PARTITIONS INTO THREE TRIANGULAR NUMBERS

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Proposed Running Head: PARTITIONS INTO THREE TRIANGLES

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Abstract

A celebrated result of Gauss states that every positive integer can be represented as the sum of three triangular numbers. In this article we study $p_{3\Delta}(n)$, the number of partitions of the integer n into three triangular numbers, as well as $p^d_{3\Delta}(n)$, the number of partitions of n into three distinct triangular numbers.

Unlike $t(n)$, which counts the number of representations of n into three triangular numbers, $p_{3\Delta}(n)$ and $p^d_{3\Delta}(n)$ appear to satisfy very few arithmetic relations (apart from certain parity results). However, we shall show that, for all $n \geq 0$,

$$p_{3\Delta}(27n+12) = 3p_{3\Delta}(3n+1) \quad \text{and} \quad p^d_{3\Delta}(27n+12) = 3p^d_{3\Delta}(3n+1).$$

Two separate proofs of these results are given, one via generating function manipulations and the other by a combinatorial argument.

Keywords

partitions, triangular numbers, Gauss, generating functions

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1. Introduction

In 1796, C. F. Gauss proved his now famous result that every integer can be written as the sum of three triangular numbers. That is, if $t(n)$ is the number of **representations** of n as a sum of three triangular numbers, then $t(n) \geq 1$ for all $n \geq 0$.

When we study $t(n)$, we find that it grows quite rapidly. Consider, for example, $t(30)$. Note that 30 can be written as

$$28+1+1,\ 1+28+1,\ 1+1+28,$$

$$21+6+3,\ 21+3+6,\ 6+21+3,\ 6+3+21,\ 3+6+21,\ 3+21+6,$$

$$15+15+0,\ 15+0+15,\ 0+15+15,$$

and

$$10+10+10.$$

So we see that $t(30) = 13$.

Our goal in this paper is to study the **partitions** of n into three triangular numbers rather than the representations of n into three triangular numbers. For instance, the three representations $28+1+1$, $1+28+1$, and $1+1+28$ stem from one partition, $28+1+1$. Thus the integer 30 can be partitioned into three triangular numbers in only four ways. In this note, we will denote this partition function by $p_{3\Delta}(n)$, so that $p_{3\Delta}(30) = 4$.

Unlike $t(n)$, $p_{3\Delta}(n)$ appears to satisfy very few arithmetic relations (apart from certain parity results). This claim is based on a great deal of numerical evidence. However, while searching for congruences satisfied by $p_{3\Delta}$, we did discover the following:

Theorem 1.
Let $p_{3\Delta}(n)$ be defined as above and let $p^d_{3\Delta}(n)$ be the number of partitions of n into three distinct triangular numbers. Then, for all $n \geq 0$,

$$p_{3\Delta}(27n+12) = 3p_{3\Delta}(3n+1) \tag{1}$$

$$\text{and} \quad p^d_{3\Delta}(27n+12) = 3p^d_{3\Delta}(3n+1). \tag{2}$$

Theorem 1 (1) is reminiscent of a result proven by the authors [2] involving $t(n)$. Namely, for all $n \geq 0$,

$$t(27n+12) = 3t(3n+1). \tag{3}$$

Many other results of this type hold for $t(n)$, such as

$$t(27n+21) = 5t(3n+2), \tag{4}$$

$$t(81n+3) = 4t(9n), \tag{5}$$

$$\text{and} \quad t(81n+57) = 4t(9n+6). \tag{6}$$

However, results corresponding to (4), (5), and (6) do not exist for $p_{3\Delta}$.

In section 2, we develop the generating functions for $p_{3\Delta}(n)$ and $p^d_{3\Delta}(n)$ using techniques similar to those employed in [3]. Section 3 is then devoted to dissecting these generating functions and proving Theorem 1. Finally, we give a combinatorial proof of Theorem 1 in section 4.

