# Fibonacci connection between Huffman codes and Wythoff array

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**Abstract**. A non-decreasing sequence of positive integer weights $P = \{p_1, p_2, \ldots, p_n\}$ is called *k-ordered* if an intermediate sequence of weights produced by Huffman algorithm for initial sequence $P$ on $i$-th step satisfy the following conditions: $p_2^{(i)} = p_3^{(i)},\ i = \overline{0,k};\ p_2^{(i)} < p_3^{(i)},\ i = \overline{k+1, n-3}$. Let $T$ be a binary tree of size $n$ and $M=M(T)$ be a set of such sequences of positive *integer* weights that $\forall P \in M$ the tree $T$ is the Huffman tree of $P$ ($|P|=n$). A sequence $P_{\min}$ of $n$ positive integer weights is called a *minimizing* sequence of the binary tree $T$ in the class $M$ ($P_{\min} \in M$) if $P_{\min}$ produces the minimal Huffman cost of the tree $T$ over all sequences from $M$, i.e., $E(T,P_{\min}) \le E(T,P)\ \forall P \in M$. Fibonacci related connection between minimizing $k$-ordered sequences of the maximum height Huffman tree and the Wythoff array [Sloane, A035513] has been proved. Let $M_{n,k}$ ($k = \overline{0, n-3}$) denote the set of all $k$-ordered sequences of size $n$ for which the Huffman tree has maximum height. Let $F(i)$ denote $i$-th Fibonacci number. **Theorem**: A minimizing $k$-ordered sequence of the maximum height Huffman tree in the class $M_{n,k}$ ($k = \overline{0, n-3}$) is $Pmin_{n,k} = \{p_1, p_2, \ldots, p_n\}$, where $p_1 = 1$, $p_2 = F(1)$, …, $p_{k+2} = F(k+1)$, $p_{k+3} = F(k+2) = w_{F(k+2),0}$, $p_{k+4} = w_{F(k+2),1}$, $p_{k+5} = w_{F(k+2),2}$, …, $p_n = w_{F(k+2),n-k-3}$, $w_{i,j}$ is $(i,j)$-th element of the Wythoff array■. The cost of Huffman trees for those sequences has been computed. Several examples of minimizing ordered sequences for Huffman codes are shown.

## 1. Main Conceptions and Terminology

### 1.1. Binary Trees

A (strictly) ***binary tree*** is an oriented ordered tree where each nonleaf node has exactly two children (siblings). A binary tree is called ***elongated*** if at least one of any two sibling nodes is a leaf. An elongated binary tree of size $n$ has maximum height among all binary trees of size $n$. An elongated binary tree is called ***left-sided*** if the right node in each pair of sibling nodes is a leaf.

A binary tree is called ***labeled*** if a certain positive integer (weight) is set in correspondence with each leaf.

***Size of a tree*** is the total number of *leaves* of this tree.

**Definition**. Let $T$ be a binary tree with positive weights $P$={$p_1$,.., $p_n$} at its leaf nodes. The ***weighted external path length*** of $T$ is

$$E(T,P) = \sum_{i=1}^{n} l_i p_i$$

where $l_i$ is the length of the path from the root to leaf $i$.

## 1.2. Huffman Algorithm

**Problem definition**. Given a sequence of $n$ positive weights $P$={$p_1$,…, $p_n$}. The problem is to find binary tree $T_{min}$ with $n$ leaves labeled $p_1$,..., $p_n$ that has minimum weighted external path length over all possible binary trees of size $n$ with the same sequence of leaf weights. $T_{min}$ is called the Huffman tree of the sequence $P$; $E(T,P_{min})$ is called the Huffman cost of the tree $T$.

The problem was solved by Huffman algorithm [1]. That algorithm builds $T_{min}$ in which each leaf (weight) is associated with a (prefix free) codeword in alphabet {0, 1}.

*Note*. A code is called a prefix (free) code if no codeword is a prefix of another one.

**Algorithm description** (in the reference to the discussed issue).

Algorithm input. A non-decreasing sequence of positive weights

$$P = \{p_1, p_2, \dots, p_n\} \ (p_k \le p_{k+1}; k = \overline{1, n-1}).$$

Algorithm output. The sum of all the weights.

The algorithm is performed in $n$-1 steps. $i$-th step ($i = \overline{1, n-1}$) is as follows.

