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Abstract. A non-decreasing sequence of positive integer weights $P = \{p_1, p_2, ..., p_n\}$ $(n = N^*(m-1) + 1, N \text{ is number of non-leaves of } m\text{-ary tree}) \text{ is called } absolutely}$ <u>ordered</u> if the intermediate sequences of weights produced by m-ary Huffmanalgorithm for initial sequence P on i-th step satisfy the following conditions $p_m^{(i)} < p_{m+1}^{(i)}, i = \overline{0, N-2}$. Let T be an m-ary tree of size n and M=M(T) be a set of such sequences of positive *integer* weights that $\forall P \in M$ the tree T is the m-aryHuffman tree of P(|P|=n). A sequence P_{\min} of n positive integer weights is called a *minimizing* sequence of the m-ary tree T in the class $M(P_{\min} \in M)$ if P_{\min} produces the minimal Huffman cost of the tree T over all sequences from M, i.e., $E(T, P_{\min}) \leq E(T, P) \forall P \in M$. Theorem 1. A <u>minimizing</u> absolutely ordered sequence of size $n = N^*(m-1) + 1$ for the maximum height m-ary Huffman tree (m > 1) is $Pmin_{abs}(N, m) =$ $\{G_0(m-1), G_1(m-1), ..., G_1(m-1), G_2(m-1), ..., G_2(m-1), ..., G_N(m-1), ..., G_N(m-1)\}$

$$\{G_{0}(m-1), \underbrace{G_{1}(m-1), \dots, G_{1}(m-1)}_{(m-1) \text{ times}}, \underbrace{G_{2}(m-1), \dots, G_{2}(m-1)}_{(m-1) \text{ times}}, \dots, \underbrace{G_{N}(m-1), \dots, G_{N}(m-1)}_{(m-1) \text{ times}}\},$$

where $G_0(m) = 1$, $G_1(m) = 1$, $G_2(m) = 2$, $G_i(m) = G_{i-1}(m) + m^*G_{i-2}(m)$ when i = 2, NPolynomials $G_i(x)$ are called Fibonacci-like polynomials. <u>Theorem 2</u>. The cost of maximum height *m*-ary Huffman tree *T* of size $n = N^*(m-1) + 1$ for the minimizing absolutely ordered sequence $Pmin_{abs}(N, m)$ is

$$E(T, Pmin_{abs}(N, m)) = \frac{G_{N+4}(m-1)-2}{m-1} - (N+3). \blacksquare$$

Samples of Fibonacci-like polynomials and costs of maximum height *m*-ary Huffman trees are shown.

0. Preface

Absolutely ordered and *k*-ordered sequences for <u>binary</u> Huffman trees those have maximum height have been investigated in [1] and [2]. In this article the generalization of absolutely ordered sequences for *m*-ary Huffman trees those have maximum height is considered.

1. Main Conceptions and Terminology 1.1. m-ary trees

A (strictly) <u>m-ary tree</u> is an oriented ordered tree where each nonleaf node has exactly *m* children (siblings). <u>Size of an m-ary tree</u> is the total number of *leaves* of this tree. Let *N* be number of nonleaves (internal nodes), *n* be number of leaves of *m*-ary tree. Number of leaves in *m*-ary tree satisfies the following condition

$$n = N^*(m-1) + 1. \tag{1}$$

An *m*-tree $(m \ge 2)$ is called <u>elongated</u> if at least (m-1) of any *m* sibling nodes are leaves. An elongated binary tree of size *n* has maximum height among all binary trees of size *n*. An elongated *m*-ary tree is called <u>left-sided</u> if only the <u>left</u> node in each *m*-tuple of sibling nodes can be nonleaf.

A *m*-ary tree is called *labeled* if a certain positive integer (weight) is set in correspondence with each leaf.

Definition. Let *T* be an *m*-ary tree with positive weights $P = \{p_1, ..., p_n\}$ at its leaf nodes. The <u>weighted</u> <u>external path length</u> of *T* is

$$E(T,P) = \sum_{i=1}^{n} l_i p_i$$

where l_i is the length of the path from the root to leaf *i*.