2. The generating functions

As defined by Ramanujan, let

$$\psi(q) = \sum_{n\geq 0} q^{(n^2+n)/2}.$$

It is clear that $\sum\limits_{n\geq 0} t(n)q^n = \psi(q)^3$. However, the generating functions for $p_{3\Delta}(n)$ and $p^d_{3\Delta}(n)$ are somewhat more complicated, as we see here.

Theorem 2.

$$\sum_{n\geq 0} p_{3\Delta}(n)q^n = \frac{1}{6}\left(\psi(q)^3 + 3\psi(q)\psi(q^2) + 2\psi(q^3)\right) \tag{7}$$

$$\text{and} \quad \sum_{n\geq 0} p^d_{3\Delta}(n)q^n = \frac{1}{6}\left(\psi(q)^3 - 3\psi(q)\psi(q^2) + 2\psi(q^3)\right). \tag{8}$$

Proof: Let $\Delta_n = (n^2+n)/2$. Then $\sum_{n\geq 0} q^{\Delta_n} = \psi(q)$. In order to build the generating functions in (7) and (8), we mimic the approach utilized in [3]. We denote the generating function for the number of partitions of n of the form $n = \Delta_a + \Delta_b + \Delta_c$ by $F(\Delta_a + \Delta_b + \Delta_c, q)$, and use similar notation to define related generating functions. (So, for example, $F(\Delta_a + \Delta_a + \Delta_b, q)$ is the generating function for the number of partitions of n into twice one triangular number plus another (different) triangular number.) With the above notation in place, we have

$$F(\Delta_a, q) = \psi(q),$$

$$
\begin{aligned}
F(\Delta_a+\Delta_a,q) &= \psi(q^2),\\
F(\Delta_a+\Delta_a+\Delta_a,q) &= \psi(q^3),\\
F(\Delta_a+\Delta_b,q) &= \frac{1}{2}\left(F(\Delta_a,q)^2 - F(\Delta_a+\Delta_a,q)\right)\\
&= \frac{1}{2}\left(\psi(q)^2-\psi(q^2)\right),\\
F(\Delta_a+\Delta_a+\Delta_b,q) &= F(\Delta_a+\Delta_a,q)F(\Delta_a,q) - F(\Delta_a+\Delta_a+\Delta_a,q)\\
&= \psi(q)\psi(q^2)-\psi(q^3),\\
\text{and}\quad F(\Delta_a+\Delta_b+\Delta_c,q) &= \frac{1}{3}\left(F(\Delta_a+\Delta_b,q)F(\Delta_a,q) - F(\Delta_a+\Delta_a+\Delta_b,q)\right)\\
&= \frac{1}{6}\left(\psi(q)^3 - 3\psi(q)\psi(q^2)+2\psi(q^3)\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\sum_{n\geq 0} p_{3\Delta}(n)q^n &= F(\Delta_a+\Delta_b+\Delta_c,q)+F(\Delta_a+\Delta_a+\Delta_b,q)+F(\Delta_a+\Delta_a+\Delta_a,q)\\
&= \frac{1}{6}\left(\psi(q)^3+3\psi(q)\psi(q^2)+2\psi(q^3)\right),
\end{aligned}
$$

which is (7). Also,

$$
\sum_{n\geq 0} p^d_{3\Delta}(n)q^n = F(\Delta_a+\Delta_b+\Delta_c,q) = \frac{1}{6}\left(\psi(q)^3-3\psi(q)\psi(q^2)+2\psi(q^3)\right),
$$

which is (8). □

3. A generating function proof of Theorem 1

In order to prove Theorem 1, we must dissect the generating functions found in Theorem 2. To do so, we develop a large number of ancillary dissection results. As in Cooper and Hirschhorn [1], let

$$
\phi(q)=\sum_{n=-\infty}^{\infty} q^{n^2},\quad X(q)=\sum_{n=-\infty}^{\infty} q^{3n^2+2n},\quad P(q)=\sum_{n=-\infty}^{\infty} q^{(3n^2-n)/2},
$$

