# Fibonacci-Like Polynomials Produced by m-ary Huffman Codes for Absolutely Ordered Sequences 

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#### Abstract

A non-decreasing sequence of positive integer weights $P=\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$ ( $n=N^{*}(m-1)+1, N$ is number of non-leaves of $m$-ary tree) is called absolutely ordered if the intermediate sequences of weights produced by $m$-ary Huffman algorithm for initial sequence $P$ on $i$-th step satisfy the following conditions $p_{m}^{(i)}<p_{m+1}^{(i)}, i=\overline{0, N-2}$. Let $T$ be an $m$-ary tree of size $n$ and $M=M(T)$ be a set of such sequences of positive integer weights that $\forall P \in M$ the tree $T$ is the $m$-ary Huffman tree of $P(|P|=n)$. A sequence $P_{\min }$ of $n$ positive integer weights is called a minimizing sequence of the $m$-ary tree $T$ in the class $M\left(P_{\text {min }} \in M\right)$ if $P_{\text {min }}$ produces the minimal Huffman cost of the tree $T$ over all sequences from $M$, i.e., $E\left(T, P_{\text {min }}\right) \leq E(T, P) \forall P \in M$. Theorem 1. A minimizing absolutely ordered sequence of size $n=N^{*}(m-1)+1$ for the maximum height $m$-ary Huffman tree $(m>1)$ is $$
\begin{gathered} \quad \operatorname{Pmin}_{\mathrm{abs}}(N, m)= \\ \{G_{0}(m-1), \underbrace{G_{1}(m-1), \ldots, G_{1}(m-1)}_{(m-1) \text { times }}, \underbrace{G_{2}(m-1), \ldots, G_{2}(m-1)}_{(m-1) \text { times }}, \ldots, \underbrace{G_{N}(m-1), \ldots, G_{N}(m-1)}_{(m-1) \text { times }}\}, \end{gathered}
$$


where $G_{0}(m)=1, G_{1}(m)=1, G_{2}(m)=2, G_{i}(m)=G_{i-1}(m)+m * G_{i-2}(m)$ when $i=\overline{2, N}$ Polynomials $G_{i}(x)$ are called Fibonacci-like polynomials. Theorem 2. The cost of maximum height $m$-ary Huffman tree $T$ of size $n=N^{*}(m-1)+1$ for the minimizing absolutely ordered sequence $\operatorname{Pmin}_{\text {abs }}(N, m)$ is

$$
E\left(T, \operatorname{Pmin}_{\mathrm{abs}}(N, m)\right)=\frac{G_{N+4}(m-1)-2}{m-1}-(N+3) .
$$

Samples of Fibonacci-like polynomials and costs of maximum height $m$-ary Huffman trees are shown.

## 0. Preface

Absolutely ordered and $k$-ordered sequences for binary Huffman trees those have maximum height have been investigated in [1] and [2]. In this article the generalization of absolutely ordered sequences for $m$-ary Huffman trees those have maximum height is considered.

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## 1. Main Conceptions and Terminology <br> 1.1. m-ary trees

A (strictly) $\underline{m \text {-ary tree }}$ is an oriented ordered tree where each nonleaf node has exactly $m$ children (siblings). Size of an m-ary tree is the total number of leaves of this tree. Let $N$ be number of nonleaves (internal nodes), $n$ be number of leaves of $m$-ary tree. Number of leaves in $m$-ary tree satisfies the following condition

$$
\begin{equation*}
n=N^{*}(m-1)+1 . \tag{1}
\end{equation*}
$$

An $m$-tree ( $m \geq 2$ ) is called elongated if at least $(m-1)$ of any $m$ sibling nodes are leaves. An elongated binary tree of size $n$ has maximum height among all binary trees of size $n$. An elongated $m$-ary tree is called left-sided if only the left node in each $m$-tuple of sibling nodes can be nonleaf.

A $m$-ary tree is called labeled if a certain positive integer (weight) is set in correspondence with each leaf.

Definition. Let $T$ be an $m$-ary tree with positive weights $P=\left\{p_{1}, . ., p_{n}\right\}$ at its leaf nodes. The weighted external path length of $T$ is

$$
E(T, P)=\sum_{i=1}^{n} l_{i} p_{i}
$$

where $l_{i}$ is the length of the path from the root to leaf $i$.

### 1.2. Generalized $\boldsymbol{m}$-ary Huffman algorithm

Problem definition. Given a sequence of $n$ positive weights $P=\left\{p_{1}, \ldots, p_{n}\right\},(n-1)=0(\bmod (m-1))$. The problem is to find $m$-ary tree $T_{\min n}$ with $n$ leaves labeled $p_{1}, \ldots, p_{n}$ that has minimum weighted external path length over all possible $m$-ary trees of size $n$ with the same sequence of leaf weights. $T_{\min }$ is called the $m$-ary Huffman tree of the sequence $P ; E\left(T, P_{\text {min }}\right)$ is called the Huffman cost of the tree $T$.

