1 Introduction

Every real number in $(0,1)$ can be expanded into an *Engel's series*:

$$x = \frac{1}{q_1} + \frac{1}{q_1 q_2} + \cdots + \frac{1}{q_1 q_2 \cdots q_n} + \cdots \tag{1}$$

where the q_n are a finite or infinite sequence of positive integers verifying $2 \leq q_1 \leq q_2 \leq \ldots$, which we shall call an Engel's admissible sequence. For short we will write $x = \langle q_1, q_2, \ldots \rangle_E$ instead of (1). For more details, see the classical text by Perron, [13]. Erdös, Rényi and Szüsz, in [2] studied the metrical properties of Engel's expansions completing results announced by Borel, see [1], and Lévy, see [9]. The main trends of the subject are related to the celebrated theorem of Borel on normal numbers and to the metrical theory of continued fractions, which was initiated by Gauss, and further developed by Kuzmin, [7], Khintchine, [6], and Lévy, [8].

The alternated version of Engel's series are known as *Pierce expansions*:

$$x = \frac{1}{q_1} - \frac{1}{q_1 q_2} + \cdots + \frac{(-1)^{n+1}}{q_1 q_2 \cdots q_n} + \cdots . \tag{2}$$

In this case, the q_n form a sequence of strictly increasing positive integers, $1 \leq q_1 < q_2 < \ldots$ (a Pierce's admissible sequence). We will denote expansion (2) as $x = \langle q_1, q_2, \ldots \rangle_P$. The metrical theory of Pierce expansions was studied by Shallit in [16]. More recent contributions to the subject are to be found in [11, 12, 18, 15, 17].

In both Engel's and Pierce's cases, irrational numbers have one, and only one, infinite representation. Rational numbers have exactly two: in the case of Engel's series one is finite and the other is infinite with the elements q_n, all equal from some place onwards. This duplicity is due to the identity:

$$\langle a_1, a_2, \ldots , a_n, a_n, a_n, \ldots \rangle_E = \langle a_1, a_2, \ldots , a_{n-1}, a_n - 1 \rangle_E .$$

As to Pierce expansions, both representations of a rational number are finite, of length n the first and of length $n+1$ the second. In this case the duplicity is due to the identity

$$\langle a_1, a_2, \ldots , a_n \rangle_P = \langle a_1, a_2, \ldots , a_n, a_n + 1 \rangle_P .$$

The finite algorithms for rational numbers attracted Erdös' attention as they are intimately related to one of his favourite subjects: Egyptian fractions. See [2, 3].

In the first part of [14], a paper of 1962, Rényi provided a shorter way of reaching most of the results found in [2]. Of these, the most spectacular is originally due to Borel, [1], and states that for almost all $x \in (0,1]$, the elements of their Engel's series verify

$$\lim_{n\to\infty} \sqrt[n]{q_n(x)} = e. \tag{3}$$

In [16], Shallit proved that the same applies for Pierce expansions. These coincident results indicate that the growth of the sequence of the elements of Engel's series and Pierce expansions is about the same (with probability 1). In the second part of [14], Rényi proved that, for Engel's series:

i) *For almost all x the sequence $q_n(x)$ is strictly increasing for $n \geq n_0(x)$, where $n_0(x)$ depends on x.*

ii) *The probability of $q_n(x)$ being strictly increasing from the very beginning, is exactly* $1/2$.

In the case of Pierce expansions, as the set of partial quotients, q_n, is already strictly increasing, Rényi's problem does not properly apply, but we can ask ourselves about the measure of the set of those x in which the 'jump' between two consecutive elements is greater than a fixed positive integer, k. Of course, this generalization also applies to Engel's series.

The aim of this paper is to extend results i) and ii) above, for both, Engel and Pierce cases, in the following directions:

- Rényi's argument for result i), based on the first of the Borel–Cantelli lemmas, applies for a jump of value $k \geq 1$ (section 3).
- To compute the exact measure of the set of all x in which the jump between consecutive elements is greater than or equal to k from the very beginning (section 4).

2 Some metrical results

The results of this section together with their proofs can be found in [2, 16]. All values x, will always refer to real numbers in $(0,1]$, and $\boldsymbol{\lambda} A$ will denote the Lebesgue measure of set $A \subset [0,1]$.

The elements (or partial quotients) of the expansions, q_n, are to be considered functions of x.

