The Number of Non-Orientable Coverings of the Klein Bottle *

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Abstract

The number of non-equivalent unbranched n-fold coverings of the Klein bottle by a non-orientable surface proves to be the multiplicative function $d_{\rm odd}(n)$ which is equal to the number of divisors m of n such that m or n/m is odd. Previously this was shown by one of the authors for odd n, in which case all n-fold coverings are non-orientable and $d_{\rm odd}(n) = d(n)$, the number of all divisors.

Keywords: non-orientable surface, odd divisor, multiplicative function, fundamental group, torus

1 Introduction

In the exposition in this note we use some well-known facts from the topology of 2-dimensional manifolds (see, e.g., [2]; see also [1]). Let $\mathcal{S}, \mathcal{U}$ and $\mathcal{U}'$ be compact surfaces. Two coverings of $\mathcal{S}$, $\rho : \mathcal{U} \to \mathcal{S}$ and $\rho' : \mathcal{U}' \to \mathcal{S}$, are called *equivalent* (or isomorphic) if $\mathcal{U}$ and $\mathcal{U}'$ are homeomorphic and there exists a homeomorphism $h : \mathcal{U} \to \tilde{\mathcal{U}}'$ such that $\rho' = \rho \circ h$. It is well known that the n-sheeted coverings of the surface $\mathcal{S}$ are in one-to-one correspondence with the subgroups of index n of the fundamental group $\pi_1(\mathcal{S})$, and two coverings are equivalent if and only if their corresponding subgroups are conjugate

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in $\pi_1(\mathcal{S})$. The classical Hurwitz enumeration problem is to count non-equivalent n-fold coverings of $\mathcal{S}$ satisfying various restrictions.

Let now $\mathcal{S}$ be a closed non-orientable surface. The number of non-equivalent non-ramified coverings of $\mathcal{S}$ was enumerated in [4]. In this case, the covering surface $\mathcal{U}$ may be orientable or non-orientable, and it is natural to count both types of coverings *separately*. However the enumeration of orientable and non-orientable coverings of $\mathcal{S}$ still remains an open problem despite the corresponding subgroups of the fundamental group $\pi_1(\mathcal{S})$ can be enumerated separately (see [4]). Moreover, this problem is open even for bordered non-orientable surfaces (in which case the fundamental group is free).

As was shown by one of the present authors [3], the enumeration can be simplified significantly for the case of the Klein bottle $\mathcal{K}$: the number of all coverings of $\mathcal{K}$ can be presented as a simple combination of well-known multiplicative number-theoretic functions (see below). Clearly, all coverings are non-orientable for odd n. But for even n, both types of coverings are possible, and no formula has been obtained for the numbers of orientable and non-orientable coverings of a Klein bottle. In the present note we fill this gap by giving a simple criterion when two different subgroups of the fundamental group of a torus $\mathcal{T}$ correspond to one orientable covering of the Klein bottle.

2 Orientable coverings of $\mathcal{K}$

Recall that the fundamental group of a Klein bottle is such: $\pi_1(\mathcal{K}) = \langle a, b \mid a^2b^2 = 1\rangle$. Let $\mathcal{T}$ be the orientable double of $\mathcal{K}$. Then $\mathcal{T}$ is a torus, and by the Reidemeister–Schreier method, $\pi_1(\mathcal{T}) = \langle a^2, ab, ba\rangle$. But $a^2b^2 = 1$, that is $aabb = 1$, whence $ab = a^{-1}b^{-1} = (ba)^{-1}$, and also $a^2 \cdot ab = ab \cdot a^2$. So $\pi_1(\mathcal{T}) = \langle a^2, ba\rangle$ is a *free abelian* group of rank 2. Set $x = a^2, y = ba$. Then

$$axa^{-1} = x \quad \text{and} \quad aya^{-1} = y^{-1} \tag{1}$$

in the group $\pi_1(\mathcal{K})$.

Remark The action of the element a by conjugation on $\pi_1(\mathcal{T}) = \langle x, y\rangle$ is induced by an orientation-reversing fixed-point-free involution $\iota_a : \mathcal{T} \to \mathcal{T}$ such that $\eta \circ \iota_a = \eta$ where $\eta : \mathcal{T} \to \mathcal{K}$ is a canonical double covering of $\mathcal{K}$ by $\mathcal{T}$. $\mathcal{K} = \mathcal{T}/\langle \iota_a\rangle$.

