Enumeration of unrooted maps with given genus

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Abstract

Let $\mathcal{N}_g(f)$ denote the number of rooted maps of genus g having f edges. Exact formula for $\mathcal{N}_g(f)$ is known for $g = 0$ (Tutte 1963), $g = 1$ (Arques 1987), $g = 2, 3$ (Bender and Canfield 1991). In the present paper we derive an enumeration formula for the number $\Theta_\gamma(e)$ of unrooted maps on an orientable surface S_γ of given genus γ and given number of edges e. It has a form of a linear combination $\sum_{i,j} c_{i,j}\mathcal{N}_{g_j}(f_i)$ of numbers of rooted maps $\mathcal{N}_{g_j}(f_i)$ for some $g_j \leq \gamma$ and $f_i \leq e$. The coefficients $c_{i,j}$ are functions of γ and e. Let us consider the quotient S_γ/Z_ℓ of S_γ by a cyclic group of automorphisms Z_ℓ as a two-dimensional orbifold O. The task to determine $c_{i,j}$ requires to solve the following two subproblems:

(a) to compute the number $Epi_o(\Gamma, Z_\ell)$ of order preserving epimorphisms from the fundamental group Γ of the orbifold $O = S_\gamma/Z_\ell$ onto Z_ℓ,

(b) to calculate the number of rooted maps on the orbifold O which lifts along the branched covering $S_\gamma \to S_\gamma/Z_\ell$ to maps on S_γ with the given number e of edges.

The number $Epi_o(\Gamma, Z_\ell)$ is expressed in terms of classical number theoretical functions. The other problem is reduced onto the standard enumeration problem to determine the numbers $\mathcal{N}_g(f)$ for some $g \leq \gamma$ and $f \leq e$. It follows that $\Theta_\gamma(e)$ can be calculated whenever the numbers $\mathcal{N}_g(f)$ are known for $g \leq \gamma$ and $f \leq e$. In the end of the paper the above approach is applied do derive the functions $\Theta_\gamma(e)$ explicitly for $\gamma \leq 3$. Let us remark that the function $\Theta_\gamma(e)$ was known only for $\gamma = 0$ (Liskovets 1981). Tables containing the numbers of isomorphism classes of maps up to 30 edges for genus $\gamma = 1, 2, 3$ are produced.

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1 Introduction

By a map we mean a 2-cell decomposition of a compact connected surface. Enumeration of maps on surfaces has attracted a lot of attention last decades. As shown in monograph

[29] the enumeration problem was investigated for various classes of maps. Generally, problems of the following sort are considered:

Problem 1: How many isomorphism classes of maps of given property $\mathcal{P}$ and given number of edges (vertices, faces) there are?

Beginnings of the enumerative theory of maps are closely related with the enumeration of plane trees considered in 60-th by Tutte [35] , Harary, Prins and Tutte [15], see [16, 28] as well. Later a lot of other distinguished classes of maps including triangulations, outerplanar, cubic, Eulerian, nonseparable, simple, looples, two-face maps e.t.c. were considered. Research in these areas till year 1998 is well represented in [29]. Although there are more than 100 published papers on map enumeration, see for instance [7, 8, 11, 22, 24, 39, 40, 43], most of them deal with the enumeration of rooted maps of given property. In particular, there is a lack of results on enumeration of unrooted maps of genus ≥ 1. The present paper can be viewed as an attempt to fill in this gap. A map on an orientable surface is called oriented if one of the two global orientations is specified. Isomorphisms between oriented maps preserve the chosen orientation. The problem considered in this paper reads as follows.

Problem 2. What is the number of isomorphism classes of oriented unrooted maps of given genus g and given number of edges e?

An oriented map is called rooted of one of the darts (arcs) is distinguished as a root. By a dart of a map we mean an edge endowed with one of the two possible orientations. Isomorphisms between oriented rooted maps take root onto root. A rooted variant of Problem 2 follows.

Problem 3. What is the number of isomorphism classes of oriented rooted maps of given genus g and given number of edges e?