Given a Pierce's admissible sequence of length r, $a_1, a_2, \ldots, a_r$, we call a 'cylinder' of rank r the set:

$$C_r^{(P)}(a_1, a_2, \ldots, a_r) = \{x : q_1(x) = a_1, q_2(x) = a_2, \ldots, q_r(x) = a_r\}.$$

Cylinders are intervals with endpoints:

$$\langle a_1, a_2, \dots, a_r \rangle_P \text{ and } \langle a_1, a_2, \dots, a_r + 1 \rangle_P$$

taken in the proper order, and of length

$$\left| C_r^{(P)}(a_1, a_2, \dots, a_r) \right| = \frac{1}{a_1 a_2 \cdots a_r (a_r + 1)}.$$

(We use $|I|$ to denote the length of an interval I.)

For a given Engel's admissible sequence of length r, $b_1, b_2, \dots, b_r$, the same definitions lead to an interval with endpoints $\langle b_1, b_2, \dots, b_r \rangle_E$ and $\langle b_1, b_2, \dots, b_r - 1 \rangle_E$ and length

$$\left| C_r^{(E)}(b_1, b_2, \dots, b_r) \right| = \frac{1}{b_1 b_2 \cdots b_r (b_r - 1)}.$$

The metrical theory concerns itself with the computation of the measure of sets of numbers which can be described by means of some property which the elements of the expansion have to verify. As a general rule these sets are expressed as unions of disjoint cylinders. In the case of Pierce expansions, this computation depends on the value of sums of the form

$$A_r(m, n) = \sum_{m \le q_1 < \cdots < q_r \le n} \frac{1}{q_1 q_2 \cdots q_r},$$

where m and n are two positive integers, $(m \le n)$ and $0 \le r \le n - m$. That is, the sum extends to the set of Pierce's admissible sequences of length r, whose elements are in the range $[m, n]$. The consideration of Engel's admissible sequences gives rise to the same sort of sums, where repetitions are allowed. That is, sums of the form

$$B_r(m, n) = \sum_{m \le q_1 \le \cdots \le q_r \le n} \frac{1}{q_1 q_2 \cdots q_r}.$$

In this case, each sum depends on r, $0 \le r < \infty$.

Since for fixed m, n the numbers A_r (we drop (m, n)), are nothing but the elementary symmetric functions for the polynomial whose roots are $\left\{ \frac{1}{m}, \frac{1}{m+1} \dots, \frac{1}{n} \right\}$, the generating function of the complex variable z, for the sequence $\{A_r\}$ is:

$$\left(1 + \frac{z}{m}\right)\left(1 + \frac{z}{m+1}\right) \cdots \left(1 + \frac{z}{n}\right) = A_0 + A_1 z + A_2 z^2 + \cdots + A_{n-m+1} z^{n-m+1} \tag{4}$$

It can also be proved that in the case of an Engel's admissible sequence, the generating function for $\{B_r\}$ is the rational function

$$\frac{1}{\left(1-\frac{z}{m}\right)\left(1-\frac{z}{m+1}\right)\cdots\left(1-\frac{z}{n}\right)} = B_0 + B_1 z + B_2 z^2 + \cdots . \tag{5}$$

2.1 The shift transform

An alternative approach to the theory of Engel's and Pierce's series is based on the introduction of the *shift transforms* T_P and T_E of which we shall make a very limited use. This is the bridge which links these systems of representation of real numbers to ergodic theory and dynamical systems. If $x = \langle q_1, q_2, q_3, \ldots \rangle$ and if T refers to either of T_P and T_E, the equation

$$T\langle q_1, q_2, q_3, \ldots \rangle = \langle q_2, q_3, \ldots \rangle$$

may be taken as the definition of each of the shift transforms. It is easy to see that the restriction of the transforms to a cylinder $C_1^{()}(m)$ takes the forms $T_E(x) = mx - 1$ and $T_P(x) = 1 - mx$. Now, if $I = (x, y)$ is an open interval, $I \subset C_1^{()}(m)$, its image under T is an interval of length

$$|Ty - Tx| = m(y - x) \Longleftrightarrow |TI| = m \cdot |I| .$$

As this last equality is true for intervals it must be true for Lebesgue measurable sets. Thus for every measurable set A

$$A \subset C_1^{()}(m) \Longrightarrow \boldsymbol{\lambda} TA = m \cdot \boldsymbol{\lambda} A. \tag{6}$$

In the following, most of the details are only given for Pierce expansions.

3 Rényi's first problem

Let us consider a fixed positive integer $k, k \geq 1$.

