AN ELEMENTARY PROOF OF APÉRY'S THEOREM

Wadim Zudilin (Moscow)

E-print `math.NT/0202159`

17 February 2002

Abstract. We present a new 'elementary' proof of the irrationality of $\zeta(3)$ based on some recent 'hypergeometric' ideas of Yu. Nesterenko, T. Rivoal, and K. Ball, and on Zeilberger's algorithm of creative telescoping.

A question of an arithmetic nature of the values of Riemann's zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

at odd integral points $s = 3, 5, 7, \dots$ looks like a challenge for Number Theory. An expected answer '*each odd zeta value is transcendental*' is still far from being proved. We dispose of a particular information on the *irrationality* of odd zeta values, namely:

- $\zeta(3)$ is irrational (R. Apéry [Ap], 1978);
- infinitely many of the numbers $\zeta(3), \zeta(5), \zeta(7), \dots$ are irrational (T. Rivoal [Ri1], [BR], 2000);
- each set $\zeta(s+2), \zeta(s+4), \dots, \zeta(8s-3), \zeta(8s-1)$ with odd $s > 1$ contains at least one irrational number (this author [Zu1], [Zu2], 2001);
- at least one of the four numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational (this author [Zu3], [Zu4], 2001).

All these results have a *classical* well-poised-hypergeometric origin, and we refer the reader roused the curiosity of this terminology to the forthcoming works [Zu4], [Zu5], [RZ] for details. The aim of this note is to prove Apéry's famous result by 'elementary means'.

Key words and phrases. Zeta value, hypergeometric series, Apéry's theorem, Zeilberger's algorithm of creative telescoping.

Apéry's theorem. *The number $\zeta(3)$ is irrational.*

The idea of the following proof is due to T. Rivoal [Ri2], [Ri3], who mixed approaches of Yu. Nesterenko [Ne] and K. Ball, and our contribution here is to make a use of Zeilberger's algorithm of creative telescoping in the most elementary manner.

Our starting point is repetition of [Ne, Section 1]. For each integer $n = 0, 1, 2, \dots$ define the rational function

$$R_n(t) := \left(\frac{(t-1)\cdots(t-n)}{t(t+1)\cdots(t+n)} \right)^2$$

and denote by D_n the least common multiple of the numbers $1, 2, \dots, n$ (and $D_0 = 1$ for completeness).

Lemma 1 (cf. [Ne, Lemma 1]). *There holds the equality*

$$F_n := -\sum_{t=1}^{\infty} R_n'(t) = u_n \zeta(3) - v_n, \tag{1}$$

where $u_n \in \mathbb{Z}$, $D_n^3 v_n \in \mathbb{Z}$.

Proof. Taking square of the partial-fraction expansion

$$\frac{(t-1)\cdots(t-n)}{t(t+1)\cdots(t+n)} = \sum_{k=0}^{n} \frac{(-1)^{n-k} \binom{n+k}{n} \binom{n}{k}}{t+k}$$

with a help of the relation

$$\frac{1}{t+k} \cdot \frac{1}{t+l} = \frac{1}{l-k} \cdot \left(\frac{1}{t+k} - \frac{1}{t+l} \right) \qquad \text{for} \quad k \ne l,$$

we arrive at the formula

$$R_n(t) = \sum_{k=0}^{n} \left(\frac{A_{2k}^{(n)}}{(t+k)^2} + \frac{A_{1k}^{(n)}}{t+k} \right),$$

with $A_{jk} = A_{jk}^{(n)}$ satisfying the inclusions

$$A_{2k} = \binom{n+k}{n}^2 \binom{n}{k}^2 \in \mathbb{Z} \quad \text{and} \quad D_n A_{1k} \in \mathbb{Z}, \qquad k = 0, 1, \dots, n. \tag{2}$$