$$
A(q)=\sum_{n=-\infty}^{\infty} q^{9n^2+2n},\quad B(q)=\sum_{n=-\infty}^{\infty} q^{9n^2+4n},\quad C(q)=\sum_{n=-\infty}^{\infty} q^{9n^2+8n},
$$

$$
H(q)=\sum_{n=-\infty}^{\infty} q^{(9n^2+n)/2},\quad I(q)=\sum_{n=-\infty}^{\infty} q^{(9n^2+5n)/2},\quad J(q)=\sum_{n=-\infty}^{\infty} q^{(9n^2+7n)/2}.
$$

We need the following collection of results.

Lemma 1.

$$\psi(q) = P(q^3) + q\psi(q^9), \tag{i}$$

$$X(q) = A(q^3) + qB(q^3) + q^5C(q^3), \tag{ii}$$

$$P(q) = H(q^3) + qI(q^3) + q^2J(q^3), \tag{iii}$$

$$H(q)H(q^2) + qI(q)I(q^2) + q^2J(q)J(q^2) = \phi(q^3)P(q), \tag{iv}$$

$$H(q^2)I(q) + qI(q^2)J(q) + qJ(q^2)H(q) = X(q)P(q), \tag{v}$$

$$H(q)I(q^2) + qI(q)J(q^2) + J(q)H(q^2) = 2\psi(q^3)X(q), \tag{vi}$$

$$P(q)P(q^2) = \phi(q^9)P(q^3) + qX(q^3)P(q^3) + 2q^2\psi(q^9)X(q^3), \tag{vii}$$

$$A(q)H(q) + qB(q)J(q) + q^2C(q)J(q) = P(q)P(q^2), \tag{viii}$$

$$A(q)I(q) + B(q)H(q) + q^2C(q)J(q) = 2\psi(q^3)P(q^2), \tag{ix}$$

$$A(q)J(q) + B(q)I(q) + qC(q)H(q) = 2\psi(q^6)P(q), \tag{x}$$

and

$$X(q)P(q) = P(q^3)P(q^6) + 2q\psi(q^9)P(q^6) + 2q^2\psi(q^{18})P(q^3) \tag{xi}$$

Proof: First, (i), (ii) and (iii) are straightforward 3–dissections. Next, let

$$F_1(q) = H(q)H(q^2) + qI(q)I(q^2) + q^2J(q)J(q^2),$$

the left hand side of (iv). Then $F_1(q)$ equals

$$\sum_{m,n=-\infty}^{\infty} q^{(9m^2+m)/2+(9n^2+n)} + \sum_{m,n=-\infty}^{\infty} q^{(9m^2+5m)/2+(9n^2+5n)} + \sum_{m,n=-\infty}^{\infty} q^{(9m^2+7m)/2+(9n^2+7n)}.$$

Thus,

$$\begin{aligned} q^3F_1(q^{72}) &= \sum q^{(18m+1)^2+2(18n+1)^2} + \sum q^{(18m-5)^2+2(18n-5)^2} + \sum q^{(18m+7)^2+2(18n+7)^2} \\ &= \sum_{a\equiv b \pmod 3} q^{(6a+1)^2+2(6b+1)^2} \\ &= \sum_{r,s=-\infty}^{\infty} q^{(12r+6s+1)^2+2(6s-6r+1)^2} \\ &= q^3 \sum_{r,s=-\infty}^{\infty} q^{216r^2+108s^2+36s} \\ &= q^3\phi(q^{216})P(q^{72}), \end{aligned}$$

so that $F_1(q) = \phi(q^3)P(q)$. This yields (iv). The proofs of (v) and (vi) are similar. Indeed, let

$$F_2(q) = H(q^2)I(q) + qI(q^2)J(q) + qJ(q^2)H(q).$$

Then

$$\begin{aligned} q^{27}F_2(q^{72}) &= \sum_{a-b\equiv -1 \pmod 3} q^{(6a-1)^2+(6b+1)^2} \\ &= \sum_{r,s=-\infty}^{\infty} q^{(12r+6s-5)^2+2(6s-6r+1)^2} \\ &= q^{27} \sum_{r,s=-\infty}^{\infty} q^{216r^2-144r+108s^2-36s} \\ &= q^{27}X(q^{72})P(q^{72}), \end{aligned}$$

which implies $F_2(q) = X(q)P(q)$.

