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A PROOF OF THE ODD PERFECT NUMBER CONJECTURE

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Abstract. It is sufficient to prove that there is an excess of prime factors in the product of repunits with odd prime bases defined by the sum of divisors of the integer $N = (4k+1)^{4m+1} \prod_{i=1}^{\ell}\ q_i^{2\alpha_i}$ to establish that there do not exist any odd integers with equality between $\sigma(N)$ and $2N$. The existence of distinct prime factors in the repunits in $\sigma(N)$ follows from a theorem on the primitive divisors of the Lucas sequences $U_{2\alpha_i+1}(q_i+1, q_i)$ and $U_{2\alpha_j+1}(q_j+1, q_j)$ with $q_i, q_j, 2\alpha_i+1, 2\alpha_j+1$ being odd primes. The occurrence of new prime divisors in each quotient $\frac{(4k+1)^{4m+2}-1}{4k}$, $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$, $i = 1, ..., \ell$ also implies that the square root of the product of $2(4k+1)$ and the sequence of repunits will not be rational unless the primes are matched. Although twelve solutions to the rationality condition for the existence of odd perfect numbers are obtained, it is verified that they all satisfy $\frac{\sigma(N)}{N} \neq 2$ because the repunits in the product representing $\sigma(N)$ introduce new prime divisors. Minimization of the number of prime divisors in $\sigma(N)$ leads to an infinite set of repunits of increasing magnitude or prime equations with no integer solutions. It is then proven that there exist no odd perfect numbers.

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1. Introduction

The uniqueness of the prime decomposition of an integer allows for a comparison of its magnitude with the sum of its divisors. Since the sum of the divisors of an odd integer $N = (4k+1)^{4m+1} \prod_{i=1}^{\ell} q_i^{2\alpha_i}$ is given by the product $\frac{(4k+1)^{4m+2}-1}{4k} \prod_{i=1}^{\ell} \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$, it is sufficient to determine the properties of the prime factors of the repunits to prove that there exist no odd perfect numbers. The irrationality of $\sqrt{2(4k+1)\left[\sigma((4k+1)^{4m+1})\cdot \prod_{i=1}^{\ell} \sigma(q_i^{2\alpha_i})\right]^{\frac{1}{2}}}$ would imply that $\sigma(N)$ cannot equal $2N$. It has been proven for a large class of primes $\{4k+1; q_i\}$ and exponents $\{4m+1; 2\alpha_i\}$ that the rationality condition is not satisfied. The irrationality of the square root of the product of $2(4k+1)$ and the sequence of repunits is not valid for all sets of primes and exponents, however, and it is verified in §4 that the rationality condition holds for twelve odd integers. The factorizations of these integers have the property that the repunits have prime divisors which form interlocking rings, whereas, in general, the sequence of prime factors does not close. The presence of a sequence of primes of increasing magnitude prevents any finite odd integer from being a perfect number. To demonstrate that $\frac{\sigma(N)}{N} \neq 2$, it is necessary also to obtain a lower bound for the number of prime divisors in $\sigma(N)$. Since the number of factors of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ is minimized when $2\alpha_i+1$ is prime, the exponent is presumed to be prime throughout the discussion. It is demonstrated in Theorem 1 that different repunits with odd prime bases and exponents do not have identical sets of prime divisors. Then, either $\sigma(N)$ has an excess of prime divisors or constraints must be imposed on $\{4k+1; q_i\}$ and $\{4m+1; 2\alpha_i\}$ which have no integer solution. The non-existence of odd perfect numbers also follows from the sequence of prime factors of increasing magnitude in the factorization of $\sigma(N)$, when one of three specified relations is satisfied, and constraints on the basis and exponents otherwise.

2. The Existence of Different Prime Divisors in the Repunit Factors of the Sum of Divisors

An excess of distinct prime divisors in the quotients $\frac{(4k+1)^{4m+2}-1}{4k}$ and $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$, $i = 1, ..., \ell$ is sufficient to prove the inequality $\frac{\sigma(N)}{N} \neq 2$.

Theorem 1. The pair of integers $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ and $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$, $q_i > q_j$ do not have identical sets of prime divisors if the bases and exponents are odd primes.

Proof. There are three known positive integer solutions to the exponential Diophantine equation $\frac{x^m-1}{y^n-1}$, $m,n>1$, $2^3-1=\frac{6^2-1}{6-1}$, $31=2^5-1=\frac{5^3-1}{5-1}$ and $8191=2^{13}-1=\frac{90^3-1}{90-1}$ [9][12[13] and no equalities of this type have been established yet for odd prime bases and exponents. It will be assumed initially that

$$\frac{q_i^{2\alpha_i+1}-1}{q_i-1} \neq \frac{q_j^{2\alpha_j+1}-1}{q_j-1} \tag{2.2}$$

until this inequality is an evident consequence of the proof.

If the inequality (2.2) holds, then either the sets of primitive divisors of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ and $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ are not identical or the exponents of the prime power divisors are different, given that $\frac{q_i^{2\alpha_i+1}-1}{q_i-1} \neq \frac{q_j^{2\alpha_j+1}-1}{q_j-1}$. Let $q_i > q_j$. Since

$$\begin{aligned}\frac{q_i^{2\alpha_i+1}-1}{q_j^{2\alpha_j+1}-1} &= \frac{q_i^{2\alpha_i+1}-1}{q_i^{2\alpha_j+1}-1}\frac{q_i^{2\alpha_j+1}-1}{q_j^{2\alpha_j+1}-1} \\ &\simeq \frac{q_i^{2\alpha_i+1}-1}{q_i^{2\alpha_j+1}-1}\Bigg[\left(\frac{q_i}{q_j}\right)^{2\alpha_j+1}\left(1+\frac{q_i^{2\alpha_j+1}-q_j^{2\alpha_j+1}-1}{q_i^{2\alpha_j+1}q_j^{2\alpha_j+1}}\right) \\ &\qquad + O\left(\frac{1}{q_i^{2\alpha_j+1}q_j^{4\alpha_j+2}}\right)\Bigg]\end{aligned} \tag{2.3}$$

this equals

$$\begin{aligned}&\frac{q_i^{2\alpha_i+1}-1}{q_i^{2\alpha_j+1}-1}\left[\left(\frac{q_i}{q_j}\right)^{2\alpha_j+1}+\epsilon\right] \\ &\epsilon \simeq \left(\frac{q_i}{q_j}\right)^{2\alpha_j+1}\frac{q_i^{2\alpha_j+1}-q_j^{2\alpha_j+1}-1}{q_i^{2\alpha_j+1}q_j^{2\alpha_j+1}}\end{aligned} \tag{2.4}$$

The exact value of the remainder term is

$$\epsilon\cdot\frac{q_i^{2\alpha_i+1}-1}{q_i^{2\alpha_j+1}-1} = \frac{q_i^{2\alpha_i+1}-1}{q_j^{2\alpha_j+1}-1}\frac{(q_i^{2\alpha_j+1}-q_j^{2\alpha_j+1}-1)(q_j^{2\alpha_j+1}-1)+(q_i^{2\alpha_j+1}-1)}{q_j^{4\alpha_j+2}(q_j^{2\alpha_j+1}-1)} \tag{2.5}$$

When $q_j < q_i < q_j^{2^{\frac{2\alpha_j+1}{2\alpha_i+1}}}$, $\epsilon\cdot\frac{q_i^{2\alpha_i+1}-1}{q_i^{2\alpha_j+1}-1} < 1$.

Amongst the quotients $\frac{q_i^n-1}{q_j^n-1}$ which are integer when $n=2$, the only trivial example of repunits with the identical sets of prime divisors is $\frac{5^2-1}{5-1}=2\cdot 3$ and $\frac{11^2-1}{11-1}=2\cdot 2\cdot 3$. For odd exponents, the first example of an integer ratio $\frac{q_i^{2\alpha_i+1}-1}{q_j^{2\alpha_j+1}-1}=\frac{29^3-1}{3^3-1}=938$ and again, the quotient $\frac{29^3-1}{29-1} = 13 \cdot 67$ introduces a new prime divisor, 67. For $q_i, q_j \gg 1$, the ratio $\frac{q_i^{2\alpha_i+1}-1}{q_j^{2\alpha_i+1}-1} \to \left(\frac{q_i}{q_j}\right)^n$, which cannot be integer for odd primes q_i, q_j, implying that one of the repunits $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ and $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$, with $\alpha_i = \alpha_j$ has a distinct prime divisor.

Let $q_i - 1 = p_1^{h_{i1}}...p_s^{h_{is}}$ and $q_j - 1 = p_1^{h_{j1}}...p_s^{h_{js}}$. Then

$$\frac{(p_1^{h_{i1}}...p_s^{h_{is}}+1)^{2\alpha_i+1}-1}{p_1^{h_{1s}}...p_s^{h_{is}}} = 2\alpha_i + 1 + \alpha_i(2\alpha_i+1)(p_1^{h_{i1}}...p_s^{h_{is}}) + ... + (p_1^{h_{i1}}...p_s^{h_{is}})^{2\alpha_i}$$
$$\frac{(p_1^{h_{j1}}...p_s^{h_{js}}+1)^{2\alpha_j+1}-1}{p_1^{h_{1s}}...p_s^{h_{js}}} = 2\alpha_j + 1 + \alpha_j(2\alpha_j+1)(p_1^{h_{j1}}...p_s^{h_{is}}) + ... + (p_1^{h_{j1}}...p_s^{h_{js}})^{2\alpha_j} \tag{2.6}$$

Let P be a primitive divisor of the $q_i^{2\alpha_i+1} - 1$. Consider the congruence $c_0 + c_1x + ... + c_{2\alpha_i}x^{2\alpha_i} \equiv 0 \ (mod\ P), c_t > 0,\ t = 0, 1, 2, ..., 2\alpha_j$. There is a unique solution x_0 to the congruence relation $f(x) \equiv c_0 + c_1x + ... + c_nx^n \equiv 0 \ (mod\ P)$ within $[x_0 - \epsilon, x_0 + \epsilon]$ [11], where

$$\epsilon f'(x_0) + \frac{\epsilon^2}{2!}f''(x_0) + ... \ < \ P \tag{2.7}$$

Setting $c_t = \binom{2\alpha_i+1}{t+1}$,

$$f'(q_i - 1) \ = \ \binom{2\alpha_i+1}{2} + 2\binom{2\alpha_i+1}{3}(q_i-1) + ... + 2\alpha_i(q_i-1)^{2\alpha_i-1} \tag{2.8}$$

the constraint (2.6) implies a bound on ϵ of $\left(1 - \frac{2}{2\alpha_i+1}\right)\frac{q_i-1}{2\alpha_i}$. Conseqently, if

$$|q_i - q_j| < \left(1 - \frac{2}{2\alpha_i+1}\right)\frac{q_i-1}{2\alpha_i} \tag{2.9}$$

the prime P is not a common factor of both repunits. The number of repunits with primitive divisor P will be bounded by the number of integer solutions to the congruence relation.