The problem was solved for binary trees by Huffman algorithm [3]. That algorithm can be generalized for $m$-ary trees. A generalized Huffman algorithm builds $T_{\text {min }}$ in which each leaf (weight) of $m$-ary tree is associated with a (prefix free) codeword in alphabet $\{0,1, \ldots, m-1\}$.

Note. A code is called a prefix (free) code if no codeword is a prefix of another one.
$\underline{m}$-ary algorithm description (in the reference to the discussed issue).
Algorithm input. A non-decreasing sequence of positive weights
$P=\left\{p_{1}, p_{2}, \ldots, p_{n}\right\} \quad p_{k} \leq p_{k+1}, k=1, n-1 ; n=N^{*}(m-1)+1$, where $N$ is number of non-leaves.
Algorithm output. The sum of all the weights.
The algorithm is performed in $N$ steps. $i$-th step $(i=\overline{1, N})$ is as follows.

- $i$-th step input. A non-decreasing sequence of weights of size $n-(m-1) *(i-1)$.
$P^{(i-1)}=\left\{p_{1}^{(i-1)}, p_{2}^{(i-1)}, \ldots, p_{n-(m-1)^{*}(i-1)}^{(i-1)}\right\}\left(p_{k}^{(i-1)} \leq p_{k+1}^{(i-1)} ; k=\overline{1, n-(m-1) *(i-1)-1}\right) ;\left|P^{(i-1)}\right|=n-(m-1)^{*}(i-1)$.
- $i$-th step method. Build a sequence

$$
\left\{p_{1}^{(i-1)}+p_{2}^{(i-1)}+\ldots+p_{m}^{(i-1)}, p_{m+1}^{(i-1)}, \ldots, p_{n-(m-1)^{*}(i-1)}^{(i-1)}\right\}
$$

and sort its.

- $\quad i$-th step output. A non-decreasing sequence of weights of size $n-(m-1) * i$.

$$
P^{(i)}=\left\{p_{1}^{(i)}, p_{2}^{(i)}, \ldots, p_{n-(m-1)^{*} i}^{(i)}\right\}\left(p_{k}^{(i)} \leq p_{k+1}^{(i)} ; k=\overline{1, n-(m-1) * i-1}\right) ;\left|P^{(i)}\right|=n-(m-1)^{*} i .
$$

Note 1. $P^{(0)}$ is an input of $m$-ary Huffman algorithm, i.e.,

$$
p_{k}^{(0)}=p_{k}(k=\overline{1, n}) .
$$

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Note 2. If an input sequence on $i$-th step(s) of the algorithm satisfies condition

$$
p_{m}^{(i)}=p_{m+1}^{(i)}(0 \leq i \leq N-2),
$$

then several $m$-ary Huffman trees can result from initial sequence $P$ of weights, but the weighted external path length is the same in all these trees.

Let $P=\left\{p_{1}, p_{2}, p_{3}, \ldots, p_{n}\right\}$ be a sequence of size $n$ for which the $m$-ary Huffman tree is elongated. Then according to generalized $m$-ary Huffman algorithm

$$
\begin{equation*}
p_{1}^{(i)}+p_{2}^{(i)}+\ldots+p_{m}^{(i)} \leq p_{2 m}^{(i)}, i=\overline{0, N-2} . \tag{2}
\end{equation*}
$$

## 2. Main Results

### 2.1. Minimizing absolutely ordered sequence of the elongated $\boldsymbol{m}$-ary Huffman tree

Let $T$ be an $m$-ary tree ( $m>1$ ) of size $n$ (i.e., $n=N^{*}(m-1)+1$, where $N$ is number of non-leaves and $n$ is number of leaves) and $M=M(T)$ be a set of such sequences of positive integer weights that $\forall P \in M$ the tree $T$ is the $m$-ary Huffman tree of $P(|P|=n)$.
Definition. A sequence $P_{\text {min }}$ of $n$ positive integer weights is called a minimizing sequence of the $m$-ary tree $T$ in the class $M\left(P_{\min } \in M\right)$ if $P_{\text {min }}$ produces the minimal Huffman cost of the $m$-ary tree $T$ over all sequences from $M$, i.e.,

$$
E\left(T, P_{\min }\right) \leq E(T, P) \forall P \in M .
$$

Definition. A non-decreasing sequence of positive integer weights $P=\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$ is called absolutely ordered if the intermediate sequences of weights produced by $m$-ary Huffman algorithm for initial sequence $P$ satisfy the following conditions

$$
\begin{equation*}
p_{m}^{(i)}<p_{m+1}^{(i)}, i=\overline{0, N-2} . \tag{3}
\end{equation*}
$$

For an absolutely ordered sequence the equality-inequality relation (2) is transformed to the (strict) equality relation

$$
\begin{equation*}
p_{1}^{(i)}+p_{2}^{(i)}+\ldots+p_{m}^{(i)}<p_{2 m}^{(i)}, i=\overline{0, N-2} . \tag{4}
\end{equation*}
$$