1.2. Generalized *m*-ary Huffman algorithm

Problem definition. Given a sequence of *n* positive weights $P = \{p_1, ..., p_n\}$, (n-1) = 0 (**mod**(*m*-1)). The problem is to find *m*-ary tree T_{\min} with *n* leaves labeled $p_1, ..., p_n$ that has minimum weighted external path length over all possible *m*-ary trees of size *n* with the same sequence of leaf weights. T_{\min} is called the *m*-ary Huffman tree of the sequence *P*; $E(T, P_{\min})$ is called the Huffman cost of the tree *T*.

The problem was solved for binary trees by Huffman algorithm [3]. That algorithm can be generalized for *m*-ary trees. A generalized Huffman algorithm builds T_{\min} in which each leaf (weight) of *m*-ary tree is associated with a (prefix free) codeword in alphabet {0, 1,..., *m*-1}.

Note. A code is called a prefix (free) code if no codeword is a prefix of another one.

m-ary algorithm description (in the reference to the discussed issue).

Algorithm input. A non-decreasing sequence of positive weights

 $P = \{p_1, p_2, \dots, p_n\}$ $p_k \le p_{k+1}, k = \overline{1, n-1}; n = N^*(m-1)+1$, where N is number of non-leaves.

<u>Algorithm output</u>. The sum of all the weights.

The algorithm is performed in N steps. *i*-th step ($i = \overline{1,N}$) is as follows.

• <u>*i*-th step input</u>. A non-decreasing sequence of weights of size n-(m-1)*(i-1).

$$P^{(i-1)} = \{ p_1^{(i-1)}, p_2^{(i-1)}, \dots, p_{n-(m-1)^*(i-1)}^{(i-1)} \} (p_k^{(i-1)} \le p_{k+1}^{(i-1)}; k = \overline{1, n - (m-1)^*(i-1) - 1}); |P^{(i-1)}| = n - (m-1)^*(i-1).$$

• <u>*i*-th step method</u>. Build a sequence

$$\{ p_1^{(i-1)} + p_2^{(i-1)} + \dots + p_m^{(i-1)}, p_{m+1}^{(i-1)}, \dots, p_{n-(m-1)^*(i-1)}^{(i-1)} \}$$

and sort its.

• <u>*i*-th step output. A non-decreasing sequence of weights of size n-(m-1)*i.</u> $P^{(i)} = \{ p_1^{(i)}, p_2^{(i)}, ..., p_{n-(m-1)*i}^{(i)} \} (p_k^{(i)} \le p_{k+1}^{(i)}; k = \overline{1, n-(m-1)*i-1}); |P^{(i)}| = n-(m-1)*i.$

<u>Note 1</u>. $P^{(0)}$ is an input of *m*-ary Huffman algorithm, i.e., $p_k^{(0)} = p_k \ (k = \overline{1,n}).$

Note 2. If an input sequence on *i*-th step(s) of the algorithm satisfies condition

$$p_m^{(i)} = p_{m+1}^{(i)} (0 \le i \le N - 2),$$

then several m-ary Huffman trees can result from initial sequence P of weights, but the weighted external path length is the same in all these trees.

Let $P = \{p_1, p_2, p_3, ..., p_n\}$ be a sequence of size *n* for which the *m*-ary Huffman tree is <u>elongated</u>. Then according to generalized *m*-ary Huffman algorithm

$$p_1^{(i)} + p_2^{(i)} + \dots + p_m^{(i)} \le p_{2m}^{(i)}, i = \overline{0, N-2}.$$
(2)

2. Main Results

2.1. Minimizing absolutely ordered sequence of the elongated *m*-ary Huffman tree

Let *T* be an *m*-ary tree (m > 1) of size *n* (i.e., $n = N^*(m-1) + 1$, where *N* is number of non-leaves and *n* is number of leaves) and M = M(T) be a set of such sequences of positive *integer* weights that $\forall P \in M$ the tree *T* is the *m*-ary Huffman tree of P(|P|=n).