Next, let

$$F_3(q) = H(q)I(q^2) + qI(q)J(q^2) + J(q)H(q^2).$$

Then

$$\begin{aligned} q^{51}F_3(q^{72}) &= \sum_{a-b\equiv 1 \pmod 3} q^{(6a+1)^2+2(6b+1)^2} \\ &= \sum_{r,s=-\infty}^{\infty} q^{(12r+6s+7)^2+2(6s-6r+1)^2} \\ &= q^{51} \sum_{r,s=-\infty}^{\infty} q^{216r^2+144r+108s^2+108s} \\ &= 2q^{51}\psi(q^{216})X(q^{72}), \end{aligned}$$

which means $F_3(q) = 2\psi(q^3)X(q)$.

Thanks to (iii), (iv), (v) and (vi), we know

$$\begin{aligned} P(q)P(q^2) &= \left(H(q^3) + qI(q^3) + q^2J(q^3)\right)\left(H(q^6) + q^2I(q^6) + q^4J(q^6)\right) \\ &= \left(H(q^3)H(q^6) + q^3I(q^3)I(q^6) + q^6J(q^3)J(q^6)\right) \\ &\quad + q\left(H(q^6)I(q^3) + q^3I(q^6)J(q^3) + q^3J(q^6)H(q^3)\right) \\ &\quad + q^2\left(H(q^3)I(q^6) + q^3I(q^3)J(q^6) + J(q^3)H(q^6)\right) \end{aligned}$$

$$= \phi(q^9)P(q^3) + qX(q^3)P(q^3) + 2q^2\psi(q^9)X(q^3).$$

This is (vii). The proofs of (viii)–(xi) follow using similar techniques and are omitted here. □

With this machinery in hand, we now prove two additional theorems in preparation for our proof of Theorem 1.

Theorem 3.
Let $u(n)$ be defined by

$$\sum_{n\geq 0} u(n)q^n = \psi(q)\psi(q^2).$$

Then, for all $n \geq 0$, $u(27n+12) = 3u(3n+1)$.

Proof: From (i) we have

$$\begin{aligned}\sum_{n\geq 0} u(n)q^n &= \psi(q)\psi(q^2)\\ &= \left(P(q^3) + q\psi(q^9)\right)\left(P(q^6) + q^2\psi(q^{18})\right)\\ &= \left(P(q^3)P(q^6) + q^3\psi(q^9)\psi(q^{18})\right) + q\psi(q^9)P(q^6) + q^2\psi(q^{18})P(q^3).\end{aligned}$$

Thanks to this dissection, we immediately see that

$$\sum_{n\geq 0} u(3n+1)q^n = \psi(q^3)P(q^2).$$

Also, from the work above and (vii), we know

$$\begin{aligned}\sum_{n\geq 0} u(3n)q^n &= P(q)P(q^2) + q\psi(q^3)\psi(q^6)\\ &= \phi(q^9)P(q^3) + qX(q^3)P(q^3) + 2q^2\psi(q^9)X(q^3) + q\psi(q^3)\psi(q^6).\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n\geq 0} u(9n+3)q^n &= X(q)P(q) + \psi(q)\psi(q^2)\\ &= X(q)P(q) + \sum_{n\geq 0} u(n)q^n.\end{aligned}$$