Lemma 1. A minimizing absolutely ordered sequence of size $n=N^{*}(m-1)+1$ for the elongated $m$-ary tree $(m>1)$ is

$$
\operatorname{Pmin}_{\mathrm{abs}}(N, m)=\{Q_{0}(m), \underbrace{Q_{1}(m), \ldots, Q_{1}(m)}_{(m-1) \text { times }}, \underbrace{Q_{2}(m), \ldots, Q_{2}(m)}_{(m-1) \text { times }}, \ldots, \underbrace{Q_{N}(m), \ldots, Q_{N}(m)}_{(m-1) \text { times }}\},
$$

where $Q_{0}(m)=1, Q_{1}(m)=1, Q_{2}(m)=2, Q_{i}(m)=Q_{i-1}(m)+(m-1)^{*} Q_{i-2}(m)$ when $i=\overline{2, N}$.
Proof. Taking into account (3) (4) we obtain the following configurations of $m$-ary Huffman algorithm steps for absolutely ordered sequence of the elongated (left-sided) $m$-ary tree.

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## Step 0 (Initial):

Step 0: $\quad p_{1}, p_{2}, \ldots, p_{m,} p_{m+1}, \ldots, p_{2 m-1}, p_{2 m}, p_{2 m+1}, \ldots, p_{n} ; \quad p_{m}<p_{m+1} ;$
Steps 1-(N-2):
Step 1: $\quad p_{m+1}, \ldots, p_{2 m-1}, \sum_{j=1}^{m} p_{j}, p_{2 m}, \ldots, p_{n} ;$

$$
\sum_{j=1}^{m} p_{j}<p_{2 m}
$$

Step 2:
$p_{2 m}, \ldots, p_{3 m-2}, \sum_{j=1}^{2 m-1} p_{j}, p_{3 m-1}, \ldots, p_{n} ;$ $\sum_{j=1}^{2 m-1} p_{j}<p_{3 m-1} ;$

Step 3:

$$
p_{3 m-1}, \ldots, p_{4 m-3}, \sum_{j=1}^{3 m-2} p_{j}, p_{4 m-2}, \ldots, p_{n}
$$

$$
\sum_{j=1}^{3 m-2} p_{j}<p_{4 m-2}
$$

Step $i$ : $\quad p_{i^{*}(m-1)+2}, \ldots, p_{(i+1)^{*}(m-1)+1}, \sum_{j=1}^{i^{*}(m-1)+1} p_{j}, p_{(i+1)^{*}(m-1)+2, \ldots, p_{n} ; \quad \sum_{j=1}^{i^{*}(m-1)+1} p_{j}<p_{(i+1)^{*}(m-1)+2} ; ~ ; ~}^{\text {; }}$


## Steps (N-1), $N$ :


Step $N: \quad \sum_{j=1}^{N^{*}(m-1)+1} p_{j}=\sum_{j=1}^{n} p_{j}$.
On $\underline{\text { Step } 0} \mathbf{~ m e r g i n g ~} m$ leaves labeled by integers $p_{1}, p_{2}, \ldots, p_{m}$ are merged. Because $\operatorname{Pmin}_{\text {abs }}(N, m)=$ $\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$ is minimizing sequence of positive integer values, $p_{1}, p_{2}, \ldots, p_{m}$ should have minimal positive integer values, i.e., at least they must be equal. So, we can write as follows

$$
\begin{aligned}
p_{1} & =q_{0}, \\
p_{2}=\ldots & =p_{m}=q_{1} ; \\
q_{0} & =q_{1} .
\end{aligned}
$$

On $\underline{\text { Step } 1} \mathbf{m e r g i n g ~ ( ~} m-1$ ) leaves labeled by integers $p_{m+1}, \ldots, p_{2 m-1}$ and one nonleaf are merged. Again, because $\operatorname{Pmin}_{\text {abs }}(N, m)=\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$ is minimizing sequence of positive integer values, $p_{m+1}, \ldots, p_{2 m-1}$ should have minimal possible positive integer values, i.e., at least they must be equal. So, we can write as follows

$$
p_{m+1}=\ldots=p_{2 m-1}=q_{2} .
$$

In the same manner for Steps $2, \ldots,(\mathbf{N}-1)$ we obtain

$$
\begin{gathered}
p_{2 m}=\ldots=p_{3 m-2}=q_{3} ; \\
P_{3 m-1}=\ldots=p_{4 m-3}=q_{4} ; \\
\ldots \\
p_{i^{*}(m-1)+2}=\ldots=p_{(i+1)^{*}(m-1)+1}=q_{i+1} ; \\
\ldots \\
p_{(N-1)^{*}(m-1)+2}=\ldots=p_{N^{*}(m-1)+1}=q_{N} .
\end{gathered}
$$

So, the configurations of $m$-ary Huffman algorithm steps for absolutely ordered sequence of the elongated (left-sided) $m$-ary tree are transformed as follows.