Definition. A sequence P_{\min} of *n* positive *integer* weights is called a *minimizing* sequence of the *m*-ary tree *T* in the class $M(P_{\min} \in M)$ if P_{\min} produces the minimal Huffman cost of the *m*-ary tree *T* over all sequences from *M*, i.e.,

 $E(T, P_{\min}) \le E(T, P) \,\forall P \in M.$

Definition. A non-decreasing sequence of positive integer weights $P = \{p_1, p_2, ..., p_n\}$ is called <u>absolutely</u> <u>ordered</u> if the intermediate sequences of weights produced by *m*-ary Huffman algorithm for initial sequence *P* satisfy the following conditions

$$p_m^{(i)} < p_{m+1}^{(i)}, \ i = \overline{0, N-2}.$$
 (3)

For an absolutely ordered sequence the equality-inequality relation (2) is transformed to the (strict) equality relation

$$p_1^{(i)} + p_2^{(i)} + \dots + p_m^{(i)} < p_{2m}^{(i)}, i = \overline{0, N-2}.$$
(4)

Lemma 1. A *minimizing* absolutely ordered sequence of size $n = N^*(m-1) + 1$ for the elongated *m*-ary tree (m > 1) is

$$Pmin_{abs}(N, m) = \{ Q_0(m), \underbrace{Q_1(m), \dots, Q_1(m)}_{(m-1) \text{ times}}, \underbrace{Q_2(m), \dots, Q_2(m)}_{(m-1) \text{ times}}, \dots, \underbrace{Q_N(m), \dots, Q_N(m)}_{(m-1) \text{ times}} \}$$

where $Q_0(m) = 1, Q_1(m) = 1, Q_2(m) = 2, Q_i(m) = Q_{i-1}(m) + (m-1)^*Q_{i-2}(m)$ when $i = \overline{2, N}$.

<u>Proof</u>. Taking into account (3) (4) we obtain the following configurations of *m*-ary Huffman algorithm steps for absolutely ordered sequence of the elongated (left-sided) *m*-ary tree.

Step 0 (Initial):

Step 0: $p_1, p_2, ..., p_m, p_{m+1}, ..., p_{2m-1}, p_{2m}, p_{2m+1}, ..., p_n;$ $p_m < p_{m+1};$ Steps 1-(N-2): $p_m < p_{m+1};$

Step 1:
$$p_{m+1}, \dots, p_{2m-1}, \sum_{j=1}^{m} p_j$$
, p_{2m}, \dots, p_n ; $\sum_{j=1}^{m} p_j < p_{2m}$;Step 2: $p_{2m}, \dots, p_{3m-2}, \sum_{j=1}^{2m-1} p_j$, p_{3m-1}, \dots, p_n ; $\sum_{j=1}^{2m-1} p_j < p_{3m-1}$;Step 3: $p_{3m-1}, \dots, p_{4m-3}, \sum_{j=1}^{3m-2} p_j$, p_{4m-2}, \dots, p_n ; $\sum_{j=1}^{m} p_j < p_{4m-2}$;

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Step *i*:
$$p_{i^*(m-1)+2}, \dots, p_{(i+1)^*(m-1)+1}, \sum_{j=1}^{i^*(m-1)+1} p_j, p_{(i+1)^*(m-1)+2}, \dots, p_n; \sum_{j=1}^{i^*(m-1)+1} p_j < p_{(i+1)^*(m-1)+2};$$

Step N-2:
$$p_{(N-2)^*(m-1)+2}, \dots, p_{(N-1)^*(m-1)+1}, \sum_{j=1}^{(N-2)^*(m-1)+1} p_j, p_{(N-1)^*(m-1)+2}, \dots, p_n; \sum_{j=1}^{(N-2)^*(m-1)+1} p_j < p_{(N-1)^*(m-1)+2};$$

Steps (N-1), N:

Step N-1:
$$p_{(N-1)*(m-1)+2}, \dots, p_{N^*(m-1)+1}, \sum_{j=1}^{(N-1)*(m-1)+1} p_j$$
 (Note. According (1) $(N^*(m-1)+1) = n$);
Step N: $\sum_{j=1}^{N^*(m-1)+1} p_j = \sum_{j=1}^{n} p_j$.