Lemma 1 (Pierce). *Given positive integers m and n, $(m \leq n)$, let $X_{m,n}$ the set*

$$X_{m,n} = \{x : q_1(x) \geq m \text{ and } \exists j, (j \geq 1), q_j(x) = n \text{ and } q_{j+1}(x) < n + k\}$$

Then

$$\boldsymbol{\lambda} X_{m,n} = \frac{1}{m}\left(\frac{1}{n+1} - \frac{1}{n+k}\right).$$

Proof. $X_{m,n}$ is the disjoint union of the intervals whose endpoints are

$$\langle q_1, \ldots, q_{j-1}, n, n+1 \rangle_P \, , \langle q_1, \ldots, q_{j-1}, n, n+k \rangle_P \, ,$$

for all admissible values of $q_1, \ldots, q_{j-1}$. Consequently, the total measure is

$$\frac{1}{n}\left(\frac{1}{n+1} - \frac{1}{n+k}\right) \cdot \sum_{m \le q_1 < \cdots < q_{j-1} \le n-1} \frac{1}{q_1 \cdots q_{j-1}},$$

setting $z = 1$ in (4), this last sum is

$$\left(1 + \frac{1}{m}\right)\left(1 + \frac{1}{m+1}\right) \cdots \left(1 + \frac{1}{n-1}\right) = \frac{n}{m}.$$

Thus, the measure we seek equals

$$\frac{1}{n}\left(\frac{1}{n+1} - \frac{1}{n+k}\right)\frac{n}{m} = \frac{1}{m}\left(\frac{1}{n+1} - \frac{1}{n+k}\right).$$

□

Now, the set of x whose Pierce expansions present, from some place onwards, jumps between consecutive elements greater than or equal to k is the complement in $(0, 1]$ of the set of x whose expansions present jumps of less than k units infinitely often, that is to say: $\limsup_n X_1, n$. For all m, the series $\sum_{n=m}^{\infty} \boldsymbol{\lambda} X_{m,n}$ converges:

$$\sum_{n=m}^{\infty} \boldsymbol{\lambda} X_{m,n} = \sum_{n=m}^{\infty} \frac{1}{m}\left(\frac{1}{n+1} - \frac{1}{n+k}\right) = \frac{1}{m}\left(\frac{1}{m+1} + \cdots + \frac{1}{m+k-1}\right). \tag{7}$$

Thus, the first Borel–Cantelli lemma tells us that

$$\boldsymbol{\lambda} \limsup_n X_1, n = 0,$$

proving thus that its complement has measure one.

For Engel's series we would obtain:

Lemma 2 (Engel). *Given positive integers m and n, $(2 \le m \le n)$, let $Y_{m,n}$ be the set*

$$Y_{m,n} = \{x : q_1(x) \ge m \, and \, \exists j, (j \ge 1), q_j(x) = n \, and \, q_{j+1}(x) < n+k\}.$$

Then

$$\boldsymbol{\lambda} Y_{m,n} = \frac{1}{m} \cdot \left(1 - \frac{1}{n}\right) \cdot \left(\frac{1}{n-1} - \frac{1}{n-1+k}\right).$$

The same considerations as before would show that

$$\boldsymbol{\lambda} \limsup_n Y_2, n = 0,$$

proving thus that almost all x in $(0,1]$ have Engel's series, whose consecutive elements, from some place onwards, present jumps greater than or equal to k.

4 Rényi's second problem

When dealing with Pierce expansions, the closest one can get to the second of Rényi's problems is to ask for the measure of the set G of numbers, whose partial quotients jump more than two units from the very beginning. A direct argument based on the inclusion and exclusion formula, may be used to show that the measure of the complement of G in $(0,1]$ is $1/e$. Thus

$$\boldsymbol{\lambda} G = 1 - \frac{1}{e} \tag{8}$$

Besides being tedious in its details, the proof in terms of the inclusion and exclusion formula was impossible to extend to larger values of the jump k, so we were forced to a different route which we now take. We give the details only for Pierce expansions though any modifications needed to make them fit for Engel's series are trivial.

Let us call

$$E_m^{(k)} = \{x : q_1(x) \geq m, \forall j,\ q_{j+1}(x) \geq q_j(x) + k\}.$$

With this notation, the set G above would be written as $E_1^{(2)}$. For the measure of these sets, let us write $p_m^k = \boldsymbol{\lambda} E_m^{(k)}$.

In the following we consider k as a given positive integer and we drop the superscript $(^k)$ in p_m^k and in $E_m^{(k)}$ in order to simplify notation.