Furthermore,

$$\sum_{k=0}^{n} A_{1k} = \sum_{k=0}^{n} \operatorname{Res}_{t=-k} R_n(t) = -\operatorname{Res}_{t=\infty} R_n(t) = 0$$

since $R_n(t) = O(t^{-2})$ as $t \to \infty$, hence the quantity

$$F_n = \sum_{t=1}^{\infty} \sum_{k=0}^{n} \left(\frac{2A_{2k}}{(t+k)^3} + \frac{A_{1k}}{(t+k)^2} \right) = \sum_{k=0}^{n} \sum_{l=k+1}^{\infty} \left(\frac{2A_{2k}}{l^3} + \frac{A_{1k}}{l^2} \right)$$
$$= 2 \sum_{k=0}^{n} A_{2k} \left(\sum_{l=1}^{\infty} - \sum_{l=1}^{k} \right) \frac{1}{l^3} + \sum_{k=0}^{n} A_{1k} \left(\sum_{l=1}^{\infty} - \sum_{l=1}^{k} \right) \frac{1}{l^2}$$

has the desired form (1), with

$$u_n = 2 \sum_{k=0}^{n} A_{2k}, \qquad v_n = 2 \sum_{k=0}^{n} A_{2k} \sum_{l=1}^{k} \frac{1}{l^3} + \sum_{k=0}^{n} A_{1k} \sum_{l=1}^{k} \frac{1}{l^2}. \tag{3}$$

Finally, using the inclusions (2) and

$$D_n^j \cdot \sum_{l=1}^{k} \frac{1}{l^j} \in \mathbb{Z} \qquad \text{for} \quad k = 0, 1, \ldots, n, \quad j = 2, 3,$$

we deduce that $u_n \in \mathbb{Z}$ and $D_n^3 v_n \in \mathbb{Z}$ as required.

Since

$$R_0(t) = \frac{1}{t^2}, \qquad R_1(t) = \frac{1}{t^2} + \frac{4}{(t+1)^2} - \frac{4}{t} + \frac{4}{t+1},$$

in accordance with formulae (3) we find that

$$F_0 = 2\zeta(3) \qquad \text{and} \qquad F_1 = 10\zeta(3) - 12. \tag{4}$$

Now, with a help of Zeilberger's algorithm of creative telescoping [PWZ, Chapter 6] we get the rational function $S_n(t) := s_n(t) R_n(t)$, where

$$s_n(t) := 4(2n+1)(-2t^2 + t + (2n+1)^2), \tag{5}$$

satisfying the following property.

Lemma 2. *For each $n = 1, 2, \ldots$, there holds the identity*

$$(n+1)^3 R_{n+1}(t) - (2n+1)(17n^2+17n+5) R_n(t) + n^3 R_{n-1}(t) = S_n(t+1) - S_n(t). \tag{6}$$

'One-line' proof. Divide both sides of (6) by $R_n(t)$ and verify numerically the identity

$$(n+1)^3 \left(\frac{t-n-1}{t+n+1} \right)^2 - (2n+1)(17n^2+17n+5) + n^3 \left(\frac{t+n}{t-n} \right)^2$$
$$= s_n(t+1) \left(\frac{t^2}{(t-n)(t+n+1)} \right)^2 - s_n(t),$$

where $s_n(t)$ is given in (5).

Lemma 3. *The quantity* (1) *satisfies the difference equation*

$$(n+1)^3u_{n+1}-(2n+1)(17n^2+17n+5)u_n+n^3u_n=0 \tag{7}$$

for $n=1,2,\dots$.

Proof. Since $R'_n(t)=O(t^{-3})$ and $S'_n(t)=O(t^{-2})$, differentiating identity (6) and summing the result over $t=1,2,\dots$ we arrive at the equality

$$(n+1)^3F_{n+1}-(2n+1)(17n^2+17n+5)F_n+n^3F_{n-1}=S'_n(1).$$

It remains to note that, for $n\geqslant 1$, both functions $R_n(t)$ and $S_n(t)=s_n(t)R_n(t)$ have second-order zero at $t=1$. Thus $S'_n(1)=0$ for $n=1,2,\dots$ and we obtain the desired recurrence (7) for the quantity (1).