On <u>Step 0</u> merging *m* leaves labeled by integers $p_1, p_2, ..., p_m$ are merged. Because $Pmin_{abs}(N, m) = \{p_1, p_2, ..., p_n\}$ is <u>minimizing</u> sequence of positive <u>integer</u> values, $p_1, p_2, ..., p_m$ should have minimal positive integer values, i.e., at least they must be equal. So, we can write as follows

$$p_1 = q_0, \ p_2 = \dots = p_m = q_1; \ q_0 = q_1.$$

On <u>Step 1</u> merging (*m*-1) leaves labeled by integers $p_{m+1}, ..., p_{2m-1}$ and one nonleaf are merged. Again, because $Pmin_{abs}(N, m) = \{p_1, p_2, ..., p_n\}$ is <u>minimizing</u> sequence of positive <u>integer</u> values, $p_{m+1}, ..., p_{2m-1}$ should have minimal *possible* positive integer values, i.e., at least they must be equal. So, we can write as follows

$$p_{m+1} = \dots = p_{2m-1} = q_2.$$

In the same manner for Steps 2,..., (N-1) we obtain
$$p_{2m} = \dots = p_{3m-2} = q_3;$$
$$P_{3m-1} = \dots = p_{4m-3} = q_4;$$
$$\dots$$
$$p_{i^*(m-1)+2} = \dots = p_{(i+1)^*(m-1)+1} = q_{i+1};$$
$$\dots$$
$$p_{(N-1)^*(m-1)+2} = \dots = p_{N^*(m-1)+1} = q_N.$$

So, the configurations of *m*-ary Huffman algorithm steps for absolutely ordered sequence of the elongated (left-sided) *m*-ary tree are transformed as follows.

Step 0 (Initial):

Step 0:
$$q_0, \underbrace{q_1, ..., q_1}_{(m-1) \text{ times}}, \underbrace{q_2, ..., q_2}_{(m-1) \text{ times}}, ..., \underbrace{q_N, ..., q_N}_{(m-1) \text{ times}}$$
 $q_1 < q_2;$

Steps 1-(N-2):

Step 1:
$$\underbrace{q_2, ..., q_2}_{(m-1) \text{ times}}, q_0 + (m-1) \sum_{j=1}^{1} q_j, \underbrace{q_3, ..., q_3}_{(m-1) \text{ times}}, ..., \underbrace{q_N, ..., q_N}_{(m-1) \text{ times}}; \qquad q_0 + (m-1) \sum_{j=1}^{1} q_j < q_3;$$

Step 2:
$$\underbrace{q_3, ..., q_3}_{(m-1) \text{ times}}, q_0 + (m-1) \sum_{j=1}^{2} q_j, \underbrace{q_4, ..., q_4}_{(m-1) \text{ times}}, ..., \underbrace{q_N, ..., q_N}_{(m-1) \text{ times}}; q_0 + (m-1) \sum_{j=1}^{2} q_j < q_4;$$

Step 3:
$$\underbrace{q_4, \dots, q_4}_{(m-1) \text{ times}}, q_0 + (m-1) \sum_{j=1}^3 q_j, \underbrace{q_5, \dots, q_5}_{(m-1) \text{ times}}, \dots, \underbrace{q_N, \dots, q_N}_{(m-1) \text{ times}}; \qquad q_0 + (m-1) \sum_{j=1}^3 q_j < q_5;$$

Step *i*:
$$\underbrace{q_{i+1},...,q_{i+1}}_{(m-1) \text{ times}}, q_0 + (m-1) \sum_{j=1}^i q_j, \underbrace{q_{i+2},...,q_{i+2}}_{(m-1) \text{ times}}, ..., \underbrace{q_N,...,q_N}_{(m-1) \text{ times}}; q_0 + (m-1) \sum_{j=1}^i q_j < q_{i+2};$$

Step N-2:
$$\underbrace{q_{N-1},...,q_{N-1}}_{(m-1) \text{ times}}, q_0 + (m-1) \sum_{j=1}^{N-2} q_j, \underbrace{q_N,...,q_N}_{(m-1) \text{ times}}; q_0 + (m-1) \sum_{j=1}^{N-2} q_j < q_N;$$

Steps (N-1), N:

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Step N-1:
$$\underline{q}_N, ..., \underline{q}_N, q_0 + (m-1) \sum_{j=1}^{N-1} q_j;$$