Since for each m, $E_m - E_{m+1} \subset C_1^{(P)}(m)$, and $T_P(E_m - E_{m+1}) = E_{m+k}$ then, by (6) $\boldsymbol{\lambda} E_{m+k} = m \cdot \boldsymbol{\lambda}(E_m - E_{m+1})$. That is to say,

$$p_{m+k} = m(p_m - p_{m+1})$$

which, rendered into homogeneous form gives

$$p_{m+k} + mp_{m+1} - mp_m = 0. \tag{9}$$

This is a finite difference equation of order k, linear with polynomial coefficients which we will presently solve after establishing a few necessary lemmas.

Lemma 3. *For all m,*

$$mp_m + p_{m+1} + p_{m+2} + \cdots + p_{m+k-1} = C,$$

where C is a constant.

Proof. Let us write the recurrence (9)

$$mp_m = mp_{m+1} + p_{m+k}. \tag{10}$$

For any value of m, we consider the following equalities obtained using recursively (10):

$$\begin{aligned} mp_m + p_{m+1} + \cdots + p_{m+k-1} &= mp_{m+1} + p_{m+1} + \cdots + p_{m+k-1} + p_{m+k} \\ &= (m+1)p_{m+1} + p_{m+2} + \cdots + p_{m+k} \end{aligned}$$

Thus, for all m, $mp_m + p_{m+1} + \cdots + p_{m+k-1} = C$. □

Lemma 4. *The sequence p_m verifies: a) $\lim_{m\to\infty} p_m = 0$; b) $\lim_{m\to\infty} mp_m = 1$.*

Proof. Assertion a) is trivial as $p_m = \boldsymbol{\lambda} E_m$ and $E_m \subset (0, 1/m]$. As for b), if $q_1(x) \geq m$ and $x \notin E_m$, then, there exists a place j such that $q_j(x) = n$ and $q_{j+1}(x) < n + k$, that is $x \in X_{m,n}$ for some n. By the covering rule,

$$\boldsymbol{\lambda}\left(\left(0, \frac{1}{m}\right] - E_m\right) \leq \boldsymbol{\lambda} \bigcup_{m=n}^{\infty} X_{m,n} \leq \sum_{m=n}^{\infty} \boldsymbol{\lambda} X_{m,n}.$$

By (7), this last sum is

$$\frac{1}{m}\left(\frac{1}{m+1} + \cdots + \frac{1}{m+k-1}\right).$$

Consequently,

$$p_m \geq \frac{1}{m} - \frac{1}{m}\left(\frac{1}{m+1} + \cdots + \frac{1}{m+k-1}\right),$$

and b) follows. □

An immediate consequence of lemma 4 is:

Lemma 5. *The constant in lemma 3 is 1, that is to say, for all m we have*

$$mp_m + p_{m+1} + p_{m+2} + \cdots + p_{m+k-1} = 1. \tag{11}$$

In the case of Engel's series, for the particular values $m = 2$ and $k = 1$ the expression above gives $2p_2 = 1$, that is $\boldsymbol{\lambda} E_2^{(1)} = p_2 = 1/2$, which is result ii) of Rényi's paper (see p. 2).

We now turn to solve recurrence (9).

Lemma 6.

$$p_m = C \cdot \int_0^1 t^{m-1} \cdot e^{t+\frac{t^2}{2}+\cdots+\frac{t^{k-1}}{k-1}}\, dt$$

is a particular solution of the recurrence equation

$$p_{m+k} + mp_{m+1} - mp_m = 0, \tag{12}$$

where C is an arbitrary constant.

Proof. We will use Laplace's method (see [5, 10]) which assumes there exists a solution of the form:

$$p_m = \int_a^b t^{m-1} \cdot \omega(t)\, dt, \tag{13}$$

where the limits a and b and the function $\omega(t)$ are to be determined. Using (13) and integrating by parts we get:

$$\begin{aligned}
p_{m+k} &= \int_a^b t^{m-1} \cdot (t^k \cdot \omega(t))\, dt, \\
mp_m &= [t^m \cdot \omega(t)]_a^b - \int_a^b t^m \cdot \omega'(t)\, dt, \\
mp_{m+1} &= [t^m \cdot (t\omega(t))]_a^b - \int_a^b t^m \cdot (\omega(t) + t\omega'(t))\, dt.
\end{aligned}$$

Replacing all these expressions in the original equation (12),

$$\int_a^b t^{m-1} \cdot (t^k \cdot \omega(t))\, dt +$$

$$+ \left[t^m \cdot (t\omega(t))\right]_a^b - \int_a^b t^m \cdot (\omega(t) + t\omega'(t))\, dt -$$

$$- \left(\left[t^m \cdot \omega(t)\right]_a^b - \int_a^b t^m \cdot \omega'(t)\, dt \right) = 0.$$

