

**RECIPROCAL SUMS OF SECOND  
ORDER RECURRENT SEQUENCES**

HONG HU, ZHI-WEI SUN AND JIAN-XIN LIU

Department of Mathematics, Huaiyin Normal College,  
Huaiyin 223001, Jiangsu Province, P. R. China

Department of Mathematics, Nanjing University,  
Nanjing 210093, P. R. China. *E-mail*: zwsun@nju.edu.cn

The Fundamental Division, Nanjing Engineering College,  
Nanjing 210013, P. R. China

1. INTRODUCTION

Let  $\mathbb{Z}$  and  $\mathbb{R}$  ( $\mathbb{C}$ ) denote the ring of the integers and the field of real (complex) numbers respectively. For a field  $F$  we put  $F^* = F \setminus \{0\}$ . Fix  $A \in \mathbb{C}$  and  $B \in \mathbb{C}^*$ , and let  $\mathcal{L}(A, B)$  consist of all those second order recurrent sequences  $\{w_n\}_{n \in \mathbb{Z}}$  of complex numbers satisfying the recursion:

$$w_{n+1} = Aw_n - Bw_{n-1} \quad (\text{i.e. } Bw_{n-1} = Aw_n - w_{n+1}) \quad \text{for } n = 0, \pm 1, \pm 2, \dots \quad (1)$$

For sequences in  $\mathcal{L}(A, B)$  the corresponding characteristic equation is  $x^2 - Ax + B = 0$ , whose roots  $(A \pm \sqrt{A^2 - 4B})/2$  are denoted by  $\alpha$  and  $\beta$ . If  $A \in \mathbb{R}^*$  and  $\Delta = A^2 - 4B \geq 0$ , then we let

$$\alpha = \frac{A - \text{sg}(A)\sqrt{\Delta}}{2} \quad \text{and} \quad \beta = \frac{A + \text{sg}(A)\sqrt{\Delta}}{2} \quad (2)$$

where  $\text{sg}(A) = 1$  if  $A > 0$ , and  $\text{sg}(A) = -1$  if  $A < 0$ . In the case  $w_1 = \alpha w_0$ , it is easy to see that  $w_n = \alpha^n w_0$  for any integer  $n$ . If  $A = 0$ , then  $w_{2n} = (-B)^n w_0$  and  $w_{2n+1} = (-B)^n w_1$  for all  $n \in \mathbb{Z}$ . The Lucas sequences  $\{u_n\}_{n \in \mathbb{Z}}$  and  $\{v_n\}_{n \in \mathbb{Z}}$  in  $\mathcal{L}(A, B)$  take special values at  $n = 0, 1$ , namely

$$u_0 = 0, \quad u_1 = 1, \quad v_0 = 2, \quad v_1 = A. \quad (3)$$

It is well known that

$$(\alpha - \beta)u_n = \alpha^n - \beta^n \quad \text{and} \quad v_n = \alpha^n + \beta^n \quad \text{for } n \in \mathbb{Z}. \quad (4)$$

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If  $A = 1$  and  $B = -1$ , then those  $F_n = u_n$  and  $L_n = v_n$  are called Fibonacci numbers and Lucas numbers respectively.

Let  $m$  be a positive integer. In 1974 I.J. Good [2] showed that

$$\sum_{n=0}^m \frac{1}{F_{2^n}} = 3 - \frac{F_{2^{m-1}}}{F_{2^m}}, \quad \text{i.e.} \quad \sum_{n=0}^{m-1} \frac{(-1)^{2^n}}{F_{2^{n+1}}} = -\frac{F_{2^{m-1}}}{F_{2^m}},$$

V.E. Hoggatt, Jr. and M. Bicknell [4] extended this by evaluating  $\sum_{n=0}^m F_{k2^n}^{-1}$  where  $k$  is a positive integer. In 1977 W.E. Greig [3] was able to determine the sum  $\sum_{n=0}^m u_{k2^n}^{-1}$  with  $B = -1$ ; in 1995 R.S. Melham and A.G. Shannon [5] gave analogous results in the case  $B = 1$ . In 1990 R. André-Jeannin [1] calculated

$$\sum_{n=1}^{\infty} \frac{1}{u_{kn}u_{k(n+1)}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{v_{kn}v_{k(n+1)}}$$

in the case  $B = -1$  and  $2 \nmid k$ , using the Lambert series

$$L(x) = \sum_{n=1}^{\infty} \frac{x^n}{1-x^n} \quad (|x| < 1);$$

in 1995 Melham and Shannon [5] computed the sums in the case  $B = 1$ , in terms of  $\alpha$  and  $\beta$ .