Step N: $q_0 + (m-1) \sum_{j=1}^{N} q_j.$

Because $Pmin_{abs}(N, m) = \{ q_0, \underbrace{q_1, \dots, q_1}_{(m-1) \text{ times}}, \underbrace{q_2, \dots, q_2}_{(m-1) \text{ times}}, \dots, \underbrace{q_N, \dots, q_N}_{(m-1) \text{ times}} \}$ is <u>minimizing</u> sequence of positive

integer values, q_0 and q_1 should have minimal positive integer values, and q_2 should have minimal *possible* positive integer value. So, we have

$$q_0 = 1, \tag{5}$$

$$q_1 = 1, \tag{6}$$

and, taking into account (3)

$$q_{2} = q_{1} + 1 = 2,$$

$$q_{i} = (m-1) \sum_{j=1}^{i-2} q_{j} + 1, \text{ when } i = \overline{2, N}.$$
(7)

Consider, $q_i - q_{i-1}$ when $i = \overline{3, N}$.

$$q_i - q_{i-1} = ((m-1)\sum_{j=1}^{i-2} q_j + 1) - ((m-1)\sum_{j=1}^{i-3} q_j + 1) = (m-1)q_{i-2}.$$
(8)

i.e.

 $q_i = q_{i-1} + (m-1) q_{i-2}$, when $i = \overline{2, N}$.

From (5), (6), (7) and (8) we obtain that
$$q_i$$
 is a function of m , i.e., $q_i = Q_i(m)$ and thus

$$Q_0(m) = 1, Q_1(m) = 1, Q_2(m) = 2, Q_i(m) = Q_{i-1}(m) + (m-1)*Q_{i-2}(m)$$
 when $i = 2, N$. (9)

The lemma has been proved.■

2.2. Fibonacci-like polynomials

From Lemma 1 we can see that *m*-ary Huffman tree (m > 1) is connected with polynomials $Q_i(m)$ (9). From that we can told that (m+1)-ary Huffman tree (m > 0) is connected with polynomials

$$G_0(m) = 1, G_1(m) = 1, G_2(m) = 2, G_i(m) = G_{i-1}(m) + m^* G_{i-2}(m)$$
 when $i = \overline{2, N}$.

So, we have polynomials that are defined by the recurrence relation

 $G_i(x) = G_{i-1}(x) + x^*G_{i-2}(x)$ when i > 2;

with

$$G_0(x) = 1, G_1(x) = 1, G_2(x) = 2.$$

Thus, Lemma 1 can be reformulate as

<u>Theorem 1</u>. A *minimizing* absolutely ordered sequence of size $n = N^*(m-1) + 1$ for the elongated *m*-ary Huffman tree (m > 1) is

Huffman related polynomials $G_i(x)$ are Fibonacci-like ones in contrast to Fibonacci polynomials that are defined by another recurrence relation [4]

$$F_i(x) = x^* F_{i-1}(x) + F_{i-2}(x)$$
 when $x > 2$;

with

$$F_1(x) = 1, F_2(x) = x.$$

The first few Fibonacci-like (Huffman related) polynomials are

$$\begin{array}{l} G_0(x) = 1 \\ G_1(x) = 1 \\ G_2(x) = 2 \\ G_3(x) = x + 2 \\ G_4(x) = 3x + 2 \\ G_5(x) = x^2 + 5x + 2 \\ G_6(x) = 4x^2 + 7x + 2 \\ G_7(x) = x^3 + 9x^2 + 9x + 2 \\ G_8(x) = 5x^3 + 16x^2 + 11x + 2 \\ G_9(x) = x^4 + 14x^3 + 25x^2 + 13x + 2 \\ G_{10}(x) = 6x^4 + 30x^3 + 36x^2 + 15x + 2 \\ G_{11}(x) = x^5 + 20x^4 + 55x^3 + 49x^2 + 17x + 2 \\ G_{12}(x) = 7x^5 + 50x^4 + 91x^3 + 64x^2 + 19x + 2 \\ G_{13}(x) = x^6 + 27x^5 + 105x^4 + 140x^3 + 81x^2 + 21x + 2 \\ G_{14}(x) = 8x^6 + 77x^5 + 196x^4 + 204x^3 + 100x^2 + 23x + 2 \\ G_{15}(x) = x^7 + 35x^6 + 182x^5 + 336x^4 + 285x^3 + 121x^2 + 25x + 2 \\ G_{16}(x) = 9x^7 + 112x^6 + 378x^5 + 540x^4 + 385x^3 + 144x^2 + 27x + 2 \\ G_{17}(x) = x^8 + 44x^7 + 294x^6 + 714x^5 + 825x^4 + 506x^3 + 169x^2 + 29x + 2 \\ G_{18}(x) = 10x^8 + 156x^7 + 672x^6 + 1254x^5 + 1210x^4 + 650x^3 + 196x^2 + 31x + 2 \\ G_{19}(x) = x^9 + 54x^8 + 450x^7 + 1386x^6 + 2079x^5 + 1716x^4 + 819x^3 + 225x^2 + 33x + 2 \\ G_{20}(x) = 11x^9 + 210x^8 + 1122x^7 + 2640x^6 + 3289^{x5} + 2366x^4 + 1015x^3 + 256^{x2} + 35x + 2. \end{array}$$