Consider another rational function

$$\widetilde{R}_n(t):=n!^2(2t+n)\frac{(t-1)\cdots(t-n)\cdot(t+n+1)\cdots(t+2n)}{(t(t+1)\cdots(t+n))^4} \tag{8}$$

and the corresponding hypergeometric series

$$\widetilde{F}_n:=\sum_{t=1}^{\infty}\widetilde{R}_n(t), \tag{9}$$

proposed by K. Ball.

Lemma 4 (cf. [BR, the second proof of Lemma 3]). *For each* $n=0,1,2,\dots$, *there holds the inequality*

$$0<\widetilde{F}_n<20(n+1)^4(\sqrt{2}-1)^{4n}. \tag{10}$$

Proof. Since $\widetilde{R}_n(t)=0$ for $t=1,2,\dots,n$ and $\widetilde{R}_n(t)>0$ for $t>n$ we deduce that $\widetilde{F}_n>0$.

With a help of elementary inequality

$$\frac{1}{m}\cdot\frac{(m+1)^m}{m^{m-1}}=\left(1+\frac{1}{m}\right)^m<e<\left(1+\frac{1}{m}\right)^{m+1}=\frac{1}{m}\cdot\frac{(m+1)^{m+1}}{m^m}$$

that yields $(m+1)^m/m^{m-1}<em<(m+1)^{m+1}/m^m$ for $m=1,2,\dots$, we deduce that

$$e^{-n}\frac{(m+n)^{m+n-1}}{m^{m-1}}<m(m+1)\dots(m+n-1)<e^{-n}\frac{(m+n)^{m+n}}{m^m}.$$

Therefore, for integers $t\geqslant n+1$,

$$\begin{aligned}\widetilde{R}_n(t)\cdot\frac{(t+n)^5}{(2t+n)(t+2n)}&=n!^2\cdot\frac{(t-1)\cdots(t-n)\cdot(t+n)\cdots(t+2n-1)}{(t(t+1)\cdots(t+n-1))^4}\\&<(n+1)^{2(n+1)}\cdot\frac{t^{5t-4}(t+2n)^{t+2n}}{(t-n)^{t-n}(t+n)^{5(t+n)-4}}\end{aligned}$$

and, as a consequence,

$$\widetilde{R}_n(t)\cdot\frac{t^4(t+n)}{(2t+n)(t+2n)(n+1)^2} < (n+1)^{2n}\cdot\frac{t^{5t}(t+2n)^{t+2n}}{(t-n)^{t-n}(t+n)^{5(t+n)}} = \left(1+\frac{1}{n}\right)^{2n}\cdot e^{nf(t/n)} < e^2\cdot\left(\sup_{\tau>1} e^{f(\tau)}\right)^n, \tag{11}$$

where

$$f(\tau) := \log\frac{\tau^{5\tau}(\tau+2)^{\tau+2}}{(\tau-1)^{\tau-1}(\tau+1)^{5(\tau+1)}}.$$

The unique (real) solution τ_0 of the equation

$$f'(\tau) = \log\frac{\tau^5(\tau+2)}{(\tau-1)(\tau+1)^5} = 0$$

in the region $\tau > 1$ is the zero of the polynomial

$$\tau^5(\tau+2) - (\tau-1)(\tau+1)^5 = -\left(\tau+\frac{1}{2}\right)\left(2\left(\tau+\frac{1}{2}\right)^4 - 5\left(\tau+\frac{1}{2}\right)^2 - \frac{7}{8}\right),$$

hence we can determine it explicitly:

$$\tau_0 = -\frac{1}{2} + \sqrt{\frac{5}{4}+\sqrt{2}}.$$

Thus,

$$\sup_{\tau>1} f(\tau) = f(\tau_0) = f(\tau_0) - \tau_0 f'(\tau_0) = 2\log(\tau_0+2) + \log(\tau_0-1) - 5\log(\tau_0+1) = 4\log(\sqrt{2}-1)$$

and we can continue the estimate (11) as follows:

$$\widetilde{R}_n(t)\cdot\frac{t^4(t+n)}{(2t+n)(t+2n)} < e^2(n+1)^2(\sqrt{2}-1)^{4n}, \tag{12}$$