- $i$-th step input. A non-decreasing sequence of weights of size $n$-$i$+1.

  $$P^{(i-1)} = \{ p_1^{(i-1)}, p_2^{(i-1)}, \dots, p_{n-i+1}^{(i-1)} \} \ (p_k^{(i-1)} \le p_{k+1}^{(i-1)}; k = \overline{1, n-i}); \ |P^{(i-1)}| = n-i+1.$$

- $i$-th step method. Build a sequence { $p_1^{(i-1)} + p_2^{(i-1)}, p_3^{(i-1)}, \dots, p_{n-i+1}^{(i-1)}$ } and sort its.
- $i$-th step output. A non-decreasing sequence of weights of size $n$-$i$.

  $$P^{(i)} = \{ p_1^{(i)}, p_2^{(i)}, \dots, p_{n-i}^{(i)} \} \ (p_k^{(i)} \le p_{k+1}^{(i)}; k = \overline{1, n-i-1}); \ |P^{(i)}| = n-i.$$

*Note 1*. $P^{(0)}$ is an input of Huffman algorithm, i.e.,

$$p_k^{(0)} = p_k \ (k = \overline{1, n}). \qquad (1)$$

*Note 2*. If an input sequence on $i$-th step(s) of the algorithm satisfies condition

$$p_2^{(i)} = p_3^{(i)} (0 \le i \le n-3),$$

then several Huffman trees can result from initial sequence $P$ of weights, but the weighted external path length is the same in all these trees.

Let $P$ = {$p_1$, $p_2$, $p_3$, …, $p_n$} be a sequence of size $n$ for which the binary Huffman tree is ***elongated***. Then according to Huffman algorithm

$$p_1^{(i)} + p_2^{(i)} \le p_4^{(i)}, i = \overline{0, n-3}. \qquad (2)$$

### 1.3. Wythoff Array

The Wythoff array is shown below, to the right of the vertical line. It has many interesting properties [2, 3, 4].

| Row number | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | Note |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | Fibonacci numbers |
| Fib[2] | 1 | 3 | 4 | 7 | 11 | 18 | 29 | 47 | 76 | 123 | 199 | 322 | 521 | Lucas numbers |
| Fib[3] | 2 | 4 | 6 | 10 | 16 | 26 | 42 | 68 | 110 | 178 | 288 | 466 | 754 | |
| Fib[4] | 3 | 6 | 9 | 15 | 24 | 39 | 63 | 102 | 165 | 267 | 432 | 699 | 1131 | |
| | 4 | 8 | 12 | 20 | 32 | 52 | 84 | 136 | 220 | 356 | 576 | 932 | 1508 | |
| Fib[5] | 5 | 9 | 14 | 23 | 37 | 60 | 97 | 157 | 254 | 411 | 665 | 1076 | 1741 | |
| | 6 | 11 | 17 | 28 | 45 | 73 | 118 | 191 | 309 | 500 | 809 | 1309 | 2118 | |
| | 7 | 12 | 19 | 31 | 50 | 81 | 131 | 212 | 343 | 555 | 898 | 1453 | 2351 | |
| Fib[6] | 8 | 14 | 22 | 36 | 58 | 94 | 152 | 246 | 398 | 644 | 1042 | 1686 | 2728 | |
| | 9 | 16 | 25 | 41 | 66 | 107 | 173 | 280 | 453 | 733 | 1186 | 1919 | 3105 | |
| | 10 | 17 | 27 | 44 | 71 | 115 | 186 | 301 | 487 | 788 | 1275 | 2063 | 3338 | |
| | 11 | 19 | 30 | 49 | 79 | 128 | 207 | 335 | 542 | 877 | 1419 | 2296 | 3715 | |
| | 12 | 21 | 33 | 54 | 87 | 141 | 228 | 369 | 597 | 966 | 1563 | 2529 | 4092 | |
| Fib[7] | 13 | 22 | 35 | 57 | 92 | 149 | 241 | 390 | 631 | 1021 | 1652 | 2673 | 4325 | |

The two columns to the left of the vertical line consist respectively of the nonnegative integers $n$, and the lower Wythoff sequence whose $n$-th term is $[(n+1)*\varphi]$, where $\varphi = (1+sqrt(5))/2$ (Golden Ratio). The rows are then filled in by the Fibonacci rule that each term is the sum of the two previous terms. The entry $n$ in the first column is the index of that row.

*Note*. The Wythoff array description above has been taken from [2].