For P to be a divisor of $q_j^{2\alpha_j+1} - 1$, $p_1^{h_{j1}}p_2^{h_{j2}}...p_s^{h_{js}}$ also would have to satisfy the congruence relation when $\alpha_i = \alpha_j$. If $p_1^{h_{i1}}p_2^{h_{i2}}...p_s^{h_{is}} = p_1^{h_{j1}}p_2^{h_{j2}}...p_s^{h_{js}} + nP$, then $p_1^{h_{j1}}p_2^{h_{j2}}...p_s^{h_{js}}$ is another solution to the congruence. Therefore $gcd(q_i - 1, q_j - 1) = p_1^{min(h_{i1},h_{j1})}...p_s^{min(h_{is},h_{js})}|nP$, and, as P cannot equal any of the prime divisors p_{ik}, $gcd(q_i - 1, q_j - 1) = p_1^{min(h_{i1},h_{j1})}...p_s^{min(h_{is},h_{js})}|n$. Then, $q_i - 1 = q_j - 1 + kgcd(q_i - 1, q_j - 1)P$ for some integer k, and $P|\frac{q_i-q_j}{gcd(q_i-1,q_j-1)}$.

However, if P is a prime divisor of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ and $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$, and $\alpha_i = \alpha_j$,

$$P|(q_i - q_j)(q_i^{2\alpha_i} + (2\alpha_i+1)q_i^{2\alpha_i-1}q_j + ... + (2\alpha_i+1)q_iq_j^{2\alpha_i-1} + q_j^{2\alpha_i}) \tag{2.10}$$

Since $gcd(q_i - q_j, q_i^{2\alpha_i} + (2\alpha_i + 1)q_i^{2\alpha_i - 1}q_j + ... + (2\alpha_i + 1)q_i q_j^{2\alpha_i - 1} + q_j^{2\alpha_i}) = 1$, either $P|q_i - q_j$ or $P|q_i^{2\alpha_i} + (2\alpha_i + 1)q_i^{2\alpha_i - 1}q_j + ... + (2\alpha_i + 1)q_i q_j^{2\alpha_i - 1} + q_j^{2\alpha_i}$. When $P|q_i - q_j$, it may satisfy the divisibility condition for the congruence relation. However, there also would be a primitive divisor which is a factor of $q_i^{2\alpha_i} + q_i^{2\alpha_i - 1}q_j + ... + q_j^{2\alpha_i - 1}q_i + q_j^{2\alpha_i}$, and not $q_i - q_j$. Then $p_1^{h_{i1}} p_2^{h_{i2}} ... p_s^{h_{is}} < p_1^{h_{j1}} p_2^{h_{j2}} ... p_s^{h_{js}} + P$. If there is only one solution to the congruence $2\alpha_i + 1 + \alpha_i(2\alpha_i + 1)x + ... + x^{2\alpha_i} \equiv 0 \ (mod\ P)$ less than P, a contradiction is obtained. When one of the congruences is not satisfied for at least one primitive divisor P, this prime is not a common factor of both repunits.

Consider the identities

$$
\begin{aligned}
q_i^{2\alpha_i+1} - 1 &= \prod_{k=0}^{2\alpha_i} (q_i - \omega_{2\alpha_i+1}^k) \\
q_j^{2\alpha_j+1} - 1 &= \prod_{k'=0}^{2\alpha_j} (q_j - \omega_{2\alpha_j+1}^{k'}) \\
\omega_n &= e^{\frac{2\pi i}{n}}
\end{aligned}
\tag{2.11}
$$

Quotients of products of linear factors in $q_i^{2\alpha_i+1} - 1$ and $q_j^{2\alpha_j+1} - 1$ generally will not involve cancellation of integer divisors because products of powers of the different units of unity cannot be real unless the sums of the exponents are multiples of the primes $2\alpha_i + 1$ or $2\alpha_j + 1$. The sumsets which give rise to equal exponents can be enumerated by determining the integers $s_1, ..., s_t$ such that the sums are either congruent to 0 modulo $2\alpha_i + 1$ and $2\alpha_j + 1$. It is apparent that the entire integer sets $\{1, 2, ..., 2\alpha_i\}$ and $\{1, 2, ..., 2\alpha_j\}$ cannot be used for unequal repunits. The number of sequences satisfying the congruence relations for each repunit must equal $\sum_{\sum t_k = 2\alpha_i + 1}(2^{t_k} - 1)$ and $\sum_{\sum t'_l = 2\alpha_j + 1}(2^{t'_l} - 1)$, with sequences of integers which sum to a non-zero value giving rise to cancellation of complex numbers in the product being included.

Consider two products $\prod_{m=1}^{t_k}(q_i - \omega_{2\alpha_i+1}^{s_m})$ and $\prod_{n=1}^{t'_k}(q_j - \omega_{2\alpha_j+1}^{s'_n})$. Upper and lower bounds for these products are

$$
\begin{aligned}
(q_i - 1)^{t_k} &< \prod_{m=1}^{t_k} (q_i - \omega_{2\alpha_i+1}^{s_m}) < (q_i + 1)^{t_k} \\
(q_j - 1)^{t'_l} &< \prod_{n=1}^{t'_l} (q_j - \omega_{2\alpha_j+1}^{s'_n}) < (q_j + 1)^{t'_l}
\end{aligned}
\tag{2.12}
$$

Since the minimum difference between primes is 2, one of the inequalities $(q_i + 1)^{t_k} \le (q_j - 1)^{t'_l}$, $(q_j + 1)^{t'_l} \le (q_i - 1)^{t^k}$ $(q_i + 1)^{t_k} > (q_j - 1)^{t'_l}$ or $(q_j + 1)^{t'_l} > (q_i - 1)^{t^k}$ will be satisfied when $t'_l \ne t_k$. The first two inequalities imply that the factors of the repunits defined by the products $\prod_{m=1}^{t_k}(q_i - \omega_{2\alpha_i+1}^{s_m})$ and $\prod_{n=1}^{t'_k}(q_j - \omega_{2\alpha_j+1}^{s'_n})$ are distinct.

If either of the next two inequalities hold, the factors possibly could be equal when $(q_i-1)^{t_k} = (q_j-1)^{t'_l}$ or $(q_i+1)^{t_k} = (q_j+1)^{t'_l}$. In the first case, $t'_l = n_{ij} t_k$, $q_i - 1 = (q_j-1)^{n_{ij}}$, $n_{ij} > 2$. This relation holds for all t_k and t'_l so that $\sum_k t_k = n_{ij} \sum_l t'_l$. However, $2\alpha_i + 1$ and $2\alpha_j + 1$ are prime, so that this equation is not valid. When $q_i + 1 = (q_j+1)^{n_{ij}}$, $n_{ij} \geq 2$, the relation $\sum_k t_k = n_{ij} \sum_l t'_l$ again leads to a contradiction.

The identification of the products in equation (2.12) for each corresponding k, l is necessary, since any additional product would give rise either to a new prime factor after appropriate rescaling, or a different power of the same number, which would would prevent a cancellation of an extra prime, described later in the proof. The inequalities $(q_j-1)^{t'_l} < (q_i-1)^{t_k} < (q_j+1)^{t'_l}$ imply that

$$t'_l \, \frac{ln(q_j-1)}{ln(q_i-1)} < t_k < t'_l \frac{ln(q_j+1)}{ln(q_i-1)} \tag{2.13}$$

If these inequalities hold for two pairs of exponents (t_{k_1}, t'_{l_1}), (t_{k_2}, t'_{l_2}), the interval $\left[t'_{l_2} \frac{ln(q_j-1)}{ln(q_i-1)}, t'_{l_2} \frac{ln(q_j+1)}{ln(q_i-1)}\right]$ contains $\frac{t'_{l_2}}{t'_{l_1}} t_{k_1}$. The fractions $\frac{t_{k_m}}{t_{k_n}}$ cannot be equal to $\frac{t'_{l_m}}{t'_{l_n}}$ for all m, n as this would imply that $\frac{t_k}{t'_l} = \frac{2\alpha_i+1}{2\alpha_j+1}$, which is not possible since $\frac{2\alpha_i+1}{2\alpha_j+1}$ is an irreducible prime fraction when $\alpha_i \neq \alpha_j$.

Then

$$\left| \frac{t_{k_1}}{t'_{l_1}} - \frac{t_{k_2}}{t'_{l_2}} \right| \geq \frac{1}{t'_{l_1} t'_{l_2}} \tag{2.14}$$

Then $t_{k_2} \notin \left[t'_{l_2} \frac{ln(q_j-1)}{ln(q_i-1)}, t'_{l_2} \frac{ln(q_j+1)}{ln(q_i-1)}\right]$ if

$$ln\left(\frac{q_j+1}{q_j-1}\right) < \frac{ln(q_i-1)}{t'_{l_1} t'_{l_2}} \tag{2.15}$$

As $t'_l \leq 2\alpha_j + 1$, this inequality is valid when

$$q_j \cdot ln(q_i-1) > 3(2\alpha_j+1)^2 \tag{2.16}$$

Since the arithmetic primitive factor of $\frac{q^n-1}{q-1}$, $n \neq 6$ is $\Phi_n(q) = \prod_{\substack{1 \leq k \leq n \\ gcd(k,n)=1}} (q_i - e^{\frac{2\pi i k}{n}})$ or $\frac{\Phi_n(q)}{p}$, where p is the largest prime factor of both $\frac{n}{gcd(n,3)}$ and $\Phi_n(q)$ [1][2][3][14][15], any prime, which is a primitive divisor of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$, divides $\Phi_{2\alpha_i+1}(q_i)$. It follows that these primes can be obtained by appropriate multiplication of the products $\prod_{m=1}^{t_k}(q_i - \omega_{2\alpha_i+1}^{s_m})$ and $\prod_{n=1}^{t'_k}(q_j - \omega_{2\alpha_j+1}^{s'_n})$. Although these products may not be the primitive divisors, they are real, and multiplication of $\prod_{m=1}^{t_1}(q_i - \omega_{2\alpha_i+1}^{s_m})$ by $\kappa_1 \in \mathbf{R}$ can be compensated by multiplication of $\prod_{m'=1}^{t_2}(q_i - \omega_{2\alpha_i+1}^{s'_m})$ by κ_1^{-1} to obtain the integer factor. Since products of two complex linear factors $(q_i - e^{\frac{2\pi i k_1}{2\alpha_i+1}})(q_i - e^{\frac{-2\pi i k_1}{2\alpha_i+1}})$ are real, let

$$\begin{aligned} x_1 &= (q_i - e^{\frac{2\pi i k_1}{2\alpha_i+1}})(q_i - e^{-\frac{2\pi i k_1}{2\alpha_i+1}}) \\ x_2 &= (q_i - e^{\frac{2\pi i k_2}{2\alpha_i+1}})(q_i - e^{-\frac{2\pi i k_2}{2\alpha_i+1}}) \\ y_1 &= (q_j - e^{\frac{2\pi i k_3}{2\alpha_j+1}})(q_i - e^{-\frac{2\pi i k_3}{2\alpha_j+1}}) \\ y_2 &= (q_j - e^{\frac{2\pi i k_4}{2\alpha_j+1}})(q_i - e^{-\frac{2\pi i k_4}{2\alpha_j+1}}) \end{aligned} \tag{2.17}$$