Finally, we apply the inequality (12) to deduce the required estimate (10):

$$\begin{aligned}\widetilde{F}_n &= \sum_{t=n+1}^{\infty}\widetilde{R}_n(t) < e^2(n+1)^2(\sqrt{2}-1)^{4n}\sum_{t=n+1}^{\infty}\frac{(2t+n)(t+2n)}{t^4(t+n)}\\ &< e^2(n+1)^2(\sqrt{2}-1)^{4n}\sum_{t=n+1}^{\infty}\left(\frac{2}{t^5}+\frac{5n}{t^4}+\frac{2n^2}{t^3}\right)\\ &\leqslant e^2(n+1)^2\big(2\zeta(5)+5n\zeta(4)+2n^2\zeta(3)\big)(\sqrt{2}-1)^{4n} < 20(n+1)^4(\sqrt{2}-1)^{4n}.\end{aligned}$$

This completes the proof.

For the rational function (8) we obtain Zeilberger's certificate

$$\begin{aligned}\widetilde{S}_n(t) := \frac{\widetilde{R}_n(t)}{(2t+n)(t+2n-1)(t+2n)}\cdot\big(&-t^6-(8n-1)t^5+(4n^2+27n+5)t^4\\ &+2n(67n^2+71n+15)t^3+(358n^4+339n^3+76n^2-7n-3)t^2\\ &+(384n^5+396n^4+97n^3-29n^2-17n-2)t\\ &+n(153n^5+183n^4+50n^3-30n^2-22n-4)\big).\end{aligned} \tag{13}$$

Lemma 5. *For each $n = 1, 2, \dots$, there holds the identity*

$$(n+1)^3\widetilde{R}_{n+1}(t)-(2n+1)(17n^2+17n+5)\widetilde{R}_n(t)+n^3\widetilde{R}_{n-1}(t) = \widetilde{S}_n(t+1)-\widetilde{S}_n(t). \quad (14)$$

'One-line' proof. Divide both sides of (14) by $\widetilde{R}_n(t)$ and verify the reduced identity.

Lemma 6. *The quantity* (9) *satisfies the difference equation* (7) *for $n = 1, 2, \dots$.*

Proof. Since $\widetilde{R}_n(t) = O(t^{-5})$ and $\widetilde{S}_n(t) = O(t^{-2})$ as $t \to \infty$ for $n \geqslant 1$, summation of equalities (14) over $t = 1, 2, \dots$ yields the relation

$$(n+1)^3\widetilde{F}_{n+1} - (2n+1)(17n^2+17n+5)\widetilde{F}_n + n^3\widetilde{F}_{n-1} = -\widetilde{S}_n(1).$$

It remains to note that, for $n \geqslant 1$, both functions (8) and (13) have zero at $t = 1$. Thus $\widetilde{S}_n(1) = 0$ for $n = 1, 2, \dots$ and we obtain the desired recurrence (7) for the quantity (9).

Lemma 7. *For each $n = 0, 1, 2, \dots$, the quantities* (1) *and* (9) *coincide.*

Proof. Since both F_n and $\widetilde{F}_n$ satisfy the same second-order difference equation (7), we have to verify that $F_0 = \widetilde{F}_0$ and $F_1 = \widetilde{F}_1$. Direct calculations show that

$$\widetilde{R}_0(t) = \frac{2}{t^3}, \qquad \widetilde{R}_1(t) = -\frac{2}{t^4} + \frac{2}{(t+1)^4} + \frac{5}{t^3} + \frac{5}{(t+1)^3} - \frac{5}{t^2} + \frac{5}{(t+1)^2},$$

hence $\widetilde{F}_0 = 2\zeta(3)$ and $\widetilde{F}_1 = 10\zeta(3) - 12$, and comparison of this result with (4) yields the desired coincidence.