In the present paper we obtain the following theorems which imply all of the above.

**Theorem 1.** *Let  $m$  be a positive integer, and  $f$  a function such that  $f(n) \in \mathbb{Z}$  and  $w_{f(n)} \neq 0$  for all  $n = 0, 1, \dots, m$ . Then*

$$\sum_{n=0}^{m-1} \frac{B^{f(n)} u_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}} = \frac{B^{f(0)} u_{f(m)-f(0)}}{w_{f(0)} w_{f(m)}} \quad (5)$$

where  $\Delta f(n) = f(n+1) - f(n)$ . If  $w_1 \neq \alpha w_0$  then

$$\sum_{n=0}^{m-1} \frac{(-1)^n}{w_{f(n)}} \left( \frac{2\alpha^{f(n)}}{w_1 - \alpha w_0} - \frac{B^{f(n)} u_{\Delta f(n)}}{w_{f(n+1)}} \right) = \frac{1}{w_1 - \alpha w_0} \left( \frac{\alpha^{f(0)}}{w_{f(0)}} - (-1)^m \frac{\alpha^{f(m)}}{w_{f(m)}} \right). \quad (6)$$

**Theorem 2.** *Suppose that  $A, B \in \mathbb{R}^*$  and  $\Delta = A^2 - 4B \geq 0$ . Let  $f : \{0, 1, 2, \dots\} \rightarrow \{k \in \mathbb{Z} : w_k \neq 0\}$  be a function such that  $\lim_{n \rightarrow +\infty} f(n) = +\infty$ . If  $w_1 \neq \alpha w_0$  then we have*

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{B^{f(n)} u_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}} &= \frac{\alpha^{f(0)}}{(w_1 - \alpha w_0) w_{f(0)}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{w_{f(n)}} \left( \frac{2\alpha^{f(n)}}{w_1 - \alpha w_0} - \frac{B^{f(n)} u_{\Delta f(n)}}{w_{f(n+1)}} \right). \end{aligned} \quad (7)$$

In the next section we will derive several results from these theorems. Theorems 1 and 2 are proved in Section 3.

## 2. CONSEQUENCES OF THEOREMS 1 AND 2

**Theorem 3.** *Let  $k$  and  $l$  be integers such that  $w_{kn+l} \neq 0$  for all  $n = 0, 1, 2, \dots$ . Then*

$$u_k \sum_{n=0}^{m-1} \frac{B^{kn}}{w_{kn+l} w_{k(n+1)+l}} = \frac{u_{km}}{w_l w_{km+l}} \quad \text{for all } m = 1, 2, 3, \dots \quad (8)$$

If  $A, B \in \mathbb{R}^*$ ,  $A^2 \geq 4B$ ,  $k > 0$  and  $w_1 \neq \alpha w_0$ , then

$$\sum_{n=0}^{\infty} \frac{u_k B^{kn+l}}{w_{kn+l} w_{k(n+1)+l}} = \frac{\alpha^l}{(w_1 - \alpha w_0) w_l} \quad (9)$$

and

$$\sum_{n=0}^{\infty} \left( 2 \frac{(-\alpha^k)^n}{w_{kn+l}} - (w_1 - \alpha w_0) u_k \beta^l \frac{(-B^k)^n}{w_{kn+l} w_{k(n+1)+l}} \right) = \frac{1}{w_l}. \quad (10)$$

Proof. Simply apply Theorems 1 and 2 with  $f(n) = kn + l$ .

*Remark 1.* When  $B = 1$ ,  $l = k$  and  $\{w_n\} = \{u_n\}$  or  $\{v_n\}$ , Melham and Shannon [5] obtained (8) with the right hand side replaced by a complicated expression in terms of  $\alpha$  and  $\beta$ .