The occurrence of the same prime divisors in the two repunits $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ and $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ yields relations of the form

$$\begin{aligned} \kappa_1 x_1 &= P_1^{\ell_1} = P_1^{h_1}\kappa'_1 y_1 \\ \kappa_1^{-1} x_2 &= P_2^{\ell_2} = P_2^{h_2}\kappa_1'^{-1} y_2 \end{aligned} \tag{2.18}$$

for two prime factors implies

$$P_1^{\ell_2 h_1 - h_2 \ell_1} = \frac{x_1^{\ell_2 - h_2} x_2^{\ell_2 - \frac{h_1 \ell_2}{\ell_1}}}{y_1^{\ell_2} y_2^{\ell_2}} \tag{2.19}$$

which cannot be satisfied by the prime P_1. For three prime factors,

$$\begin{aligned} \kappa_1 x_1 &= P_1^{\ell_1} = P_1^{h_1}\kappa'_1 y_1 \\ \kappa_1^{-1}\kappa_2 x_2 &= P_2^{\ell_2} = P_2^{h_2}\kappa_1'^{-1}\kappa'_2 y_2 \\ \kappa_2^{-1} x_3 &= P_3^{\ell_3} = P_2^{h_3}\kappa_2'^{-1} y_3 \end{aligned} \tag{2.20}$$

yielding the equation

$$P_1^{\ell_2 h_1 - h_2 \ell_1} P_3^{\ell_2 h_3 - h_2 \ell_3} = \frac{(x_1 x_2 x_3)^{\ell_2 - h_2}}{y_1 y_2 y_3} \tag{2.21}$$

However, $\frac{(x_1x_2)^{\ell_2-h_2}}{y_1y_2}$ introduces a minimum of two different primes, whereas $\frac{x_3^{\ell_2-h_2}}{y_3}$ contains at least one more prime divisor. Alternatively, this fraction equals

$$\frac{(x_1x_2)^{\frac{(\ell_2-h_2)}{2}}}{(y_1y_2)^{\frac{1}{2}}} \cdot \frac{(x_2x_3)^{\frac{(\ell_2-h_2)}{2}}}{(y_2y_3)^{\frac{1}{2}}} \cdot \frac{(x_3x_1)^{\frac{(\ell_2-h_2)}{2}}}{(y_3y_1)^{\frac{1}{2}}} \tag{2.22}$$

Each term in the product contains n primes, $n \geq 2$, and cancellation between the fractions leads to the deletion of a maximimum of $n-1$ primes, since cancellation of all of the primes would imply that the ratio of a power of one term and the other term is a prime power, which is not possible based on the form of the linear complex factors. Consequently, the product (2.22) introduces at least three distinct primes and there is no solution to equation

(2.21) for the two primes P_1, P_3. The same conclusion is reached when the products of more than two linear factors rescaled to obtain the prime power. Continuing this factorization for products of more than three prime powers, it follows that there is a distinct prime divisor amongst the factors of $q_i^{2\alpha_i+1}-1$ and $q_j^{2\alpha_j+1}-1$, with $q_i, q_j, 2\alpha_i+1, 2\alpha_j+1$ being odd primes.

If $2\alpha_i+1 = 2\alpha_j+1$, $t_{k_1} = t_{k_2}$. However, $\sum_k t_k = \sum_l t'_l$. This implies that some elements of the $\{t_{k_m}\}$ are less than t_l, while other t_{k_n} are greater than t_l. However, the inequalities $(q_j-1)^{t'_l} < (q_i-1)^{t_k} < (q_j+1)^{t'_l}$ cannot be satisfied for all k, l implying that the products $\prod_{s_m=1}^{t_k}(q_i - \omega_{2\alpha_i+1})^{t'_l}$ and $\prod_{s'_n=1}^{t'_l}(q_i - \omega_{2\alpha_i+1}^{s'_n})$ are not equal for some k, l. The rescaling then cannot be done without introducing distinct prime divisors in one of the repunits.

Now let $\alpha_i \neq \alpha_j$. The exponents t'_{l_1} and t'_{l_2} can be set equal, because the products of linear factors in equation (2.11) can be paired according to magnitude. Then, $(q_j-1)^{t'_l} < (q_i-1)^{t_{k_1}} < (q_j+1)^{t'_l}$ and $t_{k_1} \in \left[t'_l \frac{ln(q_j-1)}{ln(q_i-1)}, t'_l \frac{ln(q_j+1)}{ln(q_i-1)}\right]$. For some t_{k_2}, $\frac{t_{k_1}}{t'_{l_1}} \neq \frac{t_{k_2}}{t'_{l_2}}$ since $\frac{2\alpha_i+1}{2\alpha_j+1}$ is an irreducible prime fraction, and $|t_{k_1} - t_{k_2}| \geq 1$. However, $t_{k_2} \notin \left[t'_l \frac{ln(q_j-1)}{ln(q_i-1)}, t'_l \frac{ln(q_j+1)}{ln(q_i-1)}\right]$ if $\frac{t'_l \ ln\left(\frac{q_j+1}{q_j-1}\right)}{ln(q_i-1)} < 1$, and it then not possible to equate identify product $\prod_{s_{m_2}=1}^{t_{k_2}}(q_i - \omega_{2\alpha_i+1}^{s_{m_2}}$ with $\prod_{s'_{n_2}=1}^{t'_l}(q_j - \omega_{2\alpha_j+1}^{s'_{n_2}})$.

Futhermore, the exponents t'_{l_1}, t'_{l_2} can be set equal to 2, if the products are multiplied by the power of a prime. While $P^h(q_j-1)^2 < (q_i-1)^{t_{k_1}} < P^h(q_j+1)^2$ and $t_{k_1} \in \left[\frac{2ln(q_j-1)+h \ ln \ P}{ln(q_i-1))}, \frac{2ln(q_j+1)+h \ ln \ P}{ln(q_i-1)}\right]$, there exists a corresponding t_{k_2} such that $|t_{k_2} - t_{k_1}| \geq 1$. Then $t_{k_2} \notin \left[\frac{2ln(q_j-1)+h \ ln \ P}{ln(q_i-1)}, \frac{2ln(q_j+1)+h \ ln \ P}{ln(q_i-1)}\right]$ when $\frac{2ln\left(\frac{q_j+1}{q_j-1}\right)}{ln(q_i-1)} < 1$. This inequality implies that $q_i > 5$.

The equality is also valid because $t_{k_2} = \frac{2ln(q_j+1)}{ln(q_i-1)}$ only if $P^h = \frac{(q_i-1)^{t_{k_2}}}{(q_j+1)^2}$ which implies that $q_i - 1 = (q_j+1)^{h'}$, $h' \geq 1$. The products in equation (2.11) cannot be identified since $\prod_{s_m=1}^{t_k}(q_i - \omega_{2\alpha_i+1}^{s_m}) > (q_i-1)^{t_k}$ and $\prod_{s'_n=1}^{t'_l}(q_j - \omega_{2\alpha_j+1}^{s'_n})$. A distinct prime divisor occurs in at least one of the repunits $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$, $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ for $q_i \geq 5$ and therefore all odd primes q_i, q_j, $q_i > q_j$.

For the known solutions to the equation $\frac{x^m-1}{x-1} = \frac{y^n-1}{y-1}$, one of the prime bases is 2. Since $q_j - 1 = 1$, the theorem is circumvented because the bounds (2.11) are satisfied for a larger set of primes q_i and exponents t_k. Furthermore, the integers sets $\{1, 2, ..., 2\alpha_i\}$ and $\{1, 2, ..., 2\alpha_j\}$ are used entirely in the products to give the same integer, which must then be a prime. The existence of solutions to the inequalities $1 < (q_i-1)^{2\alpha_i} < 3^{2\alpha_i}$

implies that these bounds do not exclude the equality of $2^{2\alpha_j+1}-1$ and $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ for some $2\alpha_j+1, q_i, 2\alpha_i+1$.

If the primes which divide only $q_i^{2\alpha_i+1}-1$, are factors of q_i-1 or q_j-1, they would be partially cancelled in a comparison of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ and $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$. Any divisor of q_i-1 which is a factor of $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ also must divide $2\alpha_i+1$. When $2\alpha_i+1$ and $2\alpha_j+1$ are prime, this divisor would have to be $2\alpha_i+1$ or $2\alpha_j+1$.