Proof of Apéry's theorem. Suppose, on the contrary, that $\zeta(3) = p/q$, where p and q are positive integers. Then, using a trivial bound $D_n < 3^n$, we deduce that, for each $n = 0, 1, 2, \dots$, the integer $qD_n^3F_n = D_n^3u_np - D_n^3v_nq$ satisfies the estimate

$$0 < qD_n^3F_n < 20q(n+1)^4 3^{3n}(\sqrt{2}-1)^{4n} \quad (15)$$

that is not possible since $3^3(\sqrt{2}-1)^4 = 0.7948\ldots < 1$ and the right-hand side of (15) is less than 1 for a sufficiently large integer n. This contradiction completes the proof of the theorem.

Inspite of its elementary arguments, our proof of Apéry's theorem does not look simpler than the original (also elementary) Apéry's proof well-explained in A. van der Poorten's informal report [Po], or (almost elementary) Beukers's proof [Be] by means of Legendre polynomials and multiple integrals. We want to mention that our way to deduce the recursion (7) for the sequence F_n as well as for the coefficients u_n, v_n[‡]

[‡]Hint: multiply both sides of (6) by $(t+k)^2$, substitute $t = -k$ and sum over *all* integers k to show that the sequence u_n satisfies the difference equation (7); then $v_n = u_n\zeta(3) - F_n$ also satisfies it.

slightly differs from those considered in [Po, Section 8] and [Ze, Section 13] although it is based on the same algorithm of creative telescoping. This algorithm and the above scheme allow us [Zu5], [Zu6] to obtain Apéry-like difference equations for $\zeta(4)$ and Calalan's constant.

The fact that $\widetilde{F}_n = \widetilde{u}_n\zeta(3) - \widetilde{v}_n$ with $D_n\widetilde{u}_n, D_n^4\widetilde{v}_n \in \mathbb{Z}$ was first discovered by K. Ball; the proof follows lines of the proof of Lemma 1 and vanishing the coefficients for $\zeta(4)$ and $\zeta(2)$ is due to a well-poised origin of the series (9). An open question of T. Rivoal here is to get the better inclusions $\widetilde{u}_n, D_n^3\widetilde{v}_n \in \mathbb{Z}$ by elementary means without going back to Apéry's series (1). A solution of this question accompanied with Ball's Lemma 4 can bring the 'most elementary' proof of Apéry's theorem.

Lemma 7 can be proved by specialization of Bailey's identity [Ba, Section 6.3, formula (2)]

$$\begin{aligned}
{}_7F_6\biggl(\begin{matrix} a, 1+\frac12 a, & b, & c, & d, & e, & f \\ \frac12 a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f \end{matrix}\,\biggr|\, 1\biggr) \\
= \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-f)}{\Gamma(1+a)\,\Gamma(b)\,\Gamma(c)\,\Gamma(d)\,\Gamma(1+a-b-c)\,\Gamma(1+a-b-d)} \\
\times\Gamma(1+a-c-d)\,\Gamma(1+a-e-f) \\
\times \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(b+t)\,\Gamma(c+t)\,\Gamma(d+t)\,\Gamma(1+a-e-f+t)}{\Gamma(1+a-e+t)\,\Gamma(1+a-f+t)} \\
\times\Gamma(1+a-b-c-d-t)\,\Gamma(-t)\,\mathrm{d}t,
\end{aligned} \tag{16}$$

provided that the very-well-poised hypergeometric series on the left-hand side converges. Namely, taking $a = 3n+2$ and $b = c = d = e = f = n+1$ in (16) we obtain Ball's sequence (9) on the left and Apéry's sequence (1) on the right (for the last fact see [Ne, Lemma 2]). Identity (16) can be put forward for an explanation how the permutation group from [RV] for linear forms in 1 and $\zeta(3)$ appears (see [Zu5, Sections 4 and 5 for details]).

References

[Ap] R. Apéry, *Irrationalité de $\zeta(2)$ et $\zeta(3)$*, Astérisque **61** (1979), 11–13.