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## Step 0 (Initial):

Step 0: $\quad q_{0}, \underbrace{q_{1}, \ldots, q_{1}}_{(m-1) \text { times }}, \underbrace{q_{2}, \ldots, q_{2}}_{(m-1) \text { times }}, \ldots, \underbrace{q_{N}, \ldots, q_{N}}_{(m-1) \text { times }} \quad q_{1}<q_{2} ;$

## Steps 1-(N-2):

Step 1: $\underbrace{q_{2}, \ldots, q_{2}}_{(m-1) \text { times }}, q_{0}+(m-1) \sum_{j=1}^{1} q_{j}, \underbrace{q_{3}, \ldots, q_{3}}_{(m-1) \text { times }}, \ldots, \underbrace{q_{N}, \ldots, q_{N}}_{(m-1) \text { times }} ; \quad q_{0}+(m-1) \sum_{j=1}^{1} q_{j}<q_{3} ;$
Step 2: $\underbrace{q_{3}, \ldots, q_{3}}_{(m-1) \text { times }}, q_{0}+(m-1) \sum_{j=1}^{2} q_{j}, \underbrace{q_{4}, \ldots, q_{4}}_{(m-1) \text { times }}, \ldots, \underbrace{q_{N}, \ldots, q_{N}}_{(m-1) \text { times }} ; \quad q_{0}+(m-1) \sum_{j=1}^{2} q_{j}<q_{4}$;
Step 3: $\underbrace{q_{4}, \ldots, q_{4}}_{(m-1) \text { times }}, q_{0}+(m-1) \sum_{j=1}^{3} q_{j}, \underbrace{q_{5}, \ldots, q_{5}}_{(m-1) \text { times }}, \ldots, \underbrace{q_{N}, \ldots, q_{N}}_{(m-1) \text { times }} ; \quad q_{0}+(m-1) \sum_{j=1}^{3} q_{j}<q_{5}$;

Step $i: \quad \underbrace{q_{i+1}, \ldots, q_{i+1}}_{(m-1) \text { times }}, q_{0}+(m-1) \sum_{j=1}^{i} q_{j}, \underbrace{q_{i+2}, \ldots, q_{i+2}}_{(m-1) \text { times }}, \ldots, \underbrace{q_{N}, \ldots, q_{N}}_{(m-1) \text { times }} ; q_{0}+(m-1) \sum_{j=1}^{i} q_{j}<q_{i+2} ;$
Step $N-2: \underbrace{q_{N-1}, \ldots, q_{N-1}}_{(m-1) \text { times }}, q_{0}+(m-1) \sum_{j=1}^{N-2} q_{j}, \underbrace{q_{N}, \ldots, q_{N}}_{(m-1) \text { times }} ; \quad q_{0}+(m-1) \sum_{j=1}^{N-2} q_{j}<q_{N}$;

## Steps ( $N-1$ ),$N$ :

Step $N-1: \underbrace{q_{N}, \ldots, q_{N}}_{(m-1) \text { times }}, q_{0}+(m-1) \sum_{j=1}^{N-1} q_{j} ;$
Step $N: \quad q_{0}+(m-1) \sum_{j=1}^{N} q_{j}$.
Because $\operatorname{Pmin}_{\mathrm{abs}}(N, m)=\{q_{0}, \underbrace{q_{1}, \ldots, q_{1}}_{(m-1) \text { times }}, \underbrace{q_{2}, \ldots, q_{2}}_{(m-1) \text { times }}, \ldots, \underbrace{q_{N}, \ldots, q_{N}}_{(m-1) \text { times }}\}$ is $\underline{\text { minimizing }}$ sequence of positive integer values, $q_{0}$ and $q_{1}$ should have minimal positive integer values, and $q_{2}$ should have minimal possible positive integer value. So, we have

$$
\begin{align*}
& q_{0}=1,  \tag{5}\\
& q_{1}=1, \tag{6}
\end{align*}
$$

and, taking into account (3)

$$
\begin{gather*}
q_{2}=q_{1}+1=2,  \tag{7}\\
q_{i}=(m-1) \sum_{j=1}^{i-2} q_{j}+1, \text { when } i=\overline{2, N} .
\end{gather*}
$$

Consider, $q_{i}-q_{i-1}$ when $i=\overline{3, N}$.

$$
\begin{equation*}
q_{i}-q_{i-1}=\left((m-1) \sum_{j=1}^{i-2} q_{j}+1\right)-\left((m-1) \sum_{j=1}^{i-3} q_{j}+1\right)=(m-1) q_{i-2} . \tag{8}
\end{equation*}
$$

i.e.

$$
q_{i}=q_{i-1}+(m-1) q_{i-2}, \text { when } i=\overline{2, N} .
$$

From (5), (6), (7) and (8) we obtain that $q_{i}$ is a function of $m$, i.e., $q_{i}=Q_{i}(m)$ and thus

$$
\begin{equation*}
Q_{0}(m)=1, Q_{1}(m)=1, Q_{2}(m)=2, Q_{i}(m)=Q_{i-1}(m)+(m-1) * Q_{i-2}(m) \text { when } i=\overline{2, N} . \tag{9}
\end{equation*}
$$

The lemma has been proved.