Let $w_{i,j}$ denote an $(i,j)$-th element of the Wythoff array (row number $i \geq 0$, column number $j \geq 0$).

### 1.4. Fibonacci Numbers and Auxiliary Relations

Let $F(i)$ denote $i$-th Fibonacci number, i.e., $F(0) = 0$, $F(1) = 1$, $F(i) = F(i\text{-}1) + F(i\text{-}2)$ when $i > 1$, $L(i)$ denote $i$-th Lucas number, i.e. $L(1) = 1$, $L(2) = 3$, $L(i) = L(i\text{-}1) + L(i\text{-}1)$ when $i > 2$.

Note some property of the Wythoff array that is related to the discussed issue:

$$w_{F(i),j} = F(i+j) + F(j),\ i \geq 2,\ j \geq 0. \tag{3}$$

Note also the following property of Fibonacci numbers

$$1 + \sum_{j=1}^{i} F(j) = F(i+2)\ . \tag{4}$$

## 2. Main Results

Let $T$ be a binary tree of size $n$ and $M=M(T)$ be a set of such sequences of positive *integer* weights that $\forall P \in M$ the tree $T$ is the Huffman tree of $P$ ($|P|=n$).

**Definition**. A sequence $P_{min}$ of $n$ positive *integer* weights is called a *minimizing* sequence of the binary tree $T$ in the class $M$ ($P_{min} \in M$) if $P_{min}$ produces the minimal Huffman cost of the tree $T$ over all sequences from $M$, i.e.,

$$E(T,P_{min}) \leq E(T,P)\ \forall P \in M.$$

**Definition**. A non-decreasing sequence of positive integer weights $P = \{p_1, p_2, \ldots, p_n\}$ is called *absolutely ordered* if the intermediate sequences of weights produced by Huffman algorithm for initial sequence $P$ satisfy the following conditions

$$p_2^{(i)} < p_3^{(i)},\ i = \overline{0, n-3}.$$

**Theorem 1**. [5] A *minimizing* absolutely ordered sequence of the elongated binary tree is

$$Pmin_{abs} = \{F(1), F(2), \ldots, F(n)\},$$

where $F(i)$ is $i$-th Fibonacci number.

The weighted external path length of elongated binary tree $T$ of size $n$ for the minimizing absolutely ordered sequence $Pmin_{abs}$ is

$$E(T, Pmin_{abs}) = F(n+4) - (n + 4).$$

**Proof**. The proof of Theorem 1 of [5].

**Definition**. A non-decreasing sequence of positive integer weights $P = \{p_1, p_2, \ldots, p_n\}$ is called *k-ordered* if the intermediate sequences of weights produced by Huffman algorithm for initial sequence $P$ satisfy the following conditions

$$p_2^{(i)} = p_3^{(i)},\ i = \overline{0, k}; \quad (5)$$

$$p_2^{(i)} < p_3^{(i)},\ i = \overline{k+1, n-3}. \quad (6)$$

Let $M_{n,k}$ ($k = \overline{0, n-3}$) denote the set of all $k$-ordered sequences of size $n$ for which the binary Huffman tree is *elongated*, i.e. $\forall P \in M_{n,k}$ an elongated binary tree of size $n$ is the Huffman tree of $P$.

**Theorem 2**. A *minimizing* $k$-ordered sequence of the *elongated* binary tree in the class $M_{n,k}$ ($k = \overline{0, n-3}$) is

$$Pmin_{n,k} = \{p_1, p_2, \ldots, p_n\},\ \text{where}$$

$$p_1 = 1,$$

$$p_2 = F(1), \ldots, p_{k+2} = F(k+1),$$

$$p_{k+3} = F(k+2),$$

$$p_{k+4} = F(k+3) + F(1),\ p_{k+5} = F(k+4) + F(2),\ \ldots,\ p_n = F(n-1) + F(n-k-3).$$

In other words, taking into account (3)

$$p_1 = 1,$$

$$p_2 = F(1), \ldots, p_{k+2} = F(k+1),$$

$$p_{k+3} = F(k+2) = w_{F(k+2),0},$$

$$p_{k+4} = w_{F(k+2),1},\ p_{k+5} = w_{F(k+2),2},\ \ldots,\ p_n = w_{F(k+2),n-k-3},$$

where $w_{i,j}$ is $(i,j)$-th element of the Wythoff array.