Suppose $n = 2k$. Then it is easy to see (cf. [3]) that there are $\sigma(k)$ (the sum of all divisors of n) non-equivalent k-fold coverings of the torus $\mathcal{T}$ and they can be presented by the following subgroups of index k in $\pi_1(\mathcal{T}) = \langle x, y\rangle$:

$$\Gamma = \Gamma_{\ell,m,j} = \langle x^\ell y^j, y^m\rangle, \quad \ell m = k,\ j = 0, 1, \ldots, m-1. \tag{2}$$

The exponents j are in fact considered modulo m.

Lemma 1 *Two subgroups Γ and Γ' are conjugate in $\pi_1(\mathcal{K}) = \langle a, b\rangle$ if and only if they coincide or are conjugate by a : $\Gamma' = a\Gamma a^{-1}$.*

Proof. Since $\pi_1(\mathcal{K}) = \pi_1(\mathcal{T}) + \pi_1(\mathcal{K})a$, there are only two possibilities for Γ and Γ' to be conjugate: by some $g \in \pi_1(\mathcal{T})$ or by ag. In the first case, $\Gamma' = \Gamma$ since $\pi_1(\mathcal{T})$ is abelian. In the second case, $\Gamma' = (ag)\Gamma(ag)^{-1} = a(g\Gamma g^{-1})a^{-1} = a\Gamma a^{-1} = \Gamma$. □

Corollary 1 *Two subgroups $\Gamma_{\ell,m,j}$ and $\Gamma_{\ell',m',j'}$ are conjugate in $\pi_1(\mathcal{K})$ if and only if $(\ell', m', j') = (\ell, m, j)$ or $(\ell', m', j') = (\ell, m, m - j)$.*

Proof. By Lemma 1 we have two possibilities. If $\Gamma_{\ell',m',j'} = \Gamma_{\ell,m,j}$ then $(\ell', m', j') = (\ell, m, j)$ since (2) is the full list of subgroups without repetitions. If, instead, $a\Gamma_{\ell',m',j'}a^{-1} = \Gamma_{\ell,m,j}$, then by (1), $a\langle x^\ell y^j, y^m\rangle a^{-1} = \langle x^\ell y^{-j}, y^{-m}\rangle = \langle x^\ell y^{m-j}, y^m\rangle$, and we are done. □

3 Enumeration

As a direct enumerative corollary of the previous considerations we obtain the following.

Proposition 1 *The number of conjugacy classes of orientable subgroups*[1] *of index $n = 2k$ in $\pi_1(\mathcal{K})$ is*

$$N^+_{\mathcal{K}}(2k) = \sum_{m|k} \left\lfloor \frac{m+2}{2} \right\rfloor. \tag{3}$$

$\mathcal{K}^+(n)$ is the number of orientable n-coverings of the Klein bottle $\mathcal{K}$.

Indeed, orientable subgroups of $\pi_1(\mathcal{K})$ of index $2k$ are just subgroups of $\pi_1(\mathcal{T})$ of index k. They are listed in (2) and their conjugacy in $\pi_1(\mathcal{K})$ is characterized by Corollary 1. Now if m, $m|k$, is even, then there are $1 + m/2$ pairs $(j, m - j)$, which give rise to one and the same orientable covering of $\mathcal{K}$ of index $2k$: $j \neq m - j$ in all pairs except for two pairs, $(0, 0)$ and $(m/2, m/2)$. If m is odd, then there are $(1 + m)/2$ such pairs $(j, m - j)$ including now a sole pair of identical exponents: $(0, 0)$. □

To represent $\mathcal{K}^+(n)$ in a more convenient form we need to use two multiplicative number-theoretic functions: $d(n)$, the number of divisors of n, and $\sigma(n)$, the sum of divisors of n mentioned in the previous section. Set

$$n = 2^s n_-$$

where n_- is the odd multiplicative part of n. Then $d(n) = (s + 1)d(n_-)$ and $\sigma(n) = (2^{s+1} - 1)\sigma(n_-)$. For convenience we formally set these function to vanish for a non-integer argument.