This fact, combined with (xi), implies that

$$\sum_{n\geq 0} u(9n+3)q^n - \sum_{n\geq 0} u(n)q^n = X(q)P(q)$$
$$= P(q^3)P(q^6) + 2q\psi(q^9)P(q^6) + 2q^2\psi(q^{18})P(q^3).$$

Hence,

$$\sum_{n\geq 0} u(27n+12)q^n - \sum_{n\geq 0} u(3n+1)q^n = 2\psi(q^3)P(q^2) = 2\sum_{n\geq 0} u(3n+1)q^n$$

or

$$u(27n+12) = 3u(3n+1).$$

□

Theorem 4.
Let $v(n)$ be defined by

$$\sum_{n\geq 0} v(n)q^n = \psi(q^3).$$

Then, for all $n \geq 0$, $v(27n+12) = 3v(3n+1)$.

Proof: Since

$$\sum_{n\geq 0} v(n)q^n = \psi(q^3),$$

and $\psi(q^3)$ is a power series in q^3, we know

$$\sum_{n\geq 0} v(3n+1)q^n = 0.$$

Also, by (i), we have

$$\sum_{n\geq 0} v(3n)q^n = \psi(q) = P(q^3) + q\psi(q^9),$$

so that

$$\sum_{n\geq 0} v(9n+3)q^n = \psi(q^3).$$

By the same argument then,

$$\sum_{n\geq 0} v(27n+12)q^n = 0.$$

□

We are now prepared to prove Theorem 1.

Proof of Theorem 1: Using the notation in the previous theorems, we see that

$$\sum_{n\geq 0} p_{3\Delta}(n)q^n = \frac{1}{6}\sum_{n\geq 0}\left(t(n)+3u(n)+2v(n)\right)q^n,$$

which means

$$p_{3\Delta}(n) = \frac{1}{6}\left(t(n)+3u(n)+2v(n)\right).$$

Similarly,

$$p^d_{3\Delta}(n) = \frac{1}{6}\left(t(n)-3u(n)+2v(n)\right).$$

Theorem 1 then follows from (3) and Theorems 3 and 4. □

4. A combinatorial proof of Theorem 1

We start by noting that there is a one–to–one correspondence between partitions of $3n+1$ into three triangular numbers and partitions of $24n+11$ into three odd squares. A similar correspondence can be made between the partitions of $27n+12$ into three triangular numbers and $216n+99$ into three odd squares. Hence, proving (1) is equivalent to proving that the number of partitions of $216n+99$ into three odd squares equals three times the number of partitions of $24n+11$ into three odd squares.

In order to prove the result, we shall establish a one–to–three correspondence between the two sets of partitions.

Suppose that

$$24n+11 = k^2+l^2+m^2$$

with k, l and m odd and positive. Considering this equation modulo 6 gives

$$k^2+l^2+m^2 \equiv -1 \pmod{6}.$$

The only solutions of this are (permutations of)

$$k\equiv \pm 1,\quad l\equiv \pm 1,\quad m\equiv 3 \pmod{6}.$$

Allowing k and l to go negative, we can assume without loss of generality that

$$k\equiv 1,\quad l\equiv 1,\quad m\equiv 3 \pmod{6}$$

and that

$$k \geq l \text{ and } m > 0.$$

Now set

$$x_1 = 2k + 2l - m, \quad y_1 = 2k - l + 2m, \quad z_1 = -k + 2l + 2m,$$
$$x_2 = 2k + 2l + m, \quad y_2 = 2k - l - 2m, \quad z_2 = -k + 2l - 2m,$$
$$x_3 = 3k, \quad y_3 = 3l, \quad z_3 = 3m.$$

That is,

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} k \\ l \\ m \end{pmatrix}, \quad \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} k \\ l \\ m \end{pmatrix},$$
$$\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} k \\ l \\ m \end{pmatrix}.$$