An potential counterexample to this theorem concerning the occurrence of a distinct arises if equations of the form

$$\begin{aligned} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= (2\alpha_i+1)^{\nu}\frac{q_j^{2\alpha_j+1}-1}{q_j-1} \\ (2\alpha_j+1)^{\nu}\frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= \frac{q_j^{2\alpha_j+1}-1}{q_j-1} \\ (2\alpha_j+1)^{\nu_1}\frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= (2\alpha_i+1)^{\nu_2}\frac{q_i^{2\alpha_i+1}-1}{q_i-1} \end{aligned} \tag{2.23}$$

are satisfied, as the set of prime divisors of the repunits would be identical if $2\alpha_i+1\left|\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\right.$ in the first relation, $2\alpha_j+1\left|\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\right.$ in the second condition and $2\alpha_i+1\left|\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\right.$, $2\alpha_j+1\left|\frac{q_i^{2\alpha_i+1}-1}{q_i-1}\right.$ in the third relation. The conditions in equation (2.23) follow if different powers of the other primes do not arise. However, $q_i^p-1\equiv 0\ (mod\ p^{\nu})$ only if $q_i^p-1\equiv 0\ (mod\ p)$, and since $q_i^p-q\equiv 0\ (mod\ p)$, this is possible when $p|q_i-1$. If $p|q_i-1$, then $\frac{q_i^p-1}{q_i-1}\equiv p$ and $\frac{q_i^p-1}{q_i-1}\not\equiv 0\ (mod\ p^{\nu}),\ \nu\geq 2$. Setting $\nu=1$ gives

$$\begin{aligned} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= (2\alpha_i+1)\frac{q_j^{2\alpha_j+1}-1}{q_j-1} \\ (2\alpha_j+1)\frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= \frac{q_j^{2\alpha_j+1}-1}{q_j-1} \\ (2\alpha_j+1)\frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= (2\alpha_i+1)\frac{q_i^{2\alpha_i+1}-1}{q_i-1} \end{aligned} \tag{2.24}$$

with $2\alpha_i+1\not|\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ and $2\alpha_j+1\not|\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$. It follows that either $2\alpha_i+1$ or $2\alpha_j+1$ is a prime which divides only one of the repunits.

Furthermore, q_i-1 and q_j-1 are integers which are not rescaled, so that distinct prime divisors arise in the products of the remaining linear factors. It follows also that $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$

and $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ cannot be equal, and that the original assumption of their inequality is valid. The existence of a prime divisor not common to both repunits can be deduced for all odd prime bases and exponents $(q_i, 2\alpha_i+1, q_j, 2\alpha_j+1), q_i \neq q_j$.

■

An example of congruence with more than one solution less than P is

$$(2\alpha_i+1)+\alpha_i(2\alpha_i+1)x+\frac{(2\alpha_i+1)2\alpha_i(2\alpha_i-1)}{3!}x^2+\frac{(2\alpha_i+1)2\alpha_i(2\alpha_i-1)(2\alpha_i-2)}{4!}x^3 + \frac{(2\alpha_i+1)2\alpha_i(2\alpha_i-1)(2\alpha_i-2)(2\alpha_i-3)}{5!}x^4 = 0\ (mod\ 11) \qquad (2.20)$$

which is solved by $x = 2,\ 4$ when $2\alpha_i + 1 = 5$. This is consistent with the inequality (2.6), since $\epsilon = 0.074897796$ when $x_0 = 2$. Consequently, $\frac{3^5-1}{2} = 11 \cdot 11$ and $\frac{5^5-1}{4} = 11 \cdot 71$ both have the divisor 11 and the distinct prime divisor arises in the larger repunit. Indeed, $\frac{3^5-1}{2}$ would not have a different prime factor from a repunit $\frac{q_k^{2\alpha_k+1}-1}{q_k-1}$ that has 11 as a divisor. A set of primes $\{3, q_k, ...\}$ and exponents $\{5, 2\alpha_k, ...\}$ does not represent an exception to the Theorem 1 if $\frac{3^5-1}{2}$ is chosen to be the first repunit in the product, introducing the prime divisor 11. Each subsequent repunit then would have a distinct prime factor from the previous repunit in the sequence.

3. Formulation of the Condition for Perfect Numbers in terms of the Coefficients of Repunits

Let $N = (4k+1)^{4m+1}\prod_{i=1}^{\ell} q_i^{2\alpha_i}$ [8] and the coefficients $\{a_i\}$ and $\{b_i\}$ be defined by

$$a_i\frac{q_i^{2\alpha_i+1}-1}{q_i-1} = b_i\frac{(4k+1)^{4m+2}-1}{4k} \qquad gcd(a_i, b_i) = 1 \qquad (3.1)$$

If $\frac{\sigma(N)}{N} \neq 2$,

$$\sqrt{2(4k+1)}\left[\frac{q_1^{2\alpha_1+1}-1}{q_1-1}\frac{q_2^{2\alpha_2+1}-1}{q_2-1}...\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}\frac{(4k+1)^{4m+2}-1}{4k}\right]^{\frac{1}{2}} = \sqrt{2(4k+1)}\frac{(b_1...b_\ell)^{\frac{1}{2}}}{(a_1...a_\ell)^{\frac{1}{2}}}\cdot\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)^{\frac{(\ell+1)}{2}} \neq 2(4k+1)^{2m+1}\prod_{i=1}^{\ell} q_i^{\alpha_i} \qquad (3.2)$$

or

$$\frac{b_1...b_\ell}{a_1...a_\ell} = \prod_{i=1}^{\ell} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} \cdot \left(\frac{4k}{(4k+1)^{4m+2}-1}\right)^{\ell}$$
$$\neq 2(4k+1)^{4m+1}\left[\frac{4k}{(4k+1)^{4m+2}-1}\right]^{\ell+1} \prod_{i=1}^{\ell-1} q_i^{2\alpha_i} \cdot \frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} \tag{3.3}$$

When $\ell > 5$ is odd, there exists an odd integer ℓ_o and an even integer ℓ_e such that $\ell = 3\ell_0 + 2\ell_e$, so that

$$\frac{b_1...b_\ell}{a_1...a_\ell} = \left(\frac{b_{13}}{a_{13}}\frac{a_2}{b_2}\right)\left(\frac{b_{46}}{a_{46}}\frac{a_5}{b_5}\right)...\left(\frac{b_{3\ell_o-2,3\ell_o}}{a_{3\ell_o-2,3\ell_o}}\frac{a_{3\ell_o-1}}{b_{3\ell_o-1}}\right)\left(\frac{b_{3\ell_o+1}b_{3\ell_o+2}}{a_{3\ell_o+1}a_{3\ell_o+2}}\right)$$
$$...\left(\frac{b_{\ell-1}b_\ell}{a_{\ell-1}a_\ell}\right)\cdot\frac{s^2}{t^2} \tag{3.4}$$

where $s, t \in \mathbf{Z}$. It has been proven that $\frac{b_{3\bar{i}-2,3\bar{i}}}{a_{3\bar{i}-2,3\bar{i}}}\frac{a_{3\bar{i}-1}}{b_{3\bar{i}-1}} \neq 2(4k+1)\cdot\frac{s^2}{t^2}$ any choice of $a_{3\bar{i}-2}, a_{3\bar{i}-1}, a_{3\bar{i}}$ and $b_{3\bar{i}-2}, b_{3\bar{i}-1}, b_{3\bar{i}}$ consistent with equation (1.1) [7]. Similarly, $\frac{b_\ell}{a_\ell} \neq 2(4k+1)\cdot\frac{s^2}{t^2}$ so that

$$\frac{b_{3\bar{i}-2,3\bar{i}}}{a_{3\bar{i}-2,3\bar{i}}}\frac{a_{3\bar{i}-1}}{b_{3\bar{i}-1}} \equiv 2(4k+1)\frac{\bar{\rho}_{3\bar{i}-2}}{\bar{\chi}_{3\bar{i}-2}}\cdot\frac{s^2}{t^2}$$
$$\frac{b_{3\bar{j}+1}b_{3\bar{j}+2}}{a_{3\bar{j}+1}a_{3\bar{j}+2}} = 2(4k+1)\frac{\hat{\rho}_{3\bar{j}+2}}{\hat{\chi}_{3\bar{j}+2}}\cdot\frac{s^2}{t^2} \tag{3.5}$$

where the fractions are square-free and $gcd(\bar{\rho}_{3\bar{i}-2}, \bar{\chi}_{3\bar{i}-2}) = 1$, $gcd(\hat{\rho}_{3\bar{j}+2}, \hat{\chi}_{3\bar{j}+2}) = 1$. * Then,

$$\frac{b_1...b_\ell}{a_1...a_\ell} = 2(4k+1)\cdot\frac{\bar{\rho}_1}{\bar{\chi}_1}...\frac{\bar{\rho}_{3\ell_o-2}}{\bar{\chi}_{3\ell_o-2}}\frac{\hat{\rho}_{\ell-2\ell_e+2,2}}{\hat{\chi}_{\ell-2\ell_e+2,2}}...\frac{\hat{\rho}_{\ell 2}}{\hat{\chi}_{\ell 2}}\cdot\frac{s^2}{t^2} \tag{3.6}$$

When $\ell = 2\ell_o + 3\ell_e > 4$ for odd ℓ_0 and even ℓ_e,

$$\frac{b_1...b_\ell}{a_1...a_\ell} = 2(4k+1)\left(\frac{4k+1)^{4m+2}-1}{4k}\right)\cdot\frac{\hat{\rho}_{22}}{\hat{\chi}_{22}}...\frac{\hat{\rho}_{2\ell_o,2}}{\hat{\chi}_{2\ell_o,2}}\frac{\bar{\rho}_{\ell-3\ell_e+1}}{\bar{\chi}_{\ell-3\ell_e+1}}...\frac{\bar{\rho}_{\ell-2}}{\bar{\chi}_{\ell-2}}\cdot\frac{s^2}{t^2} \tag{3.7}$$

If the products

$$\prod_{\bar{i}=1}^{\ell_o}\left(\frac{\bar{\rho}_{3\bar{i}-2}}{\bar{\chi}_{3\bar{i}-2}}\right)\prod_{\bar{j}=1}^{\ell_e}\left(\frac{\hat{\rho}_{3\ell_o+2\bar{j},2}}{\hat{\chi}_{3\ell_o+2\bar{j},2}}\right) \tag{3.8}$$

for odd ℓ and

$$\prod_{\bar{i}=1}^{\ell_o}\left(\frac{\hat{\rho}_{2\bar{i},2}}{\hat{\chi}_{2\bar{i},2}}\right)\prod_{j=1}^{\ell_e-1}\left(\frac{\bar{\rho}_{2\ell_o+3\bar{j}+1}}{\bar{\chi}_{2\ell_o+3\bar{j}+1}}\right) \tag{3.9}$$

* The notation has been changed from that of reference [7] with a different choice of index for $\bar{\rho}$, $\bar{\chi}$.

for even ℓ are not the squares of rational numbers,

$$\begin{aligned} \frac{b_1...b_\ell}{a_1...a_\ell} &\neq 2(4k+1)\cdot\frac{s^2}{t^2} & \qquad \ell \textit{ is odd} \\ \frac{b_1...b_\ell}{a_1...a_\ell} &\neq 2(4k+1)\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)\cdot\frac{s^2}{t^2} & \qquad \ell \textit{ is even} \end{aligned} \tag{3.10}$$

which would imply the inequality (3.3) and the non-existence of an odd perfect number.