**Theorem 4.** *Let  $A, B \in \mathbb{R}^*$  and  $\Delta = A^2 - 4B > 0$ . Then for any positive integer  $k$  we have*

$$\sum_{n=1}^{\infty} \frac{(-B^k)^n}{u_{kn} u_{k(n+1)}} = \frac{\alpha^k}{u_k^2} + \text{sg}(A) \frac{\sqrt{\Delta}}{u_k} \left( 4L \left( \frac{\alpha^{4k}}{B^{2k}} \right) - 2L \left( \frac{\alpha^{2k}}{B^k} \right) \right) \quad (11)$$

and

$$\sum_{n=1}^{\infty} \frac{(-B^k)^n}{v_{kn} v_{k(n+1)}} = \frac{\text{sg}(A)}{\sqrt{\Delta}} \left( \frac{\alpha^k}{u_{2k}} - \frac{2}{u_k} \left( 4L \left( \frac{\alpha^{8k}}{B^{4k}} \right) - 4L \left( \frac{\alpha^{4k}}{B^{2k}} \right) + L \left( \frac{\alpha^{2k}}{B^k} \right) \right) \right). \quad (12)$$

Proof. Clearly  $|\alpha| < |\beta|$  and  $\beta - \alpha = \text{sg}(A) \sqrt{\Delta}$ . Thus  $u_n = (\beta^n - \alpha^n)/(\beta - \alpha)$  and  $v_n = \alpha^n + \beta^n$  are nonzero for all  $n \in \mathbb{Z} \setminus \{0\}$ . Obviously  $u_1 - \alpha u_0 = 1$  and  $v_1 - \alpha v_0 = A - 2\alpha = \beta - \alpha = \text{sg}(A) \sqrt{\Delta}$ . Applying Theorem 3 with  $l = k$  and  $\{w_n\}_{n \in \mathbb{Z}} = \{u_n\}_{n \in \mathbb{Z}}$  or  $\{v_n\}_{n \in \mathbb{Z}}$ , we then obtain that

$$\sum_{n=1}^{\infty} \left( u_k \frac{(-B^k)^n}{u_{kn} u_{k(n+1)}} - 2 \frac{(-\alpha^k)^n}{u_{kn}} \right) = \frac{\alpha^k}{u_k}$$

and

$$\sum_{n=1}^{\infty} \left( u_k \frac{(-B^k)^n}{v_{kn} v_{k(n+1)}} - \frac{2}{\text{sg}(A) \sqrt{\Delta}} \cdot \frac{(-\alpha^k)^n}{v_{kn}} \right) = \frac{\alpha^k / v_k}{\text{sg}(A) \sqrt{\Delta}}.$$

Clearly

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-\alpha^k)^n}{u_{kn}} &= \sum_{n=1}^{\infty} (\beta - \alpha) \frac{(-\alpha^k)^n}{\beta^{kn} - \alpha^{kn}} = (\beta - \alpha) \sum_{n=1}^{\infty} \frac{(-1)^n (\alpha/\beta)^{kn}}{1 - (\alpha/\beta)^{kn}} \\
&= (\beta - \alpha) \left( 2 \sum_{\substack{n=1 \\ 2|n}}^{\infty} \frac{(\alpha/\beta)^{kn}}{1 - (\alpha/\beta)^{kn}} - \sum_{n=1}^{\infty} \frac{(\alpha/\beta)^{kn}}{1 - (\alpha/\beta)^{kn}} \right) \\
&= (\beta - \alpha) \left( 2L\left(\frac{\alpha^{2k}}{\beta^{2k}}\right) - L\left(\frac{\alpha^k}{\beta^k}\right) \right) = \text{sg}(A) \sqrt{\Delta} \left( 2L\left(\frac{\alpha^{4k}}{B^{2k}}\right) - L\left(\frac{\alpha^{2k}}{B^k}\right) \right).
\end{aligned}$$