Then

$$x_1^2 + y_1^2 + z_1^2 = x_2^2 + y_2^2 + z_2^2 = x_3^2 + y_3^2 + z_3^2 = 216n + 99$$

and, modulo 6,

$$(x_1, y_1, z_1) \equiv (1, 1, 1), \quad (x_2, y_2, z_2) \equiv (1, 1, 1), \quad (x_3, y_3, z_3) \equiv (3, 3, 3).$$

It is clear that the partition given by (x_3, y_3, z_3) is different from the other two. We now show that the partitions given by (x_1, y_1, z_1) and (x_2, y_2, z_2) are different from one another. If indeed they are the same then one of the following six situations pertains.

$$x_1 = x_2, \quad y_1 = y_2, \quad z_1 = z_2,$$
$$x_1 = x_2, \quad y_1 = z_2, \quad z_1 = y_2,$$
$$x_1 = y_2, \quad y_1 = x_2, \quad z_1 = z_2,$$
$$x_1 = y_2, \quad y_1 = z_2, \quad z_1 = x_2,$$
$$x_1 = z_2, \quad y_1 = x_2, \quad z_1 = y_2,$$
$$\text{or } x_1 = z_2, \quad y_1 = y_2, \quad z_1 = x_2.$$

In every one of these cases it follows that $m = 0$, but this is false as m is odd.

So we see that for each partition of $24n + 11$ into three odd squares, there are three partitions of $216n + 99$ into three odd squares. Corresponding to the partition of $24n + 11$

given by $\mathbf{v} = \begin{pmatrix} k \\ l \\ m \end{pmatrix}$, we have the three partitions of $216n + 99$ given by $A\mathbf{v}$, $B\mathbf{v}$ and $C\mathbf{v}$, where A, B and C are the three matrices defined above. We now show that the three partitions of $216n+99$ are uniquely determined, that is, if $\mathbf{v} \neq \mathbf{v}'$ where $\mathbf{v}' = \begin{pmatrix} k' \\ l' \\ m' \end{pmatrix}$ and $(k', l', m') \equiv (1, 1, 3) \pmod 6$ then $\{A\mathbf{v}, B\mathbf{v}, C\mathbf{v}\} \cap \{A\mathbf{v}', B\mathbf{v}', C\mathbf{v}'\} = \{\}$. Suppose

$$C\mathbf{v} = C\mathbf{v}'.$$

Then

$$\begin{pmatrix} 3k \\ 3l \\ 3m \end{pmatrix} = \begin{pmatrix} 3k' \\ 3l' \\ 3m' \end{pmatrix}$$

and it follows that $\mathbf{v} = \mathbf{v}'$. Suppose

$$A\mathbf{v} = A\mathbf{v}'.$$

Then

$$A^2\mathbf{v} = A^2\mathbf{v}',$$

that is,

$$9\mathbf{v} = 9\mathbf{v}'$$

so $\mathbf{v} = \mathbf{v}'$. A similar result holds if $B\mathbf{v} = B\mathbf{v}'$. We simply multiply by the transpose of B, and note that $B^TB = 9I$. Next, suppose

$$A\mathbf{v} = B\mathbf{v}'.$$

Then $A^2\mathbf{v} = AB\mathbf{v}'$, or,

$$\begin{pmatrix} 9k \\ 9l \\ 9m \end{pmatrix} = \begin{pmatrix} 7k + 4l - 4m \\ 4k + l + 8m \\ 4k - 8l - m \end{pmatrix}.$$

Modulo 6, this become

$$\begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \equiv \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix},$$

clearly false. A similar result holds (by symmetry) if $B\mathbf{v} = A\mathbf{v}'$.

We need now show that each partition of $216n + 99$ into three odd squares gives rise to a partition of $24n + 11$ into three odd squares.