Corollary 2 *For even n,*

$$N^+_{\mathcal{K}}(n) = \frac{\sigma(n) + (2s - 1)d(n_-)}{2}. \tag{4}$$

[1] That is, subgroups that correspond to orientable coverings of $\mathcal{K}$.

Proof. $\sum\limits_{m|k}\lfloor\frac{m+2}{2}\rfloor = \sum\limits_{m|k\ \text{odd}}\frac{m+1}{2}+\sum\limits_{m|k\ \text{even}}\frac{m+2}{2} = [\sigma(n_-)+d(n_-)]/2+[\sigma(n/4)+d(n/4)] = [(2^s-1)\sigma(n_-)+(2s-1)d(n_-)]/2 = [\sigma(n)+(2s-1)d(n_-)]/2.$ □

According to [3], the total number of n-coverings of a Klein bottle $\mathcal{K}$ is

$$N_{\mathcal{K}}(n) = \begin{cases} d(n), & n \text{ odd} \\ \frac{3}{2}d(n)+\frac{1}{2}\sum\limits_{m|\frac{n}{2}}(m-1), & n \text{ even.} \end{cases} \tag{5}$$

So, for even n, $N_{\mathcal{K}}(n) = \frac{3}{2}d(n)+\frac{1}{2}(\sigma(n/2)-d(n/2)) = [3(s+1)d(n_-)+(2^s-1)\sigma(n_-)-sd(n_-)]/2 = [(2^s-1)\sigma(n_-)+(2s+3)d(n_-)]/2 = [\sigma(n)+(2s+3)d(n_-)]/2.$

Denote by $N^-_{\mathcal{K}}(n) = N_{\mathcal{K}}(n) - N^+_{\mathcal{K}}(n)$ the number of non-orientable n-fold coverings of a Klein bottle. Subtracting (4) from formula (5) for even n we obtain

$$N^-_{\mathcal{K}}(n) = 2d(n_-), \qquad 2|n.$$

Now, $2d(n_-) = (s+1)d(n_-)-(s-1)d(n_-) = d(n)-d(n/4)$ if $4|n$, and $2d(n_-) = d(n)$ for even n not divisible by 4. The same equality $N^-_{\mathcal{K}}(n) = d(n)$ is valid for odd n as well, in view of (5) since $N^+_{\mathcal{K}}(n) = 0$ in this case. Thus, we obtain finally

Theorem 1

$$N^-_{\mathcal{K}}(n) = \begin{cases} d(n)-d(n/4), & 4|n, \\ d(n), & \text{otherwise.} \end{cases} \tag{6}$$

In particular $N^-_{\mathcal{K}}(2^s) = 2$ *for any* $s \geq 1$. □

REMARK The right-hand side of formula (6) is a *multiplicative* number-theoretic function $d_{\text{odd}}(n)$ which can be defined as follows: this is the number of divisors m of n such that m or n/m is odd. It is also defined by the following conditions: $d_{\text{odd}}(2^s) = 2$ and $d_{\text{odd}}(p^s) = s+1$ for odd prime p. Now the assertion of Theorem 1 that

$$N^-_{\mathcal{K}}(n) = d_{\text{odd}}(n) \tag{6'}$$

has a simple topological sense: for any $m|n$ for which at least one of the numbers m and n/m is odd, it is possible to construct a non-orientable n-covering of $\mathcal{K}$. So, we conclude that this construction exhausts all non-orientable n-coverings of the Klein bottle.

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References

[1] J. H. Kwak and J. Lee, Enumeration of graph coverings, surface branched coverings and related group theory, In: *Combinatorial and Computational Mathematics: Present and Future* (ed. by S. Hong, J. H. Kwak et al.), Word Scientific, Singapore, 2001, 97–161.

[2] W. S. Massey, *Algebraic Topology: an Introduction*, Springer-Verlag, New York, 1977.

[3] A. D. Mednykh, On the number of subgroups in the fundamental group of a closed surface, *Commun. Algebra* **16**(10) (1988) 2137–2148.

[4] A. D. Mednykh and G. G. Pozdnyakova, Number of nonequivalent coverings over a nonorientable compact surface, *Siber. Math. J.* **27**(1) (1986), 99–106.