The rooted version of the problem was first considered in 1963 by Tutte [34] for $g = 0$, i.e. for the planar case. A corresponding planar case of the unrooted version (Problem 2 for $g = 0$) was settled by Liskovets [23, 25] and Wormald [42] much later. An attempt to enumerate unrooted maps of given genus $g > 0$ and given number of edges was done by Walsh and Lehman in [38, 39]. They have derived an algorithm based on a recursion formula. The algorithm is applied to enumerate maps with small number of edges. An explicit formula for number of rooted maps for $g = 1$ is obtained by D. Arquès [2].

In 1988 Bender, Canfield and Robinson [4] derived an explicit enumeration formula for the number of rooted maps on torus and projective plane. Three years later [5] Bender and Canfield determined the function $\mathcal{N}_g(e)$ of rooted maps of genus g with e edges for any genus g up to some constants. For $g = 2$ and $g = 3$ the generating functions are derived. Some refinement of these results can be found in [3].

In the present paper we shall deal with the problem of enumeration of oriented unrooted maps with given genus and given number of edges. Inspired by a fruitful concept of an orbifold used in low dimensional topology and in theory of Riemann surfaces we introduce a concept of a map on an orbifold. In the present paper by an orbifold we will mean a quotient of a surface by a finite group of automorphisms. As it will become clear to be later, cyclic orbifolds, that is the quotients of the type S_γ/Z_ℓ, where S_γ is an orientable surface of genus γ surface and Z_ℓ is a cyclic group of automorphisms of S_γ, will play a crucial role in the enumeration problem. In order to establish an explicit enumeration formula we first derive a general counting principle which allows us to decompose the problem into two subproblems (see Theorem 3.1). First one requires an enumeration of certain epimorphisms defined on Fuchsian groups (or on F-groups) onto a cyclic group. This problem is completely solved in Section 4. The other requires to enumerate rooted maps on cyclic orbifolds associated with the considered surface. Unfortunately, quotients of (ordinary) maps may have halved edges called semiedges here. In Section 5 we reduce this problem to the problem of enumeration of rooted maps without semiedges.

In order to formulate our main result we need to introduce some concepts.

Let S_γ be an orientable surface of genus γ surface and Z_ℓ is a cyclic group of automorphisms of S_γ. Denote by $[g; m_1, m_2, \ldots, m_r]$, $2 \le m_1 \le m_2 \le \ldots \le m_r \le \ell$, the signature of orbifold $O = S_\gamma/Z_\ell$. That is, the underlying space of O is an oriented surface of genus g and the regular cyclic covering $S_\gamma \to O = S_\gamma/Z_\ell$ is branched over r points of O with branch indexes $m_1, m_2, \ldots, m_r$, respectively. In 1966 W.J. Harvey [17] derived necessary and sufficient conditions for an existence of a cyclic orbifold S_γ/Z_ℓ with signature $[g; m_1, m_2, \ldots, m_r]$ (see Theorem 4.3).

Given orbifold O of the signature $[g; m_1, m_2, \ldots, m_r]$ define an orbifold fundamental group $\pi_1(O)$ to be an F-group generated by $2g$ generators $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ and by r generators e_j, $j = 1, \ldots, r$ satisfying the relations

$$\prod_{i=1}^{g}[a_i, b_i]\prod_{j=1}^{r} e_j = 1, \ e_j^{m_j} = 1 \text{ for every } \ j = 1, \ldots, r.$$

An epimorphism $\pi_1(O) \to Z_\ell$ onto a cyclic group of order ℓ is called *order preserving* if it preserves the orders of generators e_j, $j = 1, \ldots, r$. Equivalently, an order preserving epimorphism $\pi_1(O) \to Z_\ell$ has a torsion free kernel. We denote by $Epi_0(\pi_1(O), Z_\ell)$ the number of order preserving epimorphisms $\pi_1(O) \to Z_\ell$.

By a technical reason it is convenient to modify the signature of $O = S_\gamma/Z_\ell$ as follows. Let

$$[g; m_1, m_2, \ldots, m_r] = [g; \underbrace{2, \ldots, 2}_{q_2\, times}, \underbrace{3, \ldots, 3}_{q_3\, times}, \ldots, \underbrace{\ell, \ldots, \ell}_{q_\ell\, times}]$$

Then we will write $[g; 2^{q_2}, 3^{q_3}, \ldots, \ell^{q_\ell}]$ rather than $[g; m_1, m_2, \ldots, m_r]$ listing only j^{q_j} with $j > 0$.