**Proof**. Because $Pmin_{n,k} = \{p_1, p_2, \ldots, p_n\}$ is *minimizing* sequence of positive *integer* values, $p_1$ and $p_2$ should have minimal positive integer values, i.e.,

$$p_1 = p_2 = 1. \quad (7)$$

$Pmin_{n,k}$ is *k*-ordered ($k \geq 0$) sequence, so according to (5)

$$p_2^{(0)} = p_3^{(0)},$$

therefore according to (1) and (7)

$$p_3 = 1.$$

Thus

$$p_1 = 1,\ p_2 = F(1),\ p_3 = F(2). \quad (8)$$

Further, taking into account (2) and (6) we obtain the following Huffman algorithm steps for $k$-ordered ($k \geq 0$) sequence of the elongated (left-sided) binary tree.

<u>Steps 0-$k$:</u>

| | | |
|---|---|---|
| Step 0 (Initial): | $p_1, p_2, p_3, p_4, p_5, \ldots, p_{k-1}, p_k, p_{k+1}, \ldots, p_{n-1}, p_n$; | $p_2 = p_3$; |
| Step 1: | $p_3, p_1 + p_2, p_4, p_5, \ldots, p_{k-1}, p_k, p_{k+1}, \ldots, p_{n-1}, p_n$; | $p_1 + p_2 = p_4$; |
| Step 2: | $p_4, p_1 + p_2 + p_3, p_5, \ldots, p_{k-1}, p_k, p_{k+1}, \ldots, p_{n-1}, p_n$; | $p_1 + p_2 + p_3 = p_5$; |
| Step 3: | $p_5, p_1 + p_2 + p_3 + p_4, p_6, \ldots, p_{k-1}, p_k, p_{k+1}, \ldots, p_{n-1}, p_n$; | $p_1 + p_2 + p_3 + p_4 = p_6$; |
| … | | |
| Step $k$-1: | $p_{k+1}, p_1 + p_2 + \ldots + p_k, p_{k+2}, \ldots, p_{n-1}, p_n$; | $p_1 + p_2 + \ldots + p_k = p_{k+2}$; |
| Step $k$: | $p_{k+2}, p_1 + p_2 + \ldots + p_{k+1}, p_{k+3}, \ldots, p_{n-1}, p_n$; | $p_1 + p_2 + \ldots + p_{k+1} = p_{k+3}$; |

<u>Steps ($k$+1) – ($n$-3)</u>:

| | | |
|---|---|---|
| Step $k$+1: | $p_{k+3}, p_1 + p_2 + \ldots + p_{k+2}, p_{k+4}, \ldots, p_{n-1}, p_n$; | $p_1 + p_2 + \ldots + p_{k+2} < p_{k+4}$; |
| Step $k$+2: | $p_{k+4}, p_1 + p_2 + \ldots + p_{k+3}, p_{k+5}, \ldots, p_{n-1}, p_n$; | $p_1 + p_2 + \ldots + p_{k+3} < p_{k+5}$; |
| … | | |
| Step $n$-4: | $p_{n-2}, p_1 + p_2 + \ldots + p_{n-3}, p_{n-1}; p_n$; | $p_1 + p_2 + \ldots + p_{n-1} < p_{n-1}$; |
| Step $n$-3: | $p_{n-1}, p_1 + p_2 + \ldots + p_{n-2}, p_n$; | $p_1 + p_2 + \ldots + p_{n-2} < p_n$; |

<u>Steps ($n$-2), ($n$-1)</u>:

| | |
|---|---|
| Step $n$-2: | $p_n, p_1 + p_2 + \ldots + p_{n-1}$; |
| Step $n$-1: | $p_1 + p_2 + \ldots + p_n$. |

Consider two cases.

**<u>Case 1. Steps 0-*k*.</u>**

It follows from relations for steps 0-$k$ that

$$p_i = \sum_{j=1}^{i-2} p_j, i = \overline{4,k+3}.$$

Thus,

$$p_i - p_{i-1} = \sum_{j=1}^{i-2} p_j - \sum_{j=1}^{i-3} p_j = p_{j-2},\ i = \overline{4,k+3}.$$