If  $|x| < 1$  then

$$\begin{aligned}
\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{1+x^n} &= 2 \sum_{n=1}^{\infty} \frac{x^{2n}}{1+x^{2n}} - \sum_{n=1}^{\infty} \frac{x^n}{1+x^n} \\
&= 2 \sum_{n=1}^{\infty} \left( \frac{x^{2n}}{1-x^{2n}} - \frac{2x^{4n}}{1-x^{4n}} \right) - \sum_{n=1}^{\infty} \left( \frac{x^n}{1-x^n} - \frac{2x^{2n}}{1-x^{2n}} \right) \\
&= 2L(x^2) - 4L(x^4) - L(x) + 2L(x^2) = -4L(x^4) + 4L(x^2) - L(x).
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-\alpha^k)^n}{v_{kn}} &= \sum_{n=1}^{\infty} \frac{(-\alpha^k)^n}{\alpha^{kn} + \beta^{kn}} = \sum_{n=1}^{\infty} (-1)^n \frac{(\alpha/\beta)^{kn}}{1 + (\alpha/\beta)^{kn}} \\
&= -4L\left(\frac{\alpha^{4k}}{\beta^{4k}}\right) + 4L\left(\frac{\alpha^{2k}}{\beta^{2k}}\right) - L\left(\frac{\alpha^k}{\beta^k}\right) \\
&= -4L\left(\frac{\alpha^{8k}}{B^{4k}}\right) + 4L\left(\frac{\alpha^{4k}}{B^{2k}}\right) - L\left(\frac{\alpha^{2k}}{B^k}\right).
\end{aligned}$$

Combining the above and noting that  $u_k v_k = u_{2k}$ , we then obtain the desired (11) and (12).

*Remark 2.* If  $|x| < 1$  then

$$\begin{aligned}
L(-x) &= \sum_{n=1}^{\infty} \frac{x^{2n}}{1-x^{2n}} - \sum_{n=1}^{\infty} \frac{x^n}{1+x^n} + \sum_{n=1}^{\infty} \frac{x^{2n}}{1+x^{2n}} \\
&= L(x^2) - (L(x) - 2L(x^2)) + (L(x^2) - 2L(x^4)) = -2L(x^4) + 4L(x^2) - L(x).
\end{aligned}$$

Thus Theorem 2 of André-Jeannin [1] is essentially our (11) and (12) in the special case  $B = -1$  and  $2 \nmid k$ .

**Theorem 5.** Let  $k, l, m \in \mathbb{Z}$  and  $l, m > 0$ . If  $w_{\binom{k+n}{l}} \neq 0$  for all  $n = 0, 1, \dots, m$ , then

$$\sum_{n=0}^{m-1} \frac{B^{\binom{k+n}{l}} u_{\binom{k+n}{l-1}}}{w_{\binom{k+n}{l}} w_{\binom{k+n+1}{l}}} = \frac{B^{\binom{k}{l}} u_{\binom{k+m}{l} - \binom{k}{l}}}{w_{\binom{k}{l}} w_{\binom{k+m}{l}}}. \quad (13)$$

Proof. Let  $f(n) = \binom{k+n}{l}$  for  $n \in \mathbb{Z}$ . It is well known that

$$\Delta f(n) = \binom{k+n+1}{l} - \binom{k+n}{l} = \binom{k+n}{l-1}.$$

So Theorem 5 follows from Theorem 1.

*Remark 3.* In the case  $k = 0$  and  $l = 2$ , (13) says that

$$\sum_{n=0}^{m-1} \frac{u_n B^{n(n-1)/2}}{w_{n(n-1)/2} w_{n(n+1)/2}} = \frac{u_{m(m-1)/2}}{w_0 w_{m(m-1)/2}}. \quad (14)$$

**Theorem 6.** Let  $a, k$  be integers, and  $m$  a positive integer. Suppose that  $w_{ka^n} \neq 0$  for each  $n = 0, 1, \dots, m-1$ . Then

$$\sum_{n=0}^{m-1} \frac{B^{ka^n} u_{k(a-1)a^n}}{w_{ka^n} w_{ka^{n+1}}} = \frac{B^k u_{k(a^m-1)}}{w_k w_{ka^m}}. \quad (15)$$

Proof. Just put  $f(n) = ka^n$  in Theorem 1.

*Remark 4.* In the case  $a = 2$  and  $\{w_n\} = \{u_n\}$ , (15) becomes

$$\sum_{n=0}^{m-1} \frac{B^{k2^n}}{u_k 2^{n+1}} = \frac{B^k u_{k(2^m-1)}}{u_k u_{k2^m}}. \quad (16)$$

This was obtained by Melham and Shannon [5] in the case  $B = 1$  and  $k > 0$ . In the case  $a = 3$  and  $\{w_n\} = \{v_n\}$ , (15) turns out to be

$$\sum_{n=0}^{m-1} \frac{B^{k3^n} u_{k3^n}}{v_k 3^{n+1}} = \frac{B^k u_{k(3^m-1)}}{v_k v_{k3^m}} \quad (17)$$

since  $u_{2h} = u_h v_h$  for  $h \in \mathbb{Z}$ .