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### 2.2. Fibonacci-like polynomials

From Lemma 1 we can see that $m$-ary Huffman tree ( $m>1$ ) is connected with polynomials $Q_{i}(m)$ (9). From that we can told that $(m+1)$-ary Huffman tree $(m>0)$ is connected with polynomials

$$
G_{0}(m)=1, G_{1}(m)=1, G_{2}(m)=2, G_{i}(m)=G_{i-1}(m)+m^{*} G_{i-2}(m) \text { when } i=\overline{2, N} .
$$

So, we have polynomials that are defined by the recurrence relation

$$
G_{i}(x)=G_{i-1}(x)+x^{*} G_{i-2}(x) \text { when } i>2
$$

with

$$
G_{0}(x)=1, G_{1}(x)=1, G_{2}(x)=2 .
$$

Thus, Lemma 1 can be reformulate as
Theorem 1. A $\underline{\text { minimizing }}$ absolutely ordered sequence of size $n=N^{*}(m-1)+1$ for the elongated $m$-ary Huffman tree $(m>1)$ is
$\operatorname{Pmin}_{\mathrm{abs}}(N, m)=\{G_{0}(m-1), \underbrace{G_{1}(m-1), \ldots, G_{1}(m-1)}_{(m-1) \text { times }}, \underbrace{G_{2}(m-1), \ldots, G_{2}(m-1)}_{(m-1) \text { times }}, \ldots, \underbrace{G_{N}(m-1), \ldots, G_{N}(m-1)}_{(m-1) \text { times }}\}$,
where $G_{0}(m)=1, G_{1}(m)=1, G_{2}(m)=2, G_{i}(m)=G_{i-1}(m)+m^{*} G_{i-2}(m)$ when $i=\overline{2, N}$.
Huffman related polynomials $G_{i}(x)$ are Fibonacci-like ones in contrast to Fibonacci polynomials that are defined by another recurrence relation [4]

$$
F_{i}(x)=x^{*} F_{i-1}(x)+F_{i-2}(x) \text { when } x>2 \text {; }
$$

with

$$
F_{1}(x)=1, F_{2}(x)=x .
$$

The first few Fibonacci-like (Huffman related) polynomials are

$$
\begin{aligned}
& G_{0}(x)=1 \\
& G_{1}(x)=1 \\
& G_{2}(x)=2 \\
& G_{3}(x)=x+2 \\
& G_{4}(x)=3 x+2 \\
& G_{5}(x)=x^{2}+5 x+2 \\
& G_{6}(x)=4 x^{2}+7 x+2 \\
& G_{7}(x)=x^{3}+9 x^{2}+9 x+2 \\
& G_{8}(x)=5 x^{3}+16 x^{2}+11 x+2 \\
& G_{9}(x)=x^{4}+14 x^{3}+25 x^{2}+13 x+2 \\
& G_{10}(x)=6 x^{4}+30 x^{3}+36 x^{2}+15 x+2 \\
& G_{11}(x)=x^{5}+20 x^{4}+55 x^{3}+49 x^{2}+17 x+2 \\
& G_{12}(x)=7 x^{5}+50 x^{4}+91 x^{3}+64 x^{2}+19 x+2 \\
& G_{13}(x)=x^{6}+27 x^{5}+105 x^{4}+140 x^{3}+81 x^{2}+21 x+2 \\
& G_{14}(x)=8 x^{6}+77 x^{5}+196 x^{4}+204 x^{3}+100 x^{2}+23 x+2 \\
& G_{15}(x)=x^{7}+35 x^{6}+182 x^{5}+336 x^{4}+285 x^{3}+121 x^{2}+25 x+2 \\
& G_{16}(x)=9 x^{7}+112 x^{6}+378 x^{5}+540 x^{4}+385 x^{3}+144 x^{2}+27 x+2 \\
& G_{17}(x)=x^{8}+44 x^{7}+294 x^{6}+714 x^{5}+825 x^{4}+506 x^{3}+169 x^{2}+29 x+2 \\
& G_{18}(x)=10 x^{8}+156 x^{7}+672 x^{6}+1254 x^{5}+1210 x^{4}+650 x^{3}+196 x^{2}+31 x+2 \\
& G_{19}(x)=x^{9}+54 x^{8}+450 x^{7}+1386 x^{6}+2079 x^{5}+1716 x^{4}+819 x^{3}+225 x^{2}+33 x+2 \\
& G_{20}(x)=11 x^{9}+210 x^{8}+1122 x^{7}+2640 x^{6}+3289^{x 5}+2366 x^{4}+1015 x^{3}+256^{x 2}+35 x+2 .
\end{aligned}
$$

The Fibonacci-like polynomials $G_{i}(x)$ are normalized, i.e.

$$
\begin{equation*}
\mathrm{G}_{i}(1)=F i b_{i+1}, \tag{10}
\end{equation*}
$$

where $F i b_{i}$ is $i$-th Fibonacci number.