So, we have

$$p_i = p_{i-1} - p_{i-2},\ i = \overline{4,k+3}.$$

Taking into account (8), we obtain

$$p_i = F(i\text{-}1),\ i = \overline{2,k+3}. \qquad (9)$$

In particular,

$$p_{k+3} = \mathrm{F}(k+2) = F(k+2) + F(0). \qquad (10)$$

**<u>Case 2. Steps (*k*+1) – (*n*-3).</u>**

Because $Pmin_{n,k} = \{p_1, p_2, \ldots, p_n\}$ is <u>*minimizing*</u> sequence of positive <u>*integer*</u> values, inequalities for steps ($k$+1) – ($n$-3) are transformed to the following equalities:

| | | |
|---|---|---|
| Step $k$+1: | $p_{k+3}, p_1 + p_2 + \ldots + p_{k+2}, p_{k+4}, \ldots, p_{n-1}, p_n$; | $p_1 + p_2 + \ldots + p_{k+2} + 1 = p_{k+4}$; |
| Step $k$+2: | $p_{k+4}, p_1 + p_2 + \ldots + p_{k+3}, p_{k+5}, \ldots, p_{n-1}, p_n$; | $p_1 + p_2 + \ldots + p_{k+3} + 1 = p_{k+5}$; |
| … | | |
| Step $n$-4: | $p_{n-2}, p_1 + p_2 + \ldots + p_{n-3}, p_{n-1}; p_n$; | $p_1 + p_2 + \ldots + p_{n-1} + 1 = p_{n-1}$; |
| Step $n$-3: | $p_{n-1}, p_1 + p_2 + \ldots + p_{n-2}, p_n$; | $p_1 + p_2 + \ldots + p_{n-2} + 1 = p_n$; |

From the equality for step ($k$+1), (9), (7) and (4) results

$$p_{k+4} = F(k+3) + 1 = F(k+3) + F(1). \qquad (11)$$

Further, it follows from relations with equalities for steps ($k$+1) – ($n$-3) that

$$p_i = 1 + \sum_{j=1}^{i-2} p_j, i = \overline{k+5, n}.$$

Thus,

$$p_i - p_{i-1} = (1 + \sum_{j=1}^{i-2} p_j) - (1 + \sum_{j=1}^{i-3} p_j) = p_{j-2},\ i = \overline{k+5, n}.$$

So, we have

$$p_i = p_{i-1} - p_{i-2},\ i = \overline{k+5, n}.$$

Therefore, taking into account (10) and (11), we have

$$p_i = F(i - 1) + F(i - k - 3),\ i = \overline{k+4, n}.$$

From this and (3) it follows that

$$p_i = w_{F(k+2),i-k-3},\ i = \overline{k+4, n}, \tag{12}$$

where $w_{i,j}$ is ($i$,$j$)-th element of the Wythoff array.

The statement of the theorem follows from (7), (9) and (12).

**Corollary 1**. A minimizing 0-ordered sequence of size $n$ for the elongated binary tree in is the Lucas sequence shifted two places right, i.e. {1, 1, $L$(1), $L$(2), …, $L$($n$-2)}, where $L$($i$) is $i$-th Lucas number.

**Corollary 2**. A minimizing ($n$-3)-ordered sequence of size $n$ for the elongated binary tree in is the Fibonacci sequence shifted one place right, i.e., {1, $F$(1), $F$(2), …, $F$($n$-1)}, where $F$($i$) is $i$-th Fibonacci number.

Note that *normalized* ($n$ – 3)-ordered sequence of size $n$

$$\{1/F(n+1), F(1)/F(n+1), F(2)/F(n+1), ..., F(n-1)/F(n+1)\}$$

has maximum weighted external path length over all possible normalized sequences of size $n$ for which Huffman tree is elongated[6].

**Theorem 3**. The weighted external path length of the elongated binary tree $T$ of size $n$ for the minimizing $k$-ordered sequence $Pmin_{n,k}$ is

$$E(T, Pmin_{n,k}) = F(n + 3) + F(n – k + 1) – (n – k + 3).$$

**Proof**. Let $Pmin_{n,k} = \{p_1, p_2, …, p_n\}$ be the minimizing $k$-ordered sequence of the elongated binary tree $T$ of size $n$.

According to Theorem 2

$$Pmin_{n,k} = \{1, F(1), F(2), …, F(k+2), F(k+3) + F(1), F(k+4) + F(2), …, F(n-1) + F(n-k-3)\}.$$

Weighted external path length $E(T, Pmin_{n,k})$ is

$$E(T, Pmin_{n,k}) = \sum_{i=1}^{n} l_i p_i .$$

where $l_i$ is the length of the path from the root to leaf $i$.