**Theorem 7.** Let  $k$  be an integer and  $m$  a positive integers. If  $w_{k(2^n-1)} \neq 0$  for each  $n = 0, 1, \dots, m-1$ , then

$$\sum_{n=0}^{m-1} \frac{B^{k(2^n-1)} u_{k2^n}}{w_{k(2^n-1)} w_{k(2^{n+1}-1)}} = \frac{u_{k(2^m-1)}}{w_0 w_{k(2^m-1)}}. \quad (18)$$

Proof. Just apply Theorem 1 with  $f(n) = k(2^n - 1)$ .

## 3. PROOFS OF THEOREMS 1 AND 2

**Lemma 1.** For  $k, l, m \in \mathbb{Z}$  we have

$$w_k u_{l+m} - w_{k+m} u_l = B^l w_{k-l} u_m \quad (19)$$

and

$$w_k \alpha^l - w_l \alpha^k = (w_1 - \alpha w_0) B^l u_{k-l}. \quad (20)$$

Proof. i) Fix  $k, l \in \mathbb{Z}$ . Observe that

$$\begin{aligned} \begin{pmatrix} w_{k+1} & w_k \\ u_{l+1} & u_l \end{pmatrix} &= \begin{pmatrix} w_k & w_{k-1} \\ u_l & u_{l-1} \end{pmatrix} \begin{pmatrix} A & 1 \\ -B & 0 \end{pmatrix} \\ &= \begin{pmatrix} w_{k-1} & w_{k-2} \\ u_{l-1} & u_{l-2} \end{pmatrix} \begin{pmatrix} A & 1 \\ -B & 0 \end{pmatrix}^2 = \cdots \\ &= \begin{pmatrix} w_{k-l+1} & w_{k-l} \\ u_1 & u_0 \end{pmatrix} \begin{pmatrix} A & 1 \\ -B & 0 \end{pmatrix}^l. \end{aligned}$$

Taking the determinants we then get that

$$\begin{vmatrix} w_{k+1} & w_k \\ u_{l+1} & u_l \end{vmatrix} = \begin{vmatrix} w_{k-l+1} & w_{k-l} \\ 1 & 0 \end{vmatrix} \times \begin{vmatrix} A & 1 \\ -B & 0 \end{vmatrix}^l,$$

i.e.,

$$w_k u_{l+1} - w_{k+1} u_l = B^l w_{k-l}.$$

Thus (19) holds for  $m = 0, 1$ .

Each side of (19) can be viewed as a sequence in  $\mathcal{L}(A, B)$  with respect to the index  $m$ . By induction (19) is valid for every  $m = 0, 1, 2, \dots$ ; also (19) holds for each  $m = -1, -2, -3, \dots$ . Therefore (19) holds for any  $m \in \mathbb{Z}$ .

ii) By induction on  $l$  we find that  $w_{l+1} - \alpha w_l = (w_1 - \alpha w_0) \beta^l$ . Clearly both sides of (20) lie in  $\mathcal{L}(A, B)$  with respect to the index  $k$ . Note that if  $k = l$  then both sides of (20) are zero. As

$$(w_1 - \alpha w_0) B^l = (w_1 - \alpha w_0) \beta^l \alpha^l = (w_{l+1} - \alpha w_l) \alpha^l = \alpha^l w_{l+1} - \alpha^{l+1} w_l$$

(20) also holds for  $k = l + 1$ . Therefore (20) is always valid. We are done.

**Proof of Theorem 1.** Let  $d \in \mathbb{Z}$ . In view of Lemma 1, for  $n = 0, 1, \dots, m - 1$  we have

$$\begin{aligned} \frac{u_{d+f(n+1)}}{w_{f(n+1)}} - \frac{u_{d+f(n)}}{w_{f(n)}} &= \frac{u_{d+f(n+1)} w_{f(n)} - u_{d+f(n)} w_{f(n+1)}}{w_{f(n)} w_{f(n+1)}} \\ &= \frac{w_{f(n)} u_{d+f(n)+\Delta f(n)} - w_{f(n)+\Delta f(n)} u_{d+f(n)}}{w_{f(n)} w_{f(n+1)}} \\ &= \frac{B^{d+f(n)} w_{-d} u_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}}. \end{aligned}$$