[Ba] W. N. Bailey, *Generalized hypergeometric series*, Cambridge Math. Tracts, vol. 32, Cambridge Univ. Press, Cambridge, 1935; 2nd reprinted edition, Stechert-Hafner, New York–London, 1964.

[BR] K. Ball and T. Rivoal, *Irrationalité d'une infinité de valeurs de la fonction zêta aux entiers impairs*, Invent. Math. **146** (2001), no. 1, 193–207.

[Be] F. Beukers, *A note on the irrationality of $\zeta(2)$ and $\zeta(3)$*, Bull. London Math. Soc. **11** (1979), no. 3, 268–272.

[Ne] Yu. V. Nesterenko, *A few remarks on $\zeta(3)$*, Mat. Zametki [Math. Notes] **59** (1996), no. 6, 865–880.

[PWZ] M. Petkovšek, H. S. Wilf, and D. Zeilberger, $A = B$, A. K. Peters, Ltd., Wellesley, M.A., 1997.

[Po] A. van der Poorten, *A proof that Euler missed... Apéry's proof of the irrationality of $\zeta(3)$*, An informal report, Math. Intelligencer **1** (1978/79), no. 4, 195–203.

[RV] G. Rhin and C. Viola, *The group structure for* $\zeta(3)$, Acta Arith. **97** (2001), no. 3, 269–293.

[Ri1] T. Rivoal, *La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), no. 4, 267–270; E-print `math.NT/0008051`.

[Ri2] T. Rivoal, *Propriétés diophantinnes des valeurs de la fonction zêta de Riemann aux entiers impairs*, Thèse de Doctorat, Univ. de Caen, Caen, 2001.

[Ri3] T. Rivoal, *Séries hypergéométriques et irrationalité des valeurs de la fonction zêta*, Journées arithmétiques (Lille, July, 2001), 2002 (to appear).

[RZ] T. Rivoal and W. Zudilin, *Diophantine properties of numbers related to Catalan's constant*, Prépublication de l'Institut de Math. de Jussieu, no. 315 (Janvier 2002), submitted for publication.

[Ze] D. Zeilberger, *Closed form (pun intended!)*, A tribute to Emil Crosswald "Number theory and related analysis", Contemporary Math. (M. Knopp and M. Sheingorn, eds.), vol. 143, Amer. Math. Soc., Providence, R.I., 1993, pp. 579–607.

[Zu1] W. Zudilin, *Irrationality of values of zeta-function*, Contemporary Research in Mathematics and Mechanics, Proceedings of the XXIII Conference of Young Scientists of the Department of Mechanics and Mathematics (Moscow State University, April 9–14, 2001), Publ. Dept. Mech. Math. MSU, Moscow, 2001, Part 2, pp. 127–135; English transl., E-print `math.NT/0104249`.

[Zu2] W. Zudilin, *Irrationality of values of the Riemann zeta function*, Izv. Ross. Akad. Nauk Ser. Mat. [Russian Acad. Sci. Izv. Math.] **66** (2002), no. 3.

[Zu3] W. V. Zudilin, *One of the numbers* $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ *is irrational*, Uspekhi Mat. Nauk [Russian Math. Surveys] **56** (2001), no. 4, 149–150.

[Zu4] W. Zudilin, *Arithmetic of linear forms involving odd zeta values*, Preprint (August 2001), submitted for publication.

[Zu5] W. Zudilin, *Difference equation and permutation group for* $\zeta(4)$, Actes des 12èmes rencontres arithmétiques de Caen (June 29–30, 2001), J. Théorie Nombres Bordeaux (2002) (to appear).

[Zu6] W. Zudilin, *Apéry-like difference equation for Catalan's constant*, Preprint (January 2002), E-print `math.NT/0201024`, submitted for publication.

Moscow Lomonosov State University
Department of Mechanics and Mathematics
Vorobiovy Gory, Moscow 119899 RUSSIA
URL: `http://wain.mi.ras.ru/index.html`
E-mail address: `wadim@ips.ras.ru`