Denote by $Orb(S_\gamma/Z_\ell)$ the set of ℓ-tuples $[g; 2^{q_2}, 3^{q_3}, \ldots, \ell^{q_\ell}]$ which are the signatures of cyclic orbifolds of the type S_γ/Z_ℓ, for some S_γ and Z_ℓ. By the definition, the fundamental group $\pi_1(O)$ is uniquely determined by the signature of the orbifold O. Hence, for any $O \in Orb(S_\gamma/Z_\ell)$, $O = [g; 2^{q_2}, 3^{q_3}, \ldots, \ell^{q_\ell}]$, the group $\pi_1(O)$ is well defined. Main result of this paper follows.

Theorem 1.1 *The number of unrooted oriented maps with e edges on an orientable sur-*

face of genus γ is

$$\Theta_\gamma(e) = \frac{1}{2e}\sum_{\ell|e}\ \sum_{\substack{O\in Orb(S_\gamma/Z_\ell)\\ O=[g;2^{q_2},3^{q_3}\dots,\ell^{q_\ell}]}} Epi_0(\pi_1(O),Z_\ell)\sum_{s=0}^{q_2}\binom{2e/\ell}{s}\binom{\frac{e}{\ell}-\frac{s}{2}+2-2g}{q_2-s,q_3,\dots,q_\ell}\mathcal{N}_g\left(\frac{e}{\ell}-\frac{s}{2}\right),$$

where $\mathcal{N}_g(n)$ denotes the number of rooted maps with n edges on an orientable surface of genus g with a convention $\mathcal{N}_g(n) = 0$ if n is not an integer.

An explicit formula to calculate $Epi_0(\pi_1(O), Z_\ell)$ is given in Section 4, Proposition 4.2. The number $Epi_0(\pi_1(O), Z_\ell)$ is expressed in terms of classical number theoretical functions. In Sections 6 and 7 Theorem 1.1 is applied to derive explicit enumeration functions for $\gamma = 0, 1, 2, 3$. For $\gamma = 0$ we have confirmed the result of Liskovets [23], enumeration formulas for $\gamma = 1, 2$ and 3 are original. To apply the theorem for $\gamma > 1$ one needs to determine elements of $Orb(S_\gamma/Z_\ell)$ for all admissible ℓ. Since the set of cyclic orbifolds coming from S_γ can be easily determined (see Section 4), oriented unrooted maps on S_γ can be enumerated using Theorem 1.1 provided the numbers $\mathcal{N}_g(n)$ of rooted maps are known for $g \le \gamma$.

2 Maps, coverings and orbifolds

In what follows we build a part of theory of maps which reflects some well known ideas coming from topology of low dimensional manifolds.

Maps on surfaces. By *a surface* we mean a connected, orientable surface without a border. A *topological map* is a 2-cell decomposition of a surface. Standardly, maps on surfaces are described as 2-cell embeddings of graphs. A (combinatorial) graph is a 4-tuple (D, V, I, L), where D and V are disjoint sets of darts and vertices, respectively, I is an incidence function $I : D \to V$ assigning to each dart an initial vertex, and L is the dart-reversing involution. Edges of a graph are orbits of L. Note that some edges may be incident just with one vertex, such edges will be called semiedges. In what follows we shall deal with the category of *oriented maps*, that means one of the two global orientations of the underlying surface is fixed. Given oriented map M can be described combinatorially as a triple $M = (D, R, L)$, where D is the set of darts (edges endowed with an orientation), L is an involutory permutation of D called dart-reversing involution permuting darts sharing the same edge, and R is a permutation of D permuting cyclically (following the global orientation) for each vertex v the darts based at v. By connectivity the group $\langle R, L\rangle$ acts transitively on D. Vice-versa, having an abstract *combinatorial map* (D, R, L), where $Mon(M) = \langle R, L\rangle$ is a transitive group of permutations of D and $L^2 = 1$, we can construct an associated topological map as follows: the orbits of R, L and RL give rise to the vertices, edges and boundary walks of faces of the map, respectively, and the incidence relations between vertices, edges and faces is given by non-empty intersections of the respective sets of darts. If x is a vertex, edge or a face, the degree of x is the size of the respective orbit of R, L or RL. The degree of an edge is two or one. Edges of degree one will be called *semiedges*. Maps without semiedges will be called *ordinary* maps. The group $Mon(M) = \langle R, L\rangle$ will be called a *monodromy group*. Given element $w(R,L) = R^{i_1}L^{j_1}R^{i_2}L^{j_2}\dots R^{i_n}L^{j_n} \in Mon(M)$ and a dart x_0, there is an associated dart-walk formed by darts $x_0, L^{j_n}(x_0), RL^{j_n}(x_0), \dots, R^{i_n}L^{j_n}(x_0), \dots, w(R,L)(x_0)$. This walk can be topologically realized in the topological map associated with $(D; R, L)$ as a curve with the initial point at x_0 and terminal point at $w(R,L)(x_0)$. Thus the action of $Mon(M)$ has a topological meaning. In fact it gives an information of the action of the fundamental groupoid of the surface restricted to a certain class of curves.