Grouping integrated parts and integrals we obtain

$$\left[t^{m+1} \cdot \omega(t) - t^m \cdot \omega(t)\right]_a^b +$$

$$+ \int_a^b t^{m-1} \cdot \left\{ (t - t^2) \cdot \omega'(t) + (t^k - t) \cdot \omega(t) \right\}\, dt = 0.$$

The method prescribes to annihilate the integrand in order to obtain $\omega(t)$ and to annihilate the bracketed part to determine possible values of a and b. We start with the integrand,

$$(t - t^2) \cdot \omega'(t) + (t^k - t) \cdot \omega(t) = 0,$$

a linear homogeneous differential equation with separable variables,

$$\frac{\omega'}{\omega} = \frac{t - t^k}{t - t^2} = \frac{1 - t^{k-1}}{1 - t} = 1 + t + t^2 + \cdots + t^{k-2},$$

and thus

$$\omega(t) = Ce^{t + \frac{t^2}{2} + \cdots + \frac{t^{k-1}}{k-1}}, \quad (C \text{ an arbitrary constant}).$$

Now, replacing $\omega(t)$ with the value just found, we annihilate the integrated part seeking values a and b of t that make

$$t^m \cdot Ce^{t + \frac{t^2}{2} + \cdots + \frac{t^{k-1}}{k-1}} \cdot (t - 1) = 0.$$

Besides $\pm\infty$ depending on the parity of k, the only solutions are $a = 0$ and $b = 1$, which are the values we are going to use to obtain a particular solution of our equation:

$$p_m = C \cdot \int_0^1 t^{m-1} \cdot e^{t + \frac{t^2}{2} + \cdots + \frac{t^{k-1}}{k-1}}\, dt. \tag{14}$$

$\square$

We are now ready for the main result of this section:

Theorem 1. *The measure of E_m is*

$$p_m = e^{-\left(1+\frac{1}{2}+\cdots+\frac{1}{k-1}\right)} \cdot \int_0^1 t^{m-1} \cdot e^{t+\frac{t^2}{2}+\cdots+\frac{t^{k-1}}{k-1}}\, dt.$$

Proof. By (11) from lemma 5, we have the following set of initial conditions:

$$mp_m + p_{m+1} + p_{m+2} + \cdots + p_{m+k-1} = 1, \qquad (m = 1, 2, \ldots)$$

Replacing p_{m+j} by the corresponding values from (14),

$$C \cdot \int_0^1 \left(mt^{m-1} + t^m + \cdots + t^{m+k-2}\right) e^{t+\frac{t^2}{2}+\cdots+\frac{t^{k-1}}{k-1}}\, dt = 1.$$

Since

$$\frac{d}{dt}\{t^m e^{t+\frac{t^2}{2}+\cdots+\frac{t^{k-1}}{k-1}}\} = \left(mt^{m-1} + t^m + \cdots + t^{m+k-2}\right) e^{t+\frac{t^2}{2}+\cdots+\frac{t^{k-1}}{k-1}},$$

we have

$$C \cdot \int_0^1 \left(mt^{m-1} + t^m + \cdots + t^{m+k-2}\right) e^{t+\frac{t^2}{2}+\cdots+\frac{t^{k-1}}{k-1}}\, dt = e^{1+\frac{1}{2}+\cdots+\frac{1}{k-1}}.$$

Therefore, the infinite set of initial conditions above is consistent with the value of C given by

$$C = e^{-\left(1+\frac{1}{2}+\cdots+\frac{1}{k-1}\right)}.$$

The solution for this particular value of C is thus unique and corresponds to the measure we sought. $\square$

Now, as an immediate corollary of Theorem 1, making $m = 1$ and $k = 2$, and taking complements in $(0, 1]$, we obtain result (8).

5 Conclusions

We establish the measure of different sets of real numbers in $(0, 1]$ defined through properties verified by *all* the elements of their Pierce expansions. This settles questions about the problem set by A. Rényi in [14] concerning the measure of similar sets defined using Engel's series instead of Pierce expansions. Rényi's problem, though, involves only the tail of the expansion (from a place n_0 onwards). We end up with quite nice neat measures between zero and one. Specifically, see (1), the set of real numbers whose expansions present elements with a minimum jump of k units between consecutive elements, $E^{(k)}$, has measure:

$$\boldsymbol{\lambda} E^{(k)} = e^{-H_{k-1}} \cdot \int_{0} e^{\sum_{j=1}^{k-1} \frac{t^j}{j}} \, dt.$$

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