The Fibonacci-like polynomials
$$G_i(x)$$
 are normalized, i.e.
 $G_i(1) = Fib_{i+1},$ (10)

where Fib_i is *i*-th Fibonacci number.

According to Theorem 1 the sequence $G_0(m), \underbrace{G_1(m), \dots, G_1(m)}_{(m-1) \text{ times}}, \underbrace{G_2(m), \dots, G_2(m)}_{(m-1) \text{ times}}, \dots, \underbrace{G_N(m), \dots, G_N(m)}_{(m-1) \text{ times}}$ is minimizing absolutely ordered sequence of size $n = N^*(m-1) + 1$ for the elongated (m+1)-ary Huffman

tree (m > 0).

<u>Definition</u>. Sequence $G_0(m)$, $G_1(m)$, $G_2(m)$, $G_N(m)$ is called a <u>representative</u> Huffman *m*-sequence, i.e., representative sequence of the elongated (m+1)-ary Huffman tree (m > 0).

Several examples of representative Huffman *m*-sequences are shown in Table 1.

	$G_i(m): i = 0, 1, 2, \dots, 12$													
m	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	1	2	3	5	8	13	21	34	55	89	144	233	377
2	1	1	2	4	8	16	32	64	128	256	512	1024	2048	4096
3	1	1	2	5	11	26	59	137	314	725	1667	3842	8843	20369
4	1	1	2	6	14	38	94	246	622	1606	4094	10518	26894	68966
5	1	1	2	7	17	52	137	397	1082	3067	8477	23812	66197	185257
6	1	1	2	8	20	68	188	596	1724	5300	15644	47444	141308	425972
7	1	1	2	9	23	86	247	849	2578	8521	26567	86214	272183	875681
8	1	1	2	10	26	106	314	1162	3674	12970	42362	146122	485018	1653994
9	1	1	2	11	29	128	389	1541	5042	18911	64289	234488	813089	2923481
10	1	1	2	12	32	152	472	1992	6712	26632	93752	360072	1297592	4898312
11	1	1	2	13	35	178	563	2521	8714	36445	132299	533194	1988483	7853617
12	1	1	2	14	38	206	662	3134	11078	48686	181622	765854	2945318	12135566
13	1	1	2	15	41	236	769	3837	13834	63715	243557	1071852	4238093	18172169
14	1	1	2	16	44	268	884	4636	17012	81916	320084	1466908	5948084	26484796
15	1	1	2	17	47	302	1007	5537	20642	103697	413327	1968782	8168687	37700417

Table 1. Samples of representative Huffman *m*-sequences

2.3. Some properties of Fibonacci-like polynomials

Let

$$S(N,m) = \sum_{i=0}^{N} G_i(m) .$$
(11)

