1)^n \sum_{\substack{\sqrt{p_1 \cdots p_n}/p_i < m < \sqrt{p_1 \cdots p_n} \\ (m, p_i p_j) = 1,\ P(m) \leq p_n}} \mu(m),$$

where $P(m)$ denotes the largest prime factor of m and μ the Möbius function. If $n \geq 3$ is odd, then the latter quantity does not depend on i and j by Theorem 2.1 b). If $n \geq 4$ is even, the latter quantity does not depend on i by Corollary 3.2 b). The values one finds of $N_{*,*}(\alpha^{(n)})$ for $n = 5, 7, 11, 13, 15$ are, respectively, $-1, 3, 22, -53, -55$.

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