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According to Theorem 1 the sequence $G_{0}(m), \underbrace{G_{1}(m), \ldots, G_{1}(m)}_{(m-1) \text { times }}, \underbrace{G_{2}(m), \ldots, G_{2}(m), \ldots, \underbrace{G_{N}(m), \ldots, G_{N}(m)}_{(m-1) \text { times }}, ~(m+1)}_{(m-1) \text { times }}$ is minimizing absolutely ordered sequence of size $n=N^{*}(m-1)+1$ for the elongated $(m+1)$-ary Huffman tree ( $m>0$ ).
Definition. Sequence $G_{0}(m), G_{1}(m), G_{2}(m), \quad, G_{N}(m)$ is called a representative Huffman $m$-sequence, i.e., representative sequence of the elongated ( $m+1$ )-ary Huffman tree $(m>0)$.

Several examples of representative Huffman $m$-sequences are shown in Table 1.
Table 1. Samples of representative Huffman $m$-sequences

| m | $G_{i}(m): i=0,1,2, \ldots, 12$ |  |  |  |  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 1 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | 233 | 377 |
| 2 | 1 | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 | 2048 | 4096 |
| 3 | 1 | 1 | 2 | 5 | 11 | 26 | 59 | 137 | 314 | 725 | 1667 | 3842 | 8843 | 20369 |
| 4 | 1 | 1 | 2 | 6 | 14 | 38 | 94 | 246 | 622 | 1606 | 4094 | 10518 | 26894 | 68966 |
| 5 | 1 | 1 | 2 | 7 | 17 | 52 | 137 | 397 | 1082 | 3067 | 8477 | 23812 | 66197 | 185257 |
| 6 | 1 | 1 | 2 | 8 | 20 | 68 | 188 | 596 | 1724 | 5300 | 15644 | 47444 | 141308 | 425972 |
| 7 | 1 | 1 | 2 | 9 | 23 | 86 | 247 | 849 | 2578 | 8521 | 26567 | 86214 | 272183 | 875681 |
| 8 | 1 | 1 | 2 | 10 | 26 | 106 | 314 | 1162 | 3674 | 12970 | 42362 | 146122 | 485018 | 1653994 |
| 9 | 1 | 1 | 2 | 11 | 29 | 128 | 389 | 1541 | 5042 | 18911 | 64289 | 234488 | 813089 | 2923481 |
| 10 | 1 | 1 | 2 | 12 | 32 | 152 | 472 | 1992 | 6712 | 26632 | 93752 | 360072 | 1297592 | 4898312 |
| 11 | 1 | 1 | 2 | 13 | 35 | 178 | 563 | 2521 | 8714 | 36445 | 132299 | 533194 | 1988483 | 7853617 |
| 12 | 1 | 1 | 2 | 14 | 38 | 206 | 662 | 3134 | 11078 | 48686 | 181622 | 765854 | 2945318 | 12135566 |
| 13 | 1 | 1 | 2 | 15 | 41 | 236 | 769 | 3837 | 13834 | 63715 | 243557 | 1071852 | 4238093 | 18172169 |
| 14 | 1 | 1 | 2 | 16 | 44 | 268 | 884 | 4636 | 17012 | 81916 | 320084 | 1466908 | 5948084 | 26484796 |
| 15 | 1 | 1 | 2 | 17 | 47 | 302 | 1007 | 5537 | 20642 | 103697 | 413327 | 1968782 | 8168687 | 37700417 |

### 2.3. Some properties of Fibonacci-like polynomials

Let

$$
\begin{equation*}
S(N, m)=\sum_{i=0}^{N} G_{i}(m) \tag{11}
\end{equation*}
$$

Calculate $S(N, m)$. Consider

$$
\begin{gathered}
(m+1) * S(N, m)=(m+1) * \sum_{i=0}^{N} G_{i}(m) \\
=\sum_{i=0}^{N} G_{i}(m)+m^{*} \sum_{i=0}^{N} G_{i}(m) \\
=\left(G_{0}(x)+G_{1}(x)+\sum_{i=2}^{N} G_{i}(m)\right)+\left(m * G_{0}(x)+m^{*} \sum_{i=1}^{N} G_{i}(m)\right) \\
=(m+1) * G_{0}(x)+G_{1}(x)+\sum_{i=1}^{N-1}\left(G_{i+1}(m)+m^{*} G_{i}(m)\right)+m^{*} G_{N}(m) \\
=(m+1) * G_{0}(x)+G_{1}(x)+\sum_{i=1}^{N-1} G_{i+2}(m)+m^{*} G_{N}(m)
\end{gathered}
$$

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$$
\begin{gathered}
=(m+1) * G_{0}(x)+G_{1}(x)+\sum_{i=3}^{N+1} G_{i}(m)+m^{*} G_{N}(m) \\
=(m+1) * G_{0}(x)+G_{1}(x)+\sum_{i=0}^{N+1} G_{i}(m)-\left(G_{0}(x)+G_{1}(x)+G_{2}(x)\right)+m * G_{N}(m) \\
=m^{*} G_{0}(x)-G_{2}(x)+\sum_{i=0}^{N} G_{i}(m)+G_{N+1}(m)+m^{*} G_{N}(m) \\
=m^{*} 1-2+S(N, m)+G_{N+2}(m) \\
=S(N, m)+G_{N+2}(m)+m-2 .
\end{gathered}
$$