4. Examples of Integers satisfying the Rationality Condition

Integers $N = (4k+1)^{4m+2}\prod_{i=1}^{\ell}\ q_i^{2\alpha_i}$ which satisfy the condition of rationality of

$$\left[2(4k+1)\frac{q_1^{2\alpha_1+1}-1}{q_1-1}...\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}\frac{(4k+1)^{4m+2}-1}{4k}\right]^{\frac{1}{2}} \tag{4.1}$$

are given in the following list:

$$\begin{aligned} &37^5 3^2 29^2 79^2 83^2 137^2 283^2 313^2 \\ &37^5 3^2 29^2 67^2 79^2 83^2 137^2 283^2 \\ &37^5 3^2 7^2 11^2 29^2 79^2 83^2 137^2 191^2 283^2 \\ &37^5 3^2 5^2 7^2 11^2 13^2 29^2 47^2 79^2 313^2 \\ &37^5 3^2 11^2 13^2 47^2 79^2 211^2 \\ &37^5 3^2 79^2 83^2 137^2 211^2 283^2 313^2 \\ &37^5 3^2 5^2 7^2 11^2 79^2 83^2 137^2 211^2 283^2 313^2 \\ &37^5 3^2 11^2 13^2 29^2 47^2 67^2 79^2 \\ &37^5 7^2 13^2 29^2 47^2 79^2 191^2 313^2 \\ &37^5 3^2 7^2 11^2 67^2 79^2 83^2 137^2 191^2 211^2 283^2 313^2 \\ &37^5 3^2 5^2 11^2 47^2 67^2 79^2 211^2 313^2 \\ &37^5 3^2 5^2 47^2 79^2 191^2 211^2 313^2 \end{aligned} \tag{4.2}$$

None of these integers satisfy the condition $\frac{\sigma(N)}{N} = 2$. For example, the sum of divisors of the first integer is

$$\sigma(37^5 3^2 29^2 79^2 83^2 137^2 283^2 313^2) = 2\cdot 37\cdot 3^4\cdot 7^4\cdot 13^2\cdot 19^2\cdot 31^2\cdot 43^2\cdot 67^2\cdot 73^2\cdot 181^2\cdot 367^2 \tag{4.3}$$

so that new prime factors $7, 13, 19, 31, 43, 67, 73, 181, 367$ are introduced. The inclusion of these prime factors in the integer N leads to yet additional primes, and then the lack of closure of the set of prime factors renders it impossible for them to be paired to give even powers in $\sigma(N)$ with the exception of $2(4k+1)$.

It may be noted that the decompositions of repunits with prime bases of comparable magnitude and exponent 6 include factors that are too large and cannot be matched easily with factors of other repunits. It also can be established that repunits with prime bases and other exponents do not have square-free factors that can be easily matched to provide a closed sequence of such pairings. This can be ascertained from the partial list

$$\begin{aligned} U_5(6,5) &= \frac{5^5-1}{4} = 781 \\ U_6(6,5) &= 3906 = 2\cdot 7\cdot 31\cdot 3^2 \\ U_5(8,7) &= 2801 \\ U_5(12,11) &= 16105 \\ U_5(14,13) &= 30941 \\ U_5(18,17) &= 88741 \\ U_7(4,3) &= 1093 \\ U_7(6,5) &= 19531 \\ U_9(4,3) &= 9841 \\ U_3(32,31) &= 993 = 3\cdot 331 \end{aligned} \tag{4.4}$$

The first prime of the form $4k+1$ which arises as a coefficient D in the equality

$$\frac{q^3-1}{q-1} = Dy^2 \qquad\qquad q\ prime \tag{4.5}$$

is 3541. This prime is too large to be the basis for the factor $\frac{(4k+1)^{4m+2}-1}{4k}$, since it would give rise to many other unmatched prime factors in the product of repunits and the rationality condition would not be satisfied. Amongst the coefficients D which are composite, the least integer with a prime divisor of the form $4k+1$ is 183, obtained when $q = 13$. However, the repunit with base 61 still gives rise to factors which cannot be matched since

$$\frac{61^6-1}{60} = 858672906 = 2\cdot 3\cdot 7\cdot 13\cdot 31\cdot 97\cdot 523 \tag{4.6}$$

5. On the Non-Existence of Coefficients of Repunits satisfying the Perfect Number Condition

Theorem 2. There does not exist any set of primes $\{4k+1; q_1, ..., q_\ell\}$ such that the coefficients $\{a_i\}$ and $\{b_i\}$ satisfy the equation

$$\frac{b_1...b_\ell}{a_1...a_\ell} = 2(4k+1)\left[\frac{4k}{(4k+1)^{4m+2}-1}\right]^{\ell+1}\prod_{i=1}^{\ell} q_i^{2\alpha_i}$$

Proof. It has been observed that $\frac{b_1}{a_1} \neq 2(4k+1)\cdot\frac{s^2}{t^2}$ by the non-existence of multiply perfect numbers with less than four prime factors [5][6],

$$\frac{b_1 b_2}{a_1 a_2} \neq 2(4k+1)\left[\frac{4k}{(4k+1)^{4m+2}-1}\right]^3 q_1^{2\alpha_1} q_2^{2\alpha_2} \tag{5.1}$$

by the non-existence of perfect numbers with three prime divisors [10], and proven that $\frac{b_1 b_2 b_3}{a_1 a_2 a_3} \neq 2(4k+1)\cdot\frac{s^2}{t^2}$ so that the inequality is valid for $\ell = 1, 2, 3$.

Suppose that there are no odd integers N of the form $(4k+1)^{4m+1}\prod_{i=1}^{\ell-1} q_i^{2\alpha_i}$ with $\frac{\sigma(N)}{N} = 2$ so that

$$\frac{b_1...b_{\ell-1}}{a_1...a_{\ell-1}} \neq 2(4k+1)\left[\frac{4k}{(4k+1)^{4m+2}-1}\right]^{\ell}\prod_{i=1}^{\ell-1} q_i^{2\alpha_i} \tag{5.2}$$

If there exists a perfect number with prime factors $\{4k+1, q_1, ..., q_\ell\}$, then $\frac{b_1...b_\ell}{a_1...a_\ell}$ must have $1+\ell+\tau\left(\frac{(4k+1)^{4m+2}-1}{4k}\right) - \tau(U_{4m+2}(4k+2, 4k+1), \prod_{i=1} q_i)$ distinct prime factors, where $U_{4m+2}(4k+2, 4k+1)$ is the Lucas number $\frac{(4k+1)^{4m+2}-1}{4k}$ and $\tau(U_{4m+2}(4k+2, 4k+1), \prod_{i=1} q_i)$ denotes the number of common divisors of the two integers. However, equality $\sigma(N)$ and 2N also implies that $\prod_{i=1}^{\ell}\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ has at least $\ell+2-\tau\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)$ different prime divisors and a maximum of $\ell+1$ prime factors. If there is no cancellation between $\frac{4k}{(4k+1)^{4m+2}-1}$ and $\prod_{i=1}^{\ell}\ q_i^{2\alpha_i}$, multiplication of $\prod_{i=1}^{\ell} U_{2\alpha_i+1}(q_i+1, q_i) = \prod_{i=1}^{\ell}\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ by $U_{4m+2}(4k+2, 4k+1) = \left(\frac{4k}{(4k+1)^{4m+2}-1}\right)^{\ell}$ introduces $\tau\left(\frac{4k}{(4k+1)^{4m+2}-1}\right)$ new prime divisors and $\frac{b_1...b_\ell}{a_1...a_\ell}$ would have $\ell+2$ distinct prime factors. However, if $gcd(U_{4m+2}(4k+2, 4k+1), U_{2\alpha_i+1}(q_i+1, q_i)) = 1$ for all i, the repunit $U_{4m+2}(4k+2, 4k+1)$ must introduce additional prime divisors. It follows that

$$\frac{(4k+1)^{4m+2}-1}{8k} = \prod_{\bar{i}\in I}\ q_i^{2\alpha_i} \tag{5.3}$$

must be valid for some integer set I. There are no integer solutions to

$$\frac{x^n-1}{x-1} = 2y^2 \qquad x \equiv 1(mod\ 4),\ n \equiv 2(mod\ 4) \tag{5.4}$$

and there is no odd integer satisfying these conditions. A variant of this proof has been obtained by demonstrating the irrationality of $\left[\prod_{i=1}^{\ell} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} \cdot \left(\frac{8k(4k+1)}{(4k+1)^{4m+2}-1}\right)\right]^{\frac{1}{2}}$ when $gcd(U_{2\alpha+1}(q_i+1, q_i), U_{4m+2}(4k+2, 4k+1)) = 1$ [6].

When the number of prime factors of $\prod_{i=1}^{\ell} \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ is less than $\ell+1$, there must be at least two divisors of $U_{4m+2}(4k+2, 4k+1)$ which do not arise in the decomposition of this product. While one of the factors is 2, $4k+1$ is not a divisor, implying that any other divisor must be $q_{\bar{j}}$ for some $\bar{j}$. It has been assumed that there are no prime sets $\{4k+1; q_1, ..., q_{\ell-1}\}$ satisfying the perfect number condition. Either $\frac{b_1...b_{\ell-1}}{a_1...a_{\ell-1}}$ does not have $\ell+1$ factors, or if it does have $\ell+1$ factors, then $\prod_{i=1}^{\ell-1} \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ has $\ell-1$ factors but

$$\prod_{i=1}^{\ell-1} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} \neq (4k+1)^{4m+1} \prod_{i\neq \bar{i},\bar{j}} q_i^{2\alpha_i - t_i} \tag{5.5}$$

for some prime $q_{\bar{i}}$, where $q_{\bar{i}}^{t_{\bar{i}}} \parallel U_{4m+2}(4k+2, 4k+1)$. Multiplication by $U_{2\alpha_\ell+1}(q_\ell+1, q_\ell)$ must contain the factor $q_{\bar{i}}^{2\alpha_{\bar{i}}}$, because $\frac{(4k+1)^{4m+2}-1}{4k}$ only would introduce the two primes 2, $q_{\bar{j}}$ and not $q_{\bar{i}}$, since $\prod_{i=1}^{\ell-1} U_{2\alpha_i+1}(q_i+1, q_i)$ contains $\ell-1$ primes, including $4k+1$ and excluding $q_{\bar{i}}$. Interchanging the roles of the primes in the set $\{q_i, i = 1, ..., \ell\}$, it follows that the prime equations

$$\begin{aligned} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= (4k+1)^{h_i} q_{j_i}^{2\alpha_{j_i}} \\ \frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} &= (4k+1)^{h_\ell} q_{\bar{i}}^{2\alpha_{\bar{i}}} \\ \frac{(4k+1)^{4m+2}-1}{4k} &= 2 q_{\bar{j}}^{2\alpha_{\bar{j}}} \end{aligned} \tag{5.6}$$

must hold, where $h_\ell \neq 0$. Since the second equation has no integer solution, $k \geq 1$ [7], it follows that $\frac{b_1...b_{\ell-1}}{a_1...a_{\ell-1}}$ must not have $\ell+1$ prime factors.