Suppose

$$216n + 99 = x^2 + y^2 + z^2$$

with x, y and z odd. Modulo 54, this becomes

$$x^2 + y^2 + z^2 \equiv 45 \pmod{54}.$$

Consideration of all possibilities yields the 240 solutions (not counting permutations),

$$\begin{aligned}(x, y, z) \equiv\ & (\pm1, \pm1, \pm23),\ (\pm1, \pm5, \pm17),\ (\pm1, \pm7, \pm7),\ (\pm1, \pm11, \pm25),\ (\pm1, \pm13, \pm19),\\ & (\pm5, \pm5, \pm7),\ (\pm5, \pm11, \pm13),\ (\pm5, \pm19, \pm19),\ (\pm5, \pm23, \pm25),\ \pm7, \pm11, \pm19),\\ & (\pm7, \pm13, \pm23),\ (\pm7, \pm17, \pm25),\ (\pm11, \pm11, \pm17),\ (\pm11, \pm23, \pm23),\\ & (\pm13, \pm13, \pm25),\ (\pm13, \pm17, \pm17),\ (\pm17, \pm19, \pm23),\ (\pm19, \pm25, \pm25).\\ & (\pm3, \pm3, \pm9),\ (\pm3, \pm3, \pm27),\ (\pm3, \pm9, \pm15),\ (\pm3, \pm15, \pm27),\\ & (\pm3, \pm9, \pm21),\ (\pm3, \pm21, \pm27),\ (\pm9, \pm15, \pm15),\ (\pm15, \pm15, \pm27),\\ & (\pm9, \pm15, \pm21),\ (\pm15, \pm21, \pm27),\ \pm9, \pm21, \pm21),\ (\pm21, \pm21, \pm27).\end{aligned}$$

If we allow x, y and z to go negative, we can assume without loss of generality that, modulo 54, one of the following 30 possibilities holds.

$$\begin{aligned}(x, y, z) \equiv\ & (-23, 1, 1),\ (-5, 1, -17),\ (1, 7, 7),\ (1, 25, -11),\ (13, 1, 19),\ (7, -5, -5),\\ & (-11, -5, 13),\ (-5, 19, 19),\ (25, -23, -5),\ (19, -11, 7),\ (7, 13, -23),\ (-17, 7, 25),\\ & (-17, -11, -11),\ (-11, -23, -23),\ (25, 13, 13),\ (13, -17, -17),\ (-23, 19, -17),\\ & (19, 25, 25),\ (3, 3, 9),\ (3, 3, 27),\ (3, -15, 9),\ (3, -15, 27),\ (3, 21, 9),\\ & (3, 21, 27),\ (-15, -15, 9),\ (-15, -15, 27),\ (-15, 21, 9),\ (-15, 21, 27),\\ & (21, 21, 9), \text{or } (21, 21, 27).\end{aligned}$$

In the first eighteen cases, if we apply the matrix $A^{-1} = \frac{1}{9}A$ to the vector $\mathbf{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, we obtain a vector $\begin{pmatrix} k \\ l \\ m \end{pmatrix}$ that satisfies

$$k^2 + l^2 + m^2 = 24n + 11, \quad (k, l, m) \equiv (1, 1, 3) \pmod{6}.$$

If $k < l$, simply switch the second and third coordinates of $\mathbf{w}$, and then $k > l$. If $m < 0$, apply B^{-1} instead of A^{-1}, and then $m > 0$. In the latter twelve cases, if we apply the matrix $C^{-1} = \frac{1}{9}C$ to $\mathbf{w}$, we obtain a vector $\begin{pmatrix} k \\ l \\ m \end{pmatrix}$ that satisfies

$$k^2 + l^2 + m^2 = 24n + 11, \quad (k, l, m) \equiv (1, 1, 3) \pmod 6.$$

If $k < l$, switch the first two coordinates of $\mathbf{w}$, and then $k > l$. If $m < 0$, change the sign of the third coordinate of $\mathbf{w}$ and then $m > 0$.

This establishes the desired one–to–three correspondence, and completes the proof of (1). To prove (2), we need only replace non–strict inequalities where they occur with strict inequalities, and the proof goes through as before.

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