Calculate S(N, m). Consider

$$(m+1)^* S(N, m) = (m+1)^* \sum_{i=0}^{N} G_i(m)$$
$$= \sum_{i=0}^{N} G_i(m) + m^* \sum_{i=0}^{N} G_i(m)$$
$$= (G_0(x) + G_1(x) + \sum_{i=2}^{N} G_i(m)) + (m^* G_0(x) + m^* \sum_{i=1}^{N} G_i(m))$$
$$= (m+1)^* G_0(x) + G_1(x) + \sum_{i=1}^{N-1} (G_{i+1}(m) + m^* G_i(m)) + m^* G_N(m)$$
$$= (m+1)^* G_0(x) + G_1(x) + \sum_{i=1}^{N-1} G_{i+2}(m) + m^* G_N(m)$$

$$= (m+1)^*G_0(x) + G_1(x) + \sum_{i=3}^{N+1} G_i(m) + m^*G_N(m)$$

= $(m+1)^*G_0(x) + G_1(x) + \sum_{i=0}^{N+1} G_i(m) - (G_0(x) + G_1(x) + G_2(x)) + m^*G_N(m)$
= $m^*G_0(x) - G_2(x) + \sum_{i=0}^{N} G_i(m) + G_{N+1}(m) + m^*G_N(m)$
= $m^*1 - 2 + S(N, m) + G_{N+2}(m)$
= $S(N, m) + G_{N+2}(m) + m - 2$.

So,

$$(m+1)^* S(N, m) = S(N, m) + G_{N+2}(m) + m - 2$$
, i.e.,
 $m^*S(N, m) = G_{N+2}(m) + m - 2$.

Therefore

$$S(N,m) = \frac{G_{N+2}(m) + m - 2}{m} = \frac{G_{N+2}(m) - 2}{m} + 1.$$
 (12)

In particular,

$$S(N, 1) = \frac{G_{N+2}(1) + 1 - 2}{1} = G_{N+2}(1) - 1 = Fib_{N+3} - 1$$

2.4. Cost of minimizing absolutely ordered sequence of the elongated *m*-ary Huffman tree <u>Theorem 2</u>. The cost (i.e., weighted external path length) of elongated *m*-ary Huffman tree *T* of size $n = N^*(m-1) + 1$ for the minimizing absolutely ordered sequence $Pmin_{abs}(N, m)$ is

$$E(T, Pmin_{abs}(N, m)) = \frac{G_{N+4}(m-1)-2}{m-1} - (N+3) = \frac{G_{\frac{n+4m-5}{m-1}}(m-1) - (n+3m-2)}{m-1}$$

<u>Proof</u>. Let $Pmin_{abs}(N, m) = \{p_1, p_2, ..., p_n\}$ be the minimizing *k*-ordered sequence of the elongated binary tree *T* of size *n*.

According to Theorem 1

$$Pmin_{abs}(N, m) = \{G_0(m-1), \underbrace{G_1(m-1), \dots, G_1(m-1)}_{(m-1) \text{ times}}, \underbrace{G_2(m-1), \dots, G_2(m-1)}_{(m-1) \text{ times}}, \dots, \underbrace{G_N(m-1), \dots, G_N(m-1)}_{(m-1) \text{ times}}\}.$$

Weighted external path length $E(T, Pmin_{n,k})$ is

$$E(T, Pmin_{abs}(N, m)) = \sum_{i=1}^{n} l_i p_i .$$

where l_i is the length of the path from the root to leaf *i*.

T is the elongated binary tree, therefore

$$E(T, Pmin_{abs}(N, m)) = \sum_{i=1}^{n} l_i p_i$$

= $N^*G_0(m-1) + (m-1)\sum_{i=1}^{N} (N-i+1)^*G_i(m-1)$
= $N^*G_0(m-1) + (m-1)\sum_{i=0}^{N} (N-i+1)^*G_i(m-1) - (m-1)^*(N+1)^*G_0(m-1)$
= $(m-1)\sum_{i=0}^{N} (N-i+1)^*G_i(m-1) - (m-2)^*(N+1)^*G_0(m-1) - G_0(m-1)$

$$= (m-1)\sum_{i=0}^{N}\sum_{j=0}^{N-i}G_{i}(m-1) - (m-2)*(N+1)*G_{0}(m-1) - G_{0}(m-1)$$
$$= (m-1)\sum_{j=0}^{N}\sum_{i=0}^{j}G_{i}(m-1) - (m-2)*(N+1)*G_{0}(m-1) - G_{0}(m-1).$$