There cannot be less than $\ell+1$ different factors of $\frac{b_1...b_{\ell-1}}{a_1...a_{\ell-1}}$, as each new repunit $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ introduces at least one distinct prime divisor by Theorem 1 and the factorization of $\frac{(4k+1)^{4m+2}-1}{4k}$ contains at least two new primes. Consequently, this would imply $\frac{b_1...b_{\ell-1}}{a_1...a_{\ell-1}}$ has at least $\ell+2$ prime factors and $\frac{b_1...b_\ell}{a_1...a_\ell}$ has a minimum of $\ell+3$ prime divisors, which is larger than the number necessary for equality between $U_{4m+2}(4k+2, 4k+1) \prod_{i=1}^{\ell} U_{2\alpha_i+1}(q_i+1, q_i)$ and $2(4k+1) \prod_{i=1}^{\ell} q_i^{2\alpha_i}$.

If the maximum number of prime factors, $\ell+1$, is attained for $\prod_{i=1}^{\ell} U_{2\alpha_i+1}(q_i+1, q_i)$, then the only additional prime divisor arising from multiplication with $U_{4m+2}(4k+2, 4k+1)$

is 2. However, since both 2 and $2k+1$ divide the repunit with base $4k+1$, this property does not hold unless the prime factors of $2k+1$ and $\frac{1}{2k+1}\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)$ can be included in the set $\{q_i, i=1...,\ell\}$. There are then a total of $\ell+2$ prime factors in $\frac{b_1...b_\ell}{a_1...a_\ell}$ only if $\left[\frac{8k}{(4k+1)^{4m+2}-1}\right]^{\ell+1}$ can be absorbed into $\prod_{i=1}^{\ell} q_i^{2\alpha_i}$.

The repunit then can be expressed as

$$\frac{(4k+1)^{4m+2}-1}{4k} = 2\prod_{j\in\{K\}} q_j^{t_j} \tag{5.7}$$

where $\{K\}\subseteq\{1,...,q_\ell\}$. By the non-existence of odd perfect numbers with prime factors $4k+1, q_i, i=1...,\ell-1$, either $U_{4m+2}(4k+2,4k+1)\prod_{i=1}^{\ell-1}U_{2\alpha_i+1}(q_i+1,q_i)$ has less than $\ell+1$ factors or

$$\prod_{i=1}^{\ell-1}\frac{q_i^{2\alpha_i+1}-1}{q_i-1} \neq (4k+1)^{4m+1}\prod_{i\in\overline{\{K\}-\{q_{\bar{i}}\}}} q_i^{2\alpha_i}\prod_{\{K\}-\{q_{\bar{i}}\}} q_j^{2\alpha_j-t_j} \tag{5.8}$$

for some $\bar{i}\in\{1,...,\ell\}$. If the product of repunits for the prime basis $\{4k+1,q_1,...,q_{\ell-1}\}$ has less than $\ell+1$ factors, $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ introduces at least two new prime factors. Even if one of these divisors is $4k+1$, the other factor must be $q_{\bar{i}}$ for some $\bar{i}\neq\ell$. Interchange of the primes $q_i, i=1,...,\ell$ again yields the relations in equation (5.6).

If the product (5.8) includes ℓ primes, and not $q_{\bar{i}}$, then $q_{\bar{i}}^{2\alpha_{\bar{i}}}$ must be a factor of $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$. Interchanging the primes, it follows that the product (5.8) includes $\prod_{i\neq\bar{i}} q_i^{2\alpha_i}$.

The prime divisors in $U_{2\alpha_\ell+1}(q_\ell+1,q_\ell)$ either can be labelled q_{j_ℓ} for some $j_\ell\in\{1,...,\ell-1\}$ or equals $4k+1$. while $4k+1$ does not divide $U_{4m+2}(4k+2,4k+1)$, it can occur in the other repunits $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ for $i=1,...,\ell-1$. The prime $q_{j_\ell}=q_{\bar{i}}$ also may not be a divisor of $\frac{(4k+1)^{4m+2}-1}{4k}$.

One choice for q_ℓ is a prime divisor of $2k+1$. If $2k+1$ is prime, the factors of $\frac{(2k+1)^{2\alpha_\ell+1-1}}{2k}$ and $\frac{(4k+1)^{4m+2}-1}{4k}$ can be compared. Since the latter quotient is equal to $\frac{(4k+1)^{2m+1}-1}{4k}\cdot[(4k+1)^{2m+1}+1]$, consider setting m equal to α_ℓ. By Theorem 1, primitive divisor P divides both repunits if $4k=nP+2k$ or $2k=nP$ implying $n=2$, $P=k$. However, if $k\Big|\frac{(2k+1)^{2\alpha_\ell+1-1}}{2k}$, $1+(2k+1)+...+(2k+1)^{2\alpha_\ell}\equiv 2\alpha_\ell+1\equiv 0\ (mod\ k)$ and $k=2\alpha_\ell+1$ for prime exponents $2\alpha_\ell+1$. Suppose $\frac{(2k+1)^k-1}{2k}=k\cdot\prod_i\ P_i^{m_i}$ and $\frac{(4k+1)^k-1}{4k}=k\cdot\prod_j\ \tilde{P}_j^{n_j}$. If a primitive divisor $rk+1$, $r\in\mathbf{Z}$, divides both repunits, $(2k+1)^k-1=(rk+1)(x-1)$, $x\in\mathbf{Z}$, $(4k+1)^k-1=(rk+1)(x'-1)$, $x'\in\mathbf{Z}$, so that

$$\frac{(2k+1)^k-1}{x-1}=\frac{(4k+1)^k-1}{x'-1} \tag{5.9}$$

This relation would imply

$$x'(2k+1)^k - (2k+1)^k - x' = x(4k+1)^k - (4k+1)^k - x \tag{5.10}$$

and therefore $x-1 = b(4k+1)^k$, $x'-1 = a(4k+1)^k$. However, the equation is then

$$a(4k+1)^k(2k+1)^k - a(4k+1)^k \;=\; b(4k+1)^k(2k+1)^k - b(2k+1)^k \tag{5.11}$$

which cannot be satisfied by any integers a, b. Not all primitive divisors of the two repunits are identical. For the exception, $k=1$, $2\alpha_\ell+1=5$, $4m+2=10n$, $\frac{(2k+1)^{2\alpha_\ell+1}-1}{2k} = 11^2$, $\frac{(4k+1)^{4m+2}-1}{4k}$ has other prime factors in addition to 11.

If $m \neq \alpha_\ell$, it can be demonstrated that there is a primitive divisor of $\frac{(2k+1)^{2\alpha_\ell}+1-1}{2k}$ which is not a factor of $\frac{(4k+1)^{4m+2}-1}{4k}$ by using the comparison of the linear factors in the decomposition of each numerator in Theorem 1, since $4k \neq (2k)^n,\ n \geq 2$ and $4k+2 \neq (2k+2)^n,\ n \geq 2$ for any $k > 1$.

In general, either there exists a prime factor of $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ which does not divide $\frac{(4k+1)^{4m+2}-1}{4k}$ or there are more than $\ell+2$ primes in the decomposition of the product of repunits. The existence of a distinct prime divisor requires a separate demonstration for composite exponents. Let $p_1(4m+2)$, $p_2(4m+2)$ be two prime divisors of $4m+2$. Then $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ has a prime factor different from the divisors of $\frac{(4k+1)^{p_1(4m+2)}-1}{4k}$ and $\frac{(4k+1)^{p_2(4m+2)}-1}{4k}$ by Theorem 1. While the union of the sets of prime divisors of the two repunits is contained in the factorization of $\frac{(4k+1)^{p_1(4m+2)p_2(4m+2)}-1}{4k}$, the repunit with the composite exponent will have a primitive divisor, which does not belong to the union of the two sets and equals $a_1p_1p_2+1$. If this prime divides $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$, $a_1p_1p_2+1 = a_2(2\alpha_\ell+1)+1$. Either $2\alpha_\ell+1|4m+2$ or the primitive divisor equals $a_{12}(2\alpha_\ell+1)p_1p_2+1$. Suppose

$$(a_{12}(2\alpha_\ell+1)+1)(x+1) = \frac{(4k+1)^{p_1p_2}-1}{4k} \equiv p_1p_2\ (mod\ 4) \tag{5.12}$$

Since $p_1,\ p_2$ are odd primes, either $a_{12} \equiv 0\ (mod\ 4)$ or $a_{12} \equiv 2\ (mod\ 4)$, so that the primitive divisor equals $4c(2\alpha_\ell+1)p_1p_2+1$ when $p_1p_2 \equiv 1\ (mod\ 4)$ or $(4c+2)(2\alpha_\ell+1)p_1p_2+1$ if $p_1p_2 \equiv 3\ (mod\ 4)$. Let the primitive divisor P have the form $4c(2\alpha_\ell+1)p_1p_2+1$ so that

$$(4c(2\alpha_\ell+1)p_1p_2+1)(y+1) = \frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} \equiv 2\alpha_\ell+1\ (mod\ q_\ell-1) \tag{5.13}$$

which implies that $y+1 = \kappa_2[2\alpha_\ell + \chi_2(q_\ell-1)]$ is either the multiple of an imprimitive divisor, $\kappa_2'(2\alpha_\ell+1)$ or it is the multiple of a primitive divisor. Consider the equation

$$(4c(2\alpha_\ell+1)p_1p_2+1)\kappa_2(2\alpha_\ell+1) = \frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} \tag{5.14}$$

with $2\alpha_\ell | q_\ell - 1$. If

$$(4c(2\alpha_\ell+1)p_1p_2+1)\kappa_2(2\alpha_\ell+1) \equiv 2\alpha_\ell+1 \ (mod\ q_\ell - 1) \tag{5.15}$$

$q_\ell - 1|(4c(2\alpha_\ell+1)p_1p_2+1)\kappa_2 - 1$ or $(4c(2\alpha_\ell+1)p_1p_2+1)\kappa_2 - 1 \equiv \frac{q_\ell-1}{2\alpha_\ell-1}$ $(mod\ q_\ell - 1)$. Since $2\alpha_\ell+1|\kappa_2-1$, $\kappa_2 = \kappa_3(2\alpha_\ell+1)+1$. Let

$$(z(q_\ell-1)+1)(2\alpha_\ell+1) = \frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} \tag{5.16}$$