It follows that

$$\sum_{n=0}^{m-1} \frac{B^{d+f(n)} w_{-d} u_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}} = \sum_{n=0}^{m-1} \left( \frac{u_{d+f(n+1)}}{w_{f(n+1)}} - \frac{u_{d+f(n)}}{w_{f(n)}} \right) = \frac{u_{d+f(m)}}{w_{f(m)}} - \frac{u_{d+f(0)}}{w_{f(0)}}$$

and that

$$\begin{aligned} \sum_{n=0}^{m-1} (-1)^{n+1} \frac{B^{d+f(n)} w_{-d} u_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}} &= \sum_{n=0}^{m-1} \left( (-1)^{n+1} \frac{u_{d+f(n+1)}}{w_{f(n+1)}} + (-1)^n \frac{u_{d+f(n)}}{w_{f(n)}} \right) \\ &= 2 \sum_{n=0}^{m-1} (-1)^n \frac{u_{d+f(n)}}{w_{f(n)}} + (-1)^m \frac{u_{d+f(m)}}{w_{f(m)}} - (-1)^0 \frac{u_{d+f(0)}}{w_{f(0)}}. \end{aligned}$$

Putting  $d = -f(0)$  we then obtain (5) and that

$$\begin{aligned} &\sum_{n=0}^{m-1} (-1)^{n+1} w_{f(0)} \frac{B^{f(n)} u_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}} \\ &= 2 \sum_{n=0}^{m-1} (-1)^n \frac{B^{f(0)} u_{f(n)-f(0)}}{w_{f(n)}} + (-1)^m \frac{B^{f(0)} u_{f(m)-f(0)}}{w_{f(m)}}. \end{aligned}$$

By Lemma 1, for each  $n = 0, 1, \dots, m$ ,

$$\alpha^{f(0)} w_{f(n)} - \alpha^{f(n)} w_{f(0)} = (w_1 - \alpha w_0) B^{f(0)} u_{f(n)-f(0)},$$

i.e.,

$$-\frac{B^{f(0)} u_{f(n)-f(0)}}{w_{f(n)}} = \frac{\alpha^{f(n)} w_{f(0)}}{(w_1 - \alpha w_0) w_{f(n)}} - \frac{\alpha^{f(0)}}{w_1 - \alpha w_0}.$$

Thus

$$\begin{aligned} w_{f(0)} \sum_{n=0}^{m-1} (-1)^n \frac{B^{f(n)} u_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}} &= 2 \sum_{n=0}^{m-1} (-1)^n \left( \frac{w_{f(0)} \alpha^{f(n)}}{(w_1 - \alpha w_0) w_{f(n)}} - \frac{\alpha^{f(0)}}{w_1 - \alpha w_0} \right) \\ &\quad + (-1)^m \left( \frac{w_{f(0)} \alpha^{f(m)}}{(w_1 - \alpha w_0) w_{f(m)}} - \frac{\alpha^{f(0)}}{w_1 - \alpha w_0} \right) \end{aligned}$$

and hence

$$\begin{aligned} &\sum_{n=0}^{m-1} \frac{(-1)^n}{w_{f(n)}} \left( \frac{2\alpha^{f(n)}}{w_1 - \alpha w_0} - \frac{B^{f(n)} u_{\Delta f(n)}}{w_{f(n+1)}} \right) \\ &= \frac{2}{w_1 - \alpha w_0} \sum_{n=0}^{m-1} (-1)^n \frac{\alpha^{f(0)}}{w_{f(0)}} + \frac{(-1)^m}{w_1 - \alpha w_0} \left( \frac{\alpha^{f(0)}}{w_{f(0)}} - \frac{\alpha^{f(m)}}{w_{f(m)}} \right) \\ &= \frac{1}{w_1 - \alpha w_0} \left( \frac{\alpha^{f(0)}}{w_{f(0)}} - (-1)^m \frac{\alpha^{f(m)}}{w_{f(m)}} \right). \end{aligned}$$

This proves (6).