$T$ is the elongated binary tree, therefore $l_1 = 1$, $l_i = n – i + 1$ ($i = \overline{2, n}$).

Then

$$E(T, Pmin_{n,k}) = \sum_{i=1}^{n} l_i p_i = (n – 1)p_1 + \sum_{i=2}^{n} (n - i + 1) p_i$$

$$= (n – 1) + \sum_{i=2}^{k+3} (n - i + 1) F(i-1) + \sum_{i=k+4}^{n} (n - i + 1)(F(i-1) + F(i-k-3))$$

$$= (n – 1) + \sum_{i=1}^{k+2} (n - i) F(i) + \sum_{i=k+3}^{n-1} (n - i)(F(i) + F(i-k-2))$$

$$= (n-1) + \sum_{i=1}^{n-1}(n-i)F(i) + \sum_{i=k+3}^{n-1}(n-i)F(i-k-2) = (n-1) + \sum_{i=1}^{n-1}(n-i)F(i) + \sum_{i=1}^{n-k-3}(n-k-i-2)F(i)$$

$$= (n - 1) + \sum_{i=1}^{n-1}\sum_{j=1}^{n-i}F(i) + \sum_{i=1}^{n-k-3}\sum_{j=1}^{n-k-i-2}F(i) = (n-1) + \sum_{j=1}^{n-1}\sum_{i=1}^{j}F(i) + \sum_{j=1}^{n-k-3}\sum_{i=1}^{j}F(i).$$

Thus, taking into account (4), we obtain

$$E(T, Pmin_{n,k}) = (n-1) + \sum_{j=1}^{n-1}(F(j+2)-1) + \sum_{j=1}^{n-k-3}(F(j+2)-1)$$

$$= (n-1) + \sum_{j=1}^{n-1}F(j+2) - (n-1) + \sum_{j=1}^{n-k-3}F(j+2) - (n-k-3) = \sum_{j=3}^{n+1}F(j) + \sum_{j=3}^{n-k-1}F(j) + (n-k-3)$$

$$= = (\sum_{j=1}^{n+1}F(j) - F(2) - F(1)) + (\sum_{j=1}^{n-k-1}F(j) - F(2) - F(1)) - (n-k-3)$$

$$= (F(n+3) - 1 - F(2) - F(1)) + (F(n-k+1) - 1 - F(2) - F(1)) - (n-k-3)$$

$$= (F(n+3) - 3) + (F(n-k+1) - 3) - (n-k-3)$$

$$= F(n+3) + F(n-k+1) - (n-k+3).$$

The statement of the theorem proved.

**Corollary 3**. The weighted external path length of the elongated binary tree $T$ of size $n$ for the minimizing 0-ordered sequence $Pmin_{n,0}$ (the Lucas sequence shifted two places right) is

$$E(T, Pmin_{n,0}) = F(n+3) + F(n+1) - (n+3).$$

**Corollary 4**. The weighted external path length of the elongated binary tree $T$ of size $n$ for the minimizing $(n-3)$-ordered sequence $Pmin_{n,n-3}$ (the Fibonacci sequence shifted one place right) is

$$E(T, Pmin_{n,n-3}) = F(n+3) + F(n-(n-3)+1) - (n-(n-3)+3) = F(n+3) + F(4) - 6 = F(n+3) - 3.$$

## 3. Examples

Several examples of minimizing ordered sequences for Huffman codes are shown below. An underlined integer in the tables means a nonleaf node cost obtained as a result of merging two leaf nodes on the previous step of the Huffman algorithm. Shaded columns mean leaf nodes selected to be merged on the current step.