Thus, taking into account (11) and (12), we obtain

$$\begin{split} E(T, Pmin_{abs}(N, m)) &= (m-1) \sum_{j=0}^{N} \sum_{i=0}^{j} G_{i}(m-1) - (m-2)^{*}(N+1)^{*}G_{0}(m-1) - G_{0}(m-1) \\ &= (m-1) \sum_{i=0}^{N} S(i, m-1) - (m-2)^{*}(N+1)^{*}G_{0}(m-1) - G_{0}(m-1) \\ &= (m-1) \sum_{i=0}^{N} \frac{G_{i+2}(m-1) + (m-1) - 2}{m-1} - (m-2)^{*}(N+1)^{*}G_{0}(m-1) - G_{0}(m-1) \\ &= \sum_{i=0}^{N} (G_{i+2}(m-1) + (m-1) - 2) - (m-2)^{*}(N+1)^{*}G_{0}(m-1) - G_{0}(m-1) \\ &= \sum_{i=0}^{N} G_{i+2}(m-1) + (m-3)^{*}(N+1) - (m-2)^{*}(N+1)^{*}G_{0}(m-1) - G_{0}(m-1) \\ &= \sum_{i=0}^{N} G_{i+2}(m-1) + (m-3)^{*}(N+1) - (m-2)^{*}(N+1) - 1 \\ &= \sum_{i=0}^{N} G_{i+2}(m-1) + (m-3)^{*}(N+1) - (N+2) \\ &= \sum_{i=0}^{N} G_{i+2}(m-1) - (N+1) - 1 \\ &= \sum_{i=0}^{N} G_{i+2}(m-1) - (N+1) - 1 \\ &= \sum_{i=0}^{N} G_{i+2}(m-1) - (N+2) \\ &= S(N+2, m-1) - (N+2) \\ &= S(N+2, m-1) - (N+4) \\ &= \frac{G_{N+4}(m-1) - 2}{m-1} + 1 - (N+4) \\ &= \frac{G_{N+4}(m-1) - 2}{m-1} - (N+3) \end{split}$$

In particular, taking into account (10) we have for binary (i.e., m = 2, n = N+1) Huffman elongated tree

$$E(T, Pmin_{abs}(N, 2)) = \frac{G_{N+4}(1) - 2}{1} - (N+3)$$

= $G_{N+4}(1) - 2 - (N+3)$
= $G_{N+4}(1) - (N+5)$
= $G_{N+4}(1) - (N+5)$
= $Fib_{N+5} - (N+5)$
= $Fib_{n+4} - (n+4).$

Several examples of costs for several elongated *m*-ary Huffman trees are shown in Table 2.

Arity	N (number of non-leaves in an elongated Huffman tree)									
т	1	2	3	4	5	6	7	8	9	10
2	2	6	13	25	45	78	132	220	363	595
3	3	10	25	56	119	246	501	1012	2035	4082
4	4	14	39	97	233	546	1270	2936	6777	15619
5	5	18	55	148	393	1014	2619	6712	17229	44122
6	6	22	73	209	605	1686	4752	13228	37039	103235
7	7	26	93	280	875	2598	7897	23540	70983	212290
8	8	30	115	361	1209	3786	12306	38872	125085	397267
9	9	34	139	452	1613	5286	18255	60616	206737	691754
10	10	38	165	553	2093	7134	26044	90332	324819	1137907
11	11	42	193	664	2655	9366	35997	129748	489819	1787410
12	12	46	223	785	3305	12018	48462	180760	713953	2702435
13	13	50	255	916	4049	15126	63811	245432	1011285	3956602
14	14	54	289	1057	4893	18726	82440	325996	1397847	5635939
15	15	58	325	1208	5843	22854	104769	424852	1891759	7839842
16	16	62	363	1369	6905	27546	131242	544568	2513349	10682035
17	17	66	403	1540	8085	32838	162327	687880	3285273	14291530
18	18	70	445	1721	9389	38766	198516	857692	4232635	18813587
19	19	74	489	1912	10823	45366	240325	1057076	5383107	24410674
20	20	78	535	2113	12393	52674	288294	1289272	6767049	31263427
21	21	82	583	2324	14105	60726	342987	1557688	8417629	39571610

Table 2. Costs for several elongated *m*-ary Huffman trees

References

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