**Lemma 2.** *Let  $A, B \in \mathbb{R}^*$  and  $\Delta = A^2 - 4B \geq 0$ . Then*

$$\lim_{n \rightarrow +\infty} \frac{\alpha^n}{u_n} = 0. \quad (21)$$

and

$$\lim_{n \rightarrow +\infty} \frac{w_n}{u_{m+n}} = \frac{w_1 - \alpha w_0}{\beta^m} \quad \text{for any } m \in \mathbb{Z}. \quad (22)$$

Proof. When  $\Delta = 0$  (i.e.  $\alpha = \beta$ ), by induction  $u_n = n(A/2)^{n-1}$  for all  $n \in \mathbb{Z}$ , thus  $u_n \neq 0$  for  $n = \pm 1, \pm 2, \pm 3, \dots$ ,

$$\lim_{n \rightarrow +\infty} \frac{\alpha^n}{u_n} = \lim_{n \rightarrow +\infty} \frac{(A/2)^n}{n(A/2)^{n-1}} = 0$$

and

$$\lim_{n \rightarrow +\infty} \frac{u_{m+n}}{u_n} = \lim_{n \rightarrow +\infty} \frac{(m+n)(A/2)^{m+n-1}}{n(A/2)^{n-1}} = \left(\frac{A}{2}\right)^m = \beta^m.$$

In the case  $\Delta > 0$ ,  $|\alpha| < |\beta|$  and hence  $u_n = (\alpha^n - \beta^n)/(\alpha - \beta)$  is zero if and only if  $n = 0$ . Therefore

$$\lim_{n \rightarrow +\infty} \frac{\alpha^n}{u_n} = (\alpha - \beta) \lim_{n \rightarrow +\infty} \frac{1}{1 - (\beta/\alpha)^n} = 0$$

Also,

$$\lim_{n \rightarrow +\infty} \left( \frac{u_{n+1}}{u_n} - \beta \right) = \lim_{n \rightarrow +\infty} \frac{\alpha^{n+1} - \beta^{n+1} - \beta(\alpha^n - \beta^n)}{\alpha^n - \beta^n} = \lim_{n \rightarrow +\infty} \frac{(\alpha - \beta)}{1 - (\beta/\alpha)^n} = 0,$$

If  $m \in \{0, 1, 2, \dots\}$ , then

$$\lim_{n \rightarrow +\infty} \frac{u_{m+n}}{u_n} = \lim_{n \rightarrow +\infty} \prod_{0 \leq k < m} \frac{u_{k+n+1}}{u_{k+n}} = \beta^m$$

and

$$\lim_{n \rightarrow +\infty} \frac{u_{n-m}}{u_n} = \lim_{n \rightarrow +\infty} \frac{u_n}{u_{m+n}} = \beta^{-m}.$$

In view of the above, (21) always holds and  $\lim_{n \rightarrow +\infty} u_{m+n}/u_n = \beta^m$  for all  $m \in \mathbb{Z}$ .

By Lemma 1,  $w_1 u_n - w_n u_1 = B w_0 u_{n-1}$  for  $n \in \mathbb{Z}$ . Thus

$$\lim_{n \rightarrow +\infty} \frac{w_n}{u_n} = w_1 - \frac{B w_0}{\lim_{n \rightarrow +\infty} u_n / u_{n-1}} = w_1 - \frac{B w_0}{\beta} = w_1 - \alpha w_0$$

and hence (22) is valid.



**Proof of Theorem 2.** Assume that  $w_1 \neq \alpha w_0$ . In view of Lemma 2,

$$\lim_{m \rightarrow +\infty} \frac{B^{f(0)} u_{f(m)-f(0)}}{w_{f(m)}} = B^{f(0)} \frac{\beta^{-f(0)}}{w_1 - \alpha w_0} = \frac{\alpha^{f(0)}}{w_1 - \alpha w_0}$$

and

$$\lim_{m \rightarrow +\infty} \frac{\alpha^m}{w_m} = \lim_{m \rightarrow +\infty} \frac{\alpha^m}{u_m} \times \lim_{m \rightarrow +\infty} \frac{u_m}{w_m} = 0.$$

Applying Theorem 1 we immediately get (7).

*Remark 5.* On the condition of Theorem 2, if  $w_1 = \alpha w_0$  then by checking the proof of Theorem 2 we find that

$$\sum_{n=0}^{\infty} \frac{B^{f(n)} u_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}} = \infty. \quad (23)$$

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