**Example 1**. Absolutely minimizing ordered sequence $Pmin_{abs}$ of size 10.

| Step $i$ | $P^{(i)}$ | $p_1^{(i)}$ | $p_2^{(i)}$ | $p_3^{(i)}$ | $p_4^{(i)}$ | $p_5^{(i)}$ | $p_6^{(i)}$ | $p_7^{(i)}$ | $p_8^{(i)}$ | $p_9^{(i)}$ | $p_{10}^{(i)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Initial (0) | $P^{(0)}$ | 1 | **1** | **2** | 3 | 5 | 8 | 13 | 21 | 34 | 55 |
| 1 | $P^{(1)}$ | 2 | **<u>2</u>** | **3** | 5 | 8 | 13 | 21 | 34 | 55 | |
| 2 | $P^{(2)}$ | 3 | **<u>4</u>** | **5** | 8 | 13 | 21 | 34 | 55 | | |
| 3 | $P^{(3)}$ | 5 | **<u>7</u>** | **8** | 13 | 21 | 34 | 55 | | | |
| 4 | $P^{(4)}$ | 8 | **<u>12</u>** | **13** | 21 | 34 | 55 | | | | |
| 5 | $P^{(5)}$ | 13 | **<u>20</u>** | **21** | 34 | 55 | | | | | |
| 6 | $P^{(6)}$ | 21 | **<u>33</u>** | **34** | 55 | | | | | | |
| 7 | $P^{(7)}$ | 34 | **<u>54</u>** | **55** | | | | | | | |
| 8 | $P^{(8)}$ | 55 | **<u>88</u>** | | | | | | | | |
| 9 | $P^{(9)}$ | 143 | | | | | | | | | |

$\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}\}$ = {1, 1, 2, 3, 5, 8, 13, 21, 34, 55} –
Fibonacci sequence of size 10.

**Example 2**. Minimizing 0-ordered sequence $Pmin_{10,0}$

| Step $i$ | $P^{(i)}$ | $p_1^{(i)}$ | $p_2^{(i)}$ | $p_3^{(i)}$ | $p_4^{(i)}$ | $p_5^{(i)}$ | $p_6^{(i)}$ | $p_7^{(i)}$ | $p_8^{(i)}$ | $p_9^{(i)}$ | $p_{10}^{(i)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Initial (0) | $P^{(0)}$ | 1 | **1** | **1** | 3 | 4 | 7 | 11 | 18 | 29 | 47 |
| 1 | $P^{(1)}$ | 1 | **2** | **3** | 4 | 7 | 11 | 18 | 29 | 47 | |
| 2 | $P^{(2)}$ | 3 | **3** | **4** | 7 | 11 | 18 | 29 | 47 | | |
| 3 | $P^{(3)}$ | 4 | **6** | **7** | 11 | 18 | 29 | 47 | | | |
| 4 | $P^{(4)}$ | 7 | **10** | **11** | 18 | 29 | 47 | | | | |
| 5 | $P^{(5)}$ | 11 | **17** | **18** | 29 | 47 | | | | | |
| 6 | $P^{(6)}$ | 18 | **28** | **29** | 47 | | | | | | |
| 7 | $P^{(7)}$ | 29 | **46** | **47** | | | | | | | |
| 8 | $P^{(8)}$ | 47 | **75** | | | | | | | | |
| 9 | $P^{(9)}$ | 122 | | | | | | | | | |

$p_1 = 1$;
$\{p_2, p_3\} = \{1, 1\} = \{F(1), F(2)\}$ - Fibonacci sequence of size 2;
$\{p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}\} = \{1, 3, 4, 7, 11, 18, 29, 47\}$ -
Wythoff array row#1 (row#$F(2)$) sequence of size 8 (the Lucas sequence).

**Example 3**. Minimizing 1-ordered sequence $Pmin_{10,1}$

| Step $i$ | $P^{(i)}$ | $p_1^{(i)}$ | $p_2^{(i)}$ | $p_3^{(i)}$ | $p_4^{(i)}$ | $p_5^{(i)}$ | $p_6^{(i)}$ | $p_7^{(i)}$ | $p_8^{(i)}$ | $p_9^{(i)}$ | $p_{10}^{(i)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Initial (0) | $P^{(0)}$ | 1 | **1** | **1** | 2 | 4 | 6 | 10 | 16 | 26 | 42 |
| 1 | $P^{(1)}$ | 1 | **2** | **2** | 4 | 6 | 10 | 16 | 26 | 42 | |
| 2 | $P^{(2)}$ | 2 | **3** | **4** | 6 | 10 | 16 | 26 | 42 | | |
| 3 | $P^{(3)}$ | 4 | **5** | **6** | 10 | 16 | 26 | 42 | | | |
| 4 | $P^{(4)}$ | 6 | **9** | **10** | 16 | 26 | 42 | | | | |
| 5 | $P^{(5)}$ | 10 | **15** | **16** | 26 | 42 | | | | | |
| 6 | $P^{(6)}$ | 16 | **25** | **26** | 42 | | | | | | |
| 7 | $P^{(7)}$ | 26 | **41** | **42** | | | | | | | |
| 8 | $P^{(8)}$ | 42 | **67** | | | | | | | | |
| 9 | $P^{(9)}$ | 109 | | | | | | | | | |

$p_1 = 1$;
$\{p_2, p_3, p_4\} = \{1, 1, 2\} = \{F(1), F(2), F(3)\}$ - Fibonacci sequence of size 3;
$\{p_4, p_5, p_6, p_7, p_8, p_9, p_{10}\} = \{2, 4, 6, 10, 16, 26, 42\}$ –
Wythoff array row#2 (row#$F(3)$) sequence of size 7.