Then

$$z = 2\alpha_\ell + \frac{q_\ell-1}{2\alpha_\ell+1}(2\alpha_\ell - 1 + (2\alpha_\ell-2)(q_\ell+1)+...+q_\ell^{2\alpha_\ell-2}+...+1) \tag{5.17}$$

is integer and $z(q_\ell-1) = 4c(2\alpha_\ell+1)^2\kappa_3p_1p_2+\kappa_3(2\alpha_\ell+1)+4c(2\alpha_\ell+1)p_1p_2$ and $2\alpha_\ell+1|\kappa_2-1$ so that $\kappa_2 = \kappa_3(2\alpha_\ell+1)+1$. Let

$$(z(q_\ell-1)+1)(2\alpha_\ell+1) = \frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} \tag{5.18}$$

Then

$$z = 2\alpha_\ell + \frac{q_\ell-1}{2\alpha_\ell+1}(2\alpha_\ell - 1 + (2\alpha_\ell-2)(q_\ell+1)+...+q_\ell^{2\alpha_\ell-2}+...+1) \tag{5.19}$$

is integer and $z(q_\ell-1) = 4c(2\alpha_\ell+1)^2\kappa_3p_1p_2+\kappa_3(2\alpha_\ell+1)+4c(2\alpha_\ell+1)p_1p_2$

In the latter case, the product of the primitive divisors takes the form $b(2\alpha_\ell+1)+1$ so that $\kappa_2[(2\alpha_\ell+1)+\chi_2(q_\ell-1)] = b(2\alpha_\ell+1)+1$ since $2\alpha_\ell+1$ is prime. The two congruence relations

$$\begin{aligned} [(b-1)+(b(2\alpha_\ell+1)4cp_1p_2](2\alpha_\ell+1)+1 &\equiv 0 \ (mod\ q_\ell-1) \\ (b-\kappa_2)(2\alpha_\ell+1)+1 &\equiv 0 \ (mod\ q_\ell-1) \end{aligned} \tag{5.20}$$

imply

$$[\kappa_2 - 1 + (b(2\alpha_\ell+1)+1)4cp_1p_2](2\alpha_\ell+1) \equiv 0 \ (mod\ q_\ell-1) \tag{5.21}$$

When $2\alpha_i+1 \not| q_\ell - 1$, it follows that $q_\ell - 1|\kappa_2 - 1 + (b(2\alpha_\ell+1)+1)4cp_1p_2$. Since every primitive divisor is congruent to 1 modulo $2\alpha_\ell+1$, $\kappa_2 = \kappa_3(2\alpha_\ell+1)+1$ and $q_\ell - 1|\kappa_3(2\alpha_\ell+1)+(b(2\alpha_\ell+1)+1)4cp_1p_2$.

The factorizations

$$\begin{aligned} (4k+1)^{4m+2}-1 &= \prod_{k=0}^{4m+1}((4k+1)-\omega_{4m+2}^k) \\ q_\ell^{2\alpha_\ell+1}-1 &= \prod_{k'=0}^{2\alpha_i}(q_\ell-\omega_{2\alpha_i+1}^{k'}) \end{aligned} \tag{5.22}$$

yield real factors which can be identified only if $((4k+1)-1)^{t_k} = (4k)^{t_k} = (q_\ell - 1)^{t'_k}$ or $(4k+2)^{t_k} = (q_\ell + 1)^{t'_k}$. Then, $4k = (q_\ell - 1)^{n_\ell}$, $t_k = n_\ell t'_k$ or $4k+2 = (q_\ell+1)^{n_\ell}$, $t_k = n_\ell t'_k$. Since $\sum_k \ t_k = 4m+2$, $\sum_k \ t'_k = 2\alpha_i + 1$, $4m+2 = n_\ell(2\alpha_\ell+1)$. If $4m+2$ is the product of two primes p_1p_2, $2\alpha_\ell+1$ must equal one of the primes, p_2.

The congruence (5.15) becomes

$$[(4c(2\alpha_\ell+1)^2 p_1 + 1)\kappa_2 - 1](2\alpha_\ell+1) \equiv 0 \ (mod \ q_\ell - 1) \tag{5.23}$$

If $4k = (q_\ell-1)^{p_1}$, then $\frac{(4k+1)^{p_1p_2}-1}{4k} = \frac{((q_\ell-1)^{p_1}+1)^{p_1p_2}-1}{(q_\ell-1)^{p_1}} = p_1p_2 + \binom{p_1p_2}{2}(q_\ell-1) + ... + p_1p_2(q_\ell-1)^{p_1(p_1p_2-2)} + (q_\ell-1)^{p_1(p_1p_2-1)}$ When $2\alpha_i+1|q_\ell-1$, $2\alpha_i+1|p_1p_2 + \binom{p_1p_2}{2}(q_\ell-1) + ... + p_1p_2(q_\ell-1)^{p_1(p_1p_2-2)} + (q_\ell-1)^{p_1(p_1p_2-1)}$. However, $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} \equiv 2\alpha_\ell+1 \ (mod \ q_\ell-1)$, whereas $\frac{(4k+1)^{p_1p_2}-1}{4k} \equiv p_1(2\alpha_\ell+1) \ (mod \ q_\ell-1)$, $p_1 > 1$. The repunits have the same prime divisors only if $p_1 \equiv 1 \ (mod \ q_\ell-1)$, which implies that $p_1 = \rho(q_\ell-1)+1$. Then $\frac{((q_\ell-1)^{\rho(q_\ell-1)}+1)^{\rho(q_\ell-1)(2\alpha_\ell+1)-1}}{(q_\ell-1)^{\rho(q_\ell-1)}}$ includes at least seven new factors. If $4k+2 = (q_\ell+1)^{p_1}$, then $\frac{(4k+1)^{p_1p_2}-1}{4k} \equiv \frac{(2^{p_1}-1)^{p_1p_2}-1}{2^{p_1}-2} \ (mod \ q_\ell-1)$. If $2^{p_1}-1 \equiv 1 \ (mod \ q_\ell-1)$, $1 + (2^{p_1}-1) + ... + (2^{p_1}-1)^{p_1p_2-1} \equiv p_1p_2 \equiv 2\alpha_\ell+1 \ (mod \ q_\ell-1)$ only if $p_1 \equiv 1 \ (mod \ q_\ell-1)$. When $2^{p_1}-1 \equiv 2n+1 \ (mod \ q_\ell-1)$, $\frac{(2^{p_1}-1)^{p_1p_2}-1}{2^{p_1}-2} \equiv \frac{(2n+1)^{p_1p_2}-1}{2n}$ which is not divisible by $2\alpha_i+1$ if $p_2 \not| 2n$. When $2\alpha_i+1|n$, $\frac{(2n)^{p_1p_2}-1}{2n} \equiv p_1p_2 \ (mod \ 2n)$ which would be congruent to $2\alpha_i+1$ only if $p_1 \equiv 1 \ (mod \ 2n)$ so that $p_1 \geq 4\alpha_i+3$. The exponent is a product of a minimimum of three primes and introduces at least seven different divisors in the product of repunits. A similar conclusion is obtained if $4m+2 = p_1...p_s, \ s \geq 3$.

If every prime divisor of $\frac{(4k+1)^{4m+2}-1}{4k}$ is distinct from the factors of $\prod_{i=1}^{\ell} \ \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$, the following equations are obtained

$$\begin{aligned} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= (4k+1)^{h_i} q_{j_i}^{2\alpha_{j_i}} \\ \frac{(4k+1)^{4m+2}-1}{4k} &= 2 \\ \sum_{i=1}^{\ell} \ h_i &= 4m+1 \end{aligned} \tag{5.24}$$

which has only the solutions $(q_i, 2\alpha_i+1; 4k+1, 4m+2) = (q_i, 2\alpha_i+1; 1, 2)$.

The number of equal prime divisors in $\frac{q_{i_0}^{2\alpha_{i_0}+1}-1}{q_{i_0}-1}$ and $\frac{(4k+1)^{4m+2}-1}{4k}$ can be chosen to be greater than 1, but equality of $\sigma(N)$ and $2N$ implies that each distinct prime divisor q_{j_i} of the repunit $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ appears in the factorization with the exponent $2\alpha_{j_i}$. Consequently, it would be inconsistent with the perfect number condition for divisors of $\frac{q_{i_0}^{2\alpha_{i_0}+1}-1}{q_{i_0}-1}$ other than $q_{j_{i_0}}$ to exist.

If the prime divisor $q_{j_{i_0}}$ of $\frac{q_{i_0}^{2\alpha_{i_0}+1}-1}{q_{i_0}-1}$ is a factor of $\frac{(4k+1)^{4m+2}-1}{4k}$, then one formulation of the perfect number condition for the integer $N=(4k+1)^{4m+1}q_i^{2\alpha_i}$ is

$$\begin{aligned} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= (4k+1)^{h_i} q_{j_i}^{2\alpha_{j_i}} \qquad\qquad i \neq i_0 \\ \frac{q_{i_0}^{2\alpha_{i_0}+1}-1}{q_{i_0}-1} &= q_{j_{i_0}}^{h'_{j_{i_0}}} \\ \frac{(4k+1)^{4m+2}-1}{4k} &= 2q_{i_0}^{2\alpha_{j_{i_0}}-h'_{j_{i_0}}} \end{aligned} \tag{5.25}$$

The last relation in equation (5.25) has no solutions with $k \geq 1$ since $\frac{(4k+1)^{4m+2}-1}{4k}$ has at least four different prime divisors, 2,$2k+1$, the prime factors of $\frac{(4k+1)^{2m+1}-1}{4k}$ and its primitive divisors. Furthermore, as the number of prime divisors of $\frac{(4k+1)^{4m+2}-1}{4k}$ is less than seven, $\frac{q_{i_0}^{2\alpha_{i_0}+1}-1}{q_{i_0}-1}$ also should have a different prime factor which is contrary to the equation (5.25).

Since $k \geq 1$ in the decomposition $N=(4k+1)^{4m+1}\prod_{i=1}^{\ell} q_i^{2\alpha_i}$, the quotient $\frac{(4k+1)^{2m+1}-1}{4k}$, $k \geq 1$ has a distinct prime divisor from the factors of $\prod_{i=1}^{\ell} \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ because of its existence in the factorization of $\frac{(4k+1)^{p_1(2m+1)}-1}{4k}$ by Theorem 1. There would be then at least $\ell+3$ prime factors of $\sigma(N)$ implying that N is not an odd perfect number.