So,

$$
\begin{gathered}
(m+1) * S(N, m)=S(N, m)+G_{N+2}(m)+m-2, \text { i.e., } \\
m * S(N, m)=G_{N+2}(m)+m-2 .
\end{gathered}
$$

Therefore

$$
\begin{equation*}
S(N, m)=\frac{G_{N+2}(m)+m-2}{m}=\frac{G_{N+2}(m)-2}{m}+1 . \tag{12}
\end{equation*}
$$

In particular,

$$
S(N, 1)=\frac{G_{N+2}(1)+1-2}{1}=G_{N+2}(1)-1=F i b_{N+3}-1 \text {. }
$$

### 2.4. Cost of minimizing absolutely ordered sequence of the elongated $\boldsymbol{m}$-ary Huffman tree

 Theorem 2. The cost (i.e., weighted external path length) of elongated $m$-ary Huffman tree $T$ of size $n=N^{*}(m-1)+1$ for the minimizing absolutely ordered sequence $\operatorname{Pmin}_{\mathrm{abs}}(N, m)$ is$$
E\left(T, \operatorname{Pmin}_{\mathrm{abs}}(N, m)\right)=\frac{G_{N+4}(m-1)-2}{m-1}-(N+3)=\frac{{\frac{G_{n+4 m-5}}{m-1}(m-1)-(n+3 m-2)}_{m-1}^{m} . . .}{}
$$

Proof. Let $\operatorname{Pmin}_{\text {abs }}(N, m)=\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$ be the minimizing $k$-ordered sequence of the elongated binary tree $T$ of size $n$.

According to Theorem 1
$\operatorname{Pmin}_{\mathrm{abs}}(N, m)=\{G_{0}(m-1), \underbrace{G_{1}(m-1), \ldots, G_{1}(m-1)}_{(m-1) \text { times }}, \underbrace{G_{2}(m-1), \ldots, G_{2}(m-1)}_{(m-1) \text { times }}, \ldots, \underbrace{G_{N}(m-1), \ldots, G_{N}(m-1)}_{(m-1) \text { times }}\}$.
Weighted external path length $E\left(T, \operatorname{Pmin}_{n, k}\right)$ is

$$
E\left(T, \operatorname{Pmin}_{\mathrm{abs}}(N, m)\right)=\sum_{i=1}^{n} l_{i} p_{i} .
$$

where $l_{i}$ is the length of the path from the root to leaf $i$.
$T$ is the elongated binary tree, therefore

$$
\begin{gathered}
E\left(T, \operatorname{Pmin}_{\mathrm{abs}}(N, m)\right)=\sum_{i=1}^{n} l_{i} p_{i} \\
=N^{*} G_{0}(m-1)+(m-1) \sum_{i=1}^{N}(N-i+1) * G_{i}(m-1) \\
=N^{*} G_{0}(m-1)+(m-1) \sum_{i=0}^{N}(N-i+1) * G_{i}(m-1)-(m-1) *(N+1) * G_{0}(m-1) \\
=(m-1) \sum_{i=0}^{N}(N-i+1) * G_{i}(m-1)-(m-2) *(N+1) * G_{0}(m-1)-G_{0}(m-1)
\end{gathered}
$$

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$$
\begin{aligned}
& =(m-1) \sum_{i=0}^{N} \sum_{j=0}^{N-i} G_{i}(m-1)-(m-2) *(N+1) * G_{0}(m-1)-G_{0}(m-1) \\
& =(m-1) \sum_{j=0}^{N} \sum_{i=0}^{j} G_{i}(m-1)-(m-2) *(N+1) * G_{0}(m-1)-G_{0}(m-1) .
\end{aligned}
$$

Thus, taking into account (11) and (12), we obtain

$$
\begin{gathered}
E\left(T, \operatorname{Pmin}_{\mathrm{abs}}(N, m)\right)=(m-1) \sum_{j=0}^{N} \sum_{i=0}^{j} G_{i}(m-1)-(m-2) *(N+1) * G_{0}(m-1)-G_{0}(m-1) \\
=(m-1) \sum_{i=0}^{N} S(i, m-1)-(m-2) *(N+1) * G_{0}(m-1)-G_{0}(m-1) \\
=(m-1) \sum_{i=0}^{N} \frac{G_{i+2}(m-1)+(m-1)-2}{m-1}-(m-2) *(N+1) * G_{0}(m-1)-G_{0}(m-1) \\
=\sum_{i=0}^{N}\left(G_{i+2}(m-1)+(m-1)-2\right)-(m-2) *(N+1) * G_{0}(m-1)-G_{0}(m-1) \\
=\sum_{i=0}^{N} G_{i+2}(m-1)+(m-3) *(N+1)-(m-2) *(N+1) * G_{0}(m-1)-G_{0}(m-1) \\
=\sum_{i=0}^{N} G_{i+2}(m-1)+(m-3) *(N+1)-(m-2) *(N+1)-1 \\
=\sum_{i=0}^{N} G_{i+2}(m-1)-(N+1)-1 \\
=\sum_{i=0}^{N} G_{i+2}(m-1)-(N+2) \\
=\sum_{i=0}^{N+2} G_{i}(m-1)-\left(G_{0}(m-1)+G_{1}(m-1)\right)-(N+2) \\
= \\
=S(N+2, m-1)-2-(N+2) \\
=S(N+2, m-1)-(N+4) \\
= \\
=\frac{G_{N+4}(m-1)-2}{m-1}+1-(N+4) \\
=
\end{gathered}
$$