**Example 4**. Minimizing 4-ordered sequence $Pmin_{10,4}$

| Step $i$ | $P^{(i)}$ | $p_1^{(i)}$ | $p_2^{(i)}$ | $p_3^{(i)}$ | $p_4^{(i)}$ | $p_5^{(i)}$ | $p_6^{(i)}$ | $p_7^{(i)}$ | $p_8^{(i)}$ | $p_9^{(i)}$ | $p_{10}^{(i)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Initial (0) | $P^{(0)}$ | 1 | **1** | **1** | 2 | 3 | 5 | 8 | 14 | 22 | 36 |
| 1 | $P^{(1)}$ | 1 | **2** | **2** | 3 | 5 | 8 | 14 | 22 | 36 | |
| 2 | $P^{(2)}$ | 2 | **3** | **3** | 5 | 8 | 14 | 22 | 36 | | |
| 3 | $P^{(3)}$ | 3 | **5** | **5** | 8 | 14 | 22 | 36 | | | |
| 4 | $P^{(4)}$ | 5 | **8** | **8** | 14 | 22 | 36 | | | | |
| 5 | $P^{(5)}$ | 8 | **13** | **14** | 22 | 36 | | | | | |
| 6 | $P^{(6)}$ | 14 | **21** | **22** | 36 | | | | | | |
| 7 | $P^{(7)}$ | 22 | **35** | **36** | | | | | | | |
| 8 | $P^{(8)}$ | 36 | **57** | | | | | | | | |
| 9 | $P^{(9)}$ | 93 | | | | | | | | | |

$p_1 = 1$;
$\{p_2, p_3, p_4, p_5, p_6, p_7\} = \{1, 1, 2, 3, 5, 8\} = \{F(1), F(2), F(3), F(4), F(5), F(6)\}$ –
Fibonacci sequence of size 6;
$\{p_7, p_8, p_9, p_{10}\} = \{8, 14, 22, 36\}$ - Wythoff array row#8 (row#$F(6)$) sequence of size 4.

**Example 5**. Minimizing 7-ordered sequence $Pmin_{10,7}$

| Step $i$ | $P^{(i)}$ | $p_1^{(i)}$ | $p_2^{(i)}$ | $p_3^{(i)}$ | $p_4^{(i)}$ | $p_5^{(i)}$ | $p_6^{(i)}$ | $p_7^{(i)}$ | $p_8^{(i)}$ | $p_9^{(i)}$ | $p_{10}^{(i)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Initial (0) | $P^{(0)}$ | 1 | **1** | **1** | 2 | 3 | 5 | 8 | 13 | 21 | 34 |
| 1 | $P^{(1)}$ | 1 | **2** | **2** | 3 | 5 | 8 | 13 | 21 | 34 | |
| 2 | $P^{(2)}$ | 2 | **3** | **3** | 5 | 8 | 13 | 21 | 34 | | |
| 3 | $P^{(3)}$ | 3 | **5** | **5** | 8 | 13 | 21 | 34 | | | |
| 4 | $P^{(4)}$ | 5 | **8** | **8** | 13 | 21 | 34 | | | | |
| 5 | $P^{(5)}$ | 8 | **13** | **13** | 21 | 34 | | | | | |
| 6 | $P^{(6)}$ | 13 | **21** | **21** | 34 | | | | | | |
| 7 | $P^{(7)}$ | 21 | **34** | **34** | | | | | | | |
| 8 | $P^{(8)}$ | 34 | **55** | | | | | | | | |
| 9 | $P^{(9)}$ | 89 | | | | | | | | | |

$p_1 = 1$;
$\{p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}\} = \{1, 1, 2, 3, 5, 8, 13, 21, 34\} =$
$\{F(1), F(2), F(3), F(4), F(5), F(6), F(7), F(8), F(9)\}$ - Fibonacci sequence of size 9;
$\{p_{10}\} = \{34\}$ - Wythoff array row#34 (row#$F(9)$) sequence of size 1.

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