■

6. A Proof of the Odd Perfect Number Conjecture

Theorem 3. There are no odd perfect numbers.

Proof. The exponents will be assumed to be prime, since repunits with composite exponents have at least three prime divisors [7]. By Theorem 1, if the bases q_i, q_j and the exponents $2\alpha_i+1$, $2\alpha_j+1$ are odd primes, there exists a prime divisor which is not a common factor of the repunits $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ and $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$. Consequently, with the condition (2.1) on the prime bases and exponents, there will be no odd prime pairs (q_i,q_j) satisfying the exponential Diophantine equation $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}=\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$.

Additionally, it has been established that there are at most two solutions for fixed x and y if $y \geq 7$ [4]. If $a=\frac{y-1}{\delta}$, $b=\frac{x-1}{\delta}$, $c=\frac{y-x}{\delta}$, $\delta=gcd(x-1,y-1)$ and s is the least integer such that $x^s \equiv 1 \ (mod\ by^{n_1})$, where (x,y,m_1,n_1) is a presumed solution of the Diophantine equation, then m_2-m_1 is a multiple of s if (x,y,m_2,n_2) is another solution with the same bases x, y [4]. As $x < y$, x can be identified with q_j, y with with q_i and δ with $gcd(q_j - 1, q_i - 1)$. Given that the first exponents are $2\alpha_i + 1$ and $2\alpha_j + 1$, the constraint on a second exponent of q_j then would be $m_2 - (2\alpha_j + 1) = \iota\varphi\left(\left(\frac{q_j-1}{\delta}\right) q_i^{2\alpha_i+1}\right)$ where $\iota \in \mathbf{Q}$, with $\iota \in \mathbf{Z}$ when $q_j \not| q_i - 1$. This constraint can be extended to $m_1 = n_1 = 1$, since it follows from the equations $(y-1)x^{m_i} - (x-1)y^{n_i} = y - x$, $ax^{m_i} - by^{n_i} = c$, which hold also for these values of the exponents. Based on the trivial solution to these equations for an arbitrary pair of odd primes q_i, q_j, the first non-trivial solution for the exponent of q_j would be greater than or equal to than $1 + \iota(q_i - 1)\varphi\left(\frac{q_j-1}{\delta}\right)$. This exponent introduces new prime divisors which are factors of $\frac{q_j^{1+\iota(q_i-1)\varphi\left(\left(\frac{q_j-1}{\delta}\right)\right)}-1}{q_j-1}$ and therefore does not minimize the number of prime factors in the product of repunits.

Imprimitive prime divisors can be introduced into equations generically relating the two unequal repunits and minimizing the number of prime factors in the product $\frac{q_i^{2\alpha_i+1}-1}{q_i-1} \cdot \frac{q_j^{2\alpha_j+1}-1}{q_j-1}$. Given an odd integer $N = (4k+1)^{4m+1} \prod_{i=1}^{\ell} q_i^{2\alpha_i}$ the least number of unmatched prime divisors in $\sigma(N)$ will be attained if there are pairs (q_i, q_j) satisfying one of the three relations

$$\begin{aligned}\frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= (2\alpha_i+1)\cdot\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\\ (2\alpha_j+1)\cdot\frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= \frac{q_j^{2\alpha_j+1}-1}{q_j-1}\\ (2\alpha_j+1)\cdot\frac{q_i^{2\alpha_i+1}-1}{q_i-1} &= (2\alpha_i+1)\cdot\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\end{aligned} \tag{6.1}$$

As $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ would not introduce any additional prime divisors if the first relation in equation (6.1) is satisfied, the product of two pairs of repunits of this kind, with prime bases (q_i, q_j), $(q_k, q_{k'})$, yields a minimum of three distinct prime factors, if two of the repunits are primes. In the second relation, $(q_i, 2\alpha_i + 1)$ and $(q_j, 2\alpha_j + 1)$ are interchanged, whereas in the the third relation, an additional prime divisor is introduced when $2\alpha_i + 1 \neq 2\alpha_j + 1$. Equality of $\sigma(N)$ and 2N is possible only when the prime divisors of the product of repunits also arise in the decomposition of N. If the two repunits $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ and $\frac{q_{k'}^{2\alpha_{k'}+1}-1}{q_{k'}-1}$ are prime, they will be bases for new repunits

$$\frac{\left[\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\right]^{2\alpha_{n_1}+1}-1}{\frac{q_j^{2\alpha_j+1}-1}{q_j-1}-1} = \frac{q_j-1}{q_j(q_j^{2\alpha_j}-1)}\cdot\left[\left[\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\right]^{2\alpha_{n_1}+1}-1\right]$$
$$\frac{\left[\frac{q_{k'}^{2\alpha_{k'}+1}-1}{q_{k'}-1}\right]^{2\alpha_{n_2}+1}-1}{\frac{q_{k'}^{2\alpha_{k'}+1}-1}{q_{k'}-1}-1} = \frac{q_{k'}-1}{q_{k'}(q_{k'}^{2\alpha_{k'}}-1)}\cdot\left[\left[\frac{q_{k'}^{2\alpha_{k'}+1}-1}{q_{k'}-1}\right]^{2\alpha_{n_2}+1}-1\right] \quad (6.2)$$

These new quotients are either additional prime divisors or satisfy relations of the form

$$\frac{\left[\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\right]^{2\alpha_{n_1}+1}-1}{\frac{q_j^{2\alpha_j+1}-1}{q_j-1}-1} = (2\alpha_{n_1}+1)\frac{q_{t_1}^{2\alpha_{t_1}+1}-1}{q_{t_1}-1} \quad (6.3)$$

to minimize the introduction of new primes. Furthermore, $2\alpha_{n_1}+1$ is the factor of lesser magnitude,

$$\frac{q_t^{2\alpha_t+1}-1}{q_t-1} > \left[\frac{q_j-1}{q_j(q_j^{2\alpha_j}-1)}\cdot\left[\left[\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\right]^{2\alpha_{n_1}+1}-1\right]\right]^{\frac{1}{2}} \quad (6.4)$$

The number of prime divisors of N is minimized if the repunit $\frac{q_t^{2\alpha_t+1}-1}{q_t-1}$ is a prime. However, it is then the basis of another repunit

$$\frac{\left[\frac{q_t^{2\alpha_t+1}-1}{q_t-1}\right]^{2\alpha_{n_3}+1}-1}{\left[\frac{q_t^{2\alpha_t+1}-1}{q_t-1}\right]-1} > \frac{\left[\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\right]^{2\alpha_{n_1}+1}-1}{\frac{q_j^{2\alpha_j+1}-1}{q_j-1}-1} \quad (6.5)$$

since $\alpha_{n_3} \geq 1$. An infinite sequence of repunits of increasing magnitude is generated, implying the non-existence of odd integers N with prime factors satisfying one of the relations in equation (6.1).

If the third relation in equation (6.1) holds,

$$(2\alpha_t+1)\frac{\left[\frac{q_j^{2\alpha_j+1}-1}{q_j-1}\right]^{2\alpha_{n_1}+1}-1}{\frac{q_j^{2\alpha_j+1}-1}{q_j-1}-1} = (2\alpha_{n_1}+1)\frac{q_{t_1}^{2\alpha_{t_1}+1}-1}{q_{t_1}-1} \quad (6.6)$$

The inequality (6.4) is still satisfied, and again, repunits of increasing magnitude are introduced in the sum of divisors.

If the repunits $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ and $\frac{q_{\ell'}^{2\alpha_{\ell'}+1}-1}{q_{\ell'}-1}$ are not prime, then the product of the four repunits with bases $q_i, q_j, q_k, q_{k'}$ would have a minimum of five different prime divisors. The product of $\frac{(4k+1)^{4m+2}-1}{4k}$, with at least two distinct prime divisors, and $\prod_{i=1}^{\ell} \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ possesses a minimum of $\ell+3$ different prime factors and the perfect number condition cannot be satisfied.

Even when none of the three relations hold in equation (6.1) hold, by Theorem 1, there exists a primitive divisor which is not a common divisor of both repunits. It can be assumed that the last repunit $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ has a prime exponent since it will introduce at least three prime divisors if the exponent is composite. The new prime divisor may be denoted q_{j_ℓ} if it does not equal $4k+1$, and interchanging q_ℓ with $q_{\bar{i}}$ and $4k+1$, it can be deduced that $U_{4m+2}(4k+2,4k+1)\prod_{\substack{i=1 \\ i\neq\bar{i}}}^{\ell} U_{2\alpha_i+1}(q_i+1,q_i)$ will not be divisible by $j_{\bar{i}} \in 1,...,\ell$, $j_{\bar{i}} \neq \bar{i}$, with the exception of one value i_0. The prime q_{j_0} will not be a factor of $\prod_{i=1}^{\ell} U_{2\alpha_i+1}(q_i+1,q_i)$. Then

$$\begin{aligned} \frac{q_{\bar{i}}^{2\alpha_{\bar{i}}+1}-1}{q_{\bar{i}}-1} &= q_{j_{\bar{i}}}^{h_{j_{\bar{i}}}} \qquad \bar{i} \neq i_0 \\ \frac{q_{i_0}^{2\alpha_{i_0}+1}-1}{q_{i_0}-1} &= (4k+1)^{4m+1} \\ \frac{(4k+1)^{4m+2}-1}{4k} &= 2q_{j_{i_0}}^{h_{j_{i_0}}} \end{aligned} \tag{6.7}$$

The last equation is equivalent to

$$\begin{aligned} \frac{(4k+1)^{2m+1}-1}{4k} &= y_1^2 \\ \frac{(4k+1)^{2m+1}+1}{2} &= y_2^2 \\ y_1 y_2 &= q_{j_{i_0}}^{\frac{h_{j_{i_0}}}{2}} \end{aligned} \tag{6.8}$$

which has no integer solutions for k and m. There are no prime sets $\{4k+1; q_1, ..., q_\ell\}$ which satisfy these conditions with $h_{j_{\bar{i}}}$ and $h_{j_{i_0}}$ even. Thus, there are no odd integers $N = (4k+1)^{4m+1}\prod_{i=1}^{\ell} q_i^{2\alpha_i}$ such that the primes and exponents $\{q_i, 2\alpha_i+1\}$ satisfy none of the relations in equation (6.1) and equality between $\sigma(N)$ and $2N$. For all odd integers, $\frac{\sigma(N)}{N} \neq 2$.

■

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