In particular, taking into account (10) we have for binary (i.e., $m=2, n=N+1$ ) Huffman elongated tree

$$
\begin{aligned}
& E\left(T, \operatorname{Pmin}_{\mathrm{abs}}(N, 2)\right)=\frac{G_{N+4}(1)-2}{1}-(N+3) \\
&= G_{N+4}(1)-2-(N+3) \\
&= G_{N+4}(1)-(N+5) \\
&=G_{N+4}(1)-(N+5) \\
&= F i b_{N+5}-(N+5) \\
&= F i b_{n+4}-(n+4) .
\end{aligned}
$$

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Several examples of costs for several elongated $m$-ary Huffman trees are shown in Table 2.
Table 2. Costs for several elongated $m$-ary Huffman trees

| Arity <br> $\boldsymbol{m}$ | $\boldsymbol{N}$ (number of non-leaves in an elongated Huffman tree) |  |  |  |  |  |  |  |  |  |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
|  | $\mathbf{1}$ | $\mathbf{2}$ | $\mathbf{3}$ | $\mathbf{4}$ | $\mathbf{5}$ | $\mathbf{6}$ | $\mathbf{7}$ | $\mathbf{8}$ | $\mathbf{9}$ | $\mathbf{1 0}$ |
| 2 | 2 | 6 | 13 | 25 | 45 | 78 | 132 | 220 | 363 | 595 |
| 3 | 3 | 10 | 25 | 56 | 119 | 246 | 501 | 1012 | 2035 | 4082 |
| 4 | 4 | 14 | 39 | 97 | 233 | 546 | 1270 | 2936 | 6777 | 15619 |
| 5 | 5 | 18 | 55 | 148 | 393 | 1014 | 2619 | 6712 | 17229 | 44122 |
| 6 | 6 | 22 | 73 | 209 | 605 | 1686 | 4752 | 13228 | 37039 | 103235 |
| 7 | 7 | 26 | 93 | 280 | 875 | 2598 | 7897 | 23540 | 70983 | 212290 |
| 8 | 8 | 30 | 115 | 361 | 1209 | 3786 | 12306 | 38872 | 125085 | 397267 |
| 9 | 9 | 34 | 139 | 452 | 1613 | 5286 | 18255 | 60616 | 206737 | 691754 |
| 10 | 10 | 38 | 165 | 553 | 2093 | 7134 | 26044 | 90332 | 324819 | 1137907 |
| 11 | 11 | 42 | 193 | 664 | 2655 | 9366 | 35997 | 129748 | 489819 | 1787410 |
| 12 | 12 | 46 | 223 | 785 | 3305 | 12018 | 48462 | 180760 | 713953 | 2702435 |
| 13 | 13 | 50 | 255 | 916 | 4049 | 15126 | 63811 | 245432 | 1011285 | 3956602 |
| 14 | 14 | 54 | 289 | 1057 | 4893 | 18726 | 82440 | 325996 | 1397847 | 5635939 |
| 15 | 15 | 58 | 325 | 1208 | 5843 | 22854 | 104769 | 424852 | 1891759 | 7839842 |
| 16 | 16 | 62 | 363 | 1369 | 6905 | 27546 | 131242 | 544568 | 2513349 | 10682035 |
| 17 | 17 | 66 | 403 | 1540 | 8085 | 32838 | 162327 | 687880 | 3285273 | 14291530 |
| 18 | 18 | 70 | 445 | 1721 | 9389 | 38766 | 198516 | 857692 | 4232635 | 18813587 |
| 19 | 19 | 74 | 489 | 1912 | 10823 | 45366 | 240325 | 1057076 | 5383107 | 24410674 |
| 20 | 20 | 78 | 535 | 2113 | 12393 | 52674 | 288294 | 1289272 | 6767049 | 31263427 |
| 21 | 21 | 82 | 583 | 2324 | 14105 | 60726 | 342987 | 1557688 | 8417629 | 39571610 |

## References

[1] Vinokur A.B, Huffman trees and Fibonacci numbers, Kibernetika Issue 6 (1986) 9-12 (in Russian) English translation in Cybernetics 21, Issue 6 (1986), 692-696.
[2] Alex Vinokur, Fibonacci connection between Huffman codes and Wythoff array, E-print, 10 pagesArXiv - Oct. 2004. - cs.DM/0410013 - http://arxiv.org/abs/cs/0410013.
[3] Huffman D, A method for the construction of minimum redundancy codes, Proc. of the IRE 40 (1952) 1098-1101.
[4] Eric W. Weisstein. "Fibonacci Polynomial." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/FibonacciPolynomial.html