Given maps $M_i = (D_i, R_i, L_i)$, $i = 1, 2$, a *covering* $M_1 \to M_2$ is a mapping $\psi : D_1 \to D_2$ such that $\psi R_1 = R_2\psi$ and $\psi L_1 = L_2\psi$. Note that transitivity of the actions of the monodromy groups force ψ to be onto. In particular, two maps $M_i = (D, R_i, L_i)$, $i = 1, 2$, based on the same set of darts D are isomorphic if and only if there exists ψ in the symmetric group S_D such that $R_2 = R_1^{\psi}$ and $L_2 = L_1^{\psi}$. Coverings $M \to M$ form a group $Aut(M)$ of *automorphisms* of a map M. While the monodromy group is transitive on the set of darts, the automorphism group acts with trivial stabilisers, i.e. the action of $Aut(M)$ is semi-regular. More information on combinatorial maps can be found in [19].

Regular coverings. Let $\psi : M \to N$ be a covering of maps. The covering transformation group consists of automorphisms α of M satisfying the condition $\psi = \psi \circ \alpha$. A covering $\psi : M \to N$ will be called *regular* if the covering transformation group acts transitively on a fibre $\psi^{-1}(x)$ over a dart x of N. Regular coverings can be constructed by taking a subgroup $G \leq Aut(M)$, $M = (D, R, L)$, and setting $\bar{D}$ to be the set of orbits of G, $\bar{R}[x] = [Rx]$, $\bar{L}[x] = [Lx]$. Then the natural projection $x \mapsto [x]$ defines a regular covering $M \to N$, where $N = (\bar{D}, \bar{R}, \bar{L})$. Regular coverings of maps are extensively used in many considerations on maps and graph embeddings (see for instance [14, 30]).

Signatures of maps and orbifolds associated with maps. Given regular covering $\psi : M \to N$, let $x \in V(N) \cup F(N) \cup E(N)$ be a vertex, face or edge of N. The ratio of degrees $b(x) = deg(\tilde{x})/deg(x)$, where $\tilde{x} \in \psi^{-1}(x)$ is a lift of x along ψ, will be called a *branch index* of x. It is matter of routine to show that a branch index is a well-defined positive integer not depending on the choice of the lift $\tilde{x}$. In some considerations, it is important to save an information about branch indexes coming from some regular covering defined over a map N. This can be done by introducing a signature σ on M. A *signature* is a function $\sigma : x \in V(N) \cup F(N) \cup E(N) \to Z^+$ assigning a positive integer to each vertex, edge and face, with the only restriction: if x is an edge of degree 2 then $\sigma(x) = 1$, if it is of degree 1 then $\sigma(x) \in \{1, 2\}$. We say that a signature σ on N is *induced* by a covering $\psi : M \to N$ if it assigns to vertices, faces and edges of N their branch indexes with respect to ψ.

If a map $M = (D, R, L)$ is finite we can calculate the genus g of M by the well-known Euler-Poincare formula: $v(M) - e_2(M) + f(M) = 2 - 2g$, where $v(M)$ is the number of vertices, $e_2(M)$ is the number of edges of degree two, and $f(M)$ is the number of faces. Given a couple (M, σ) where M is a finite map and σ is a signature we define an *orbifold type* of (M, σ) to be an $(r+1)$-tuple of the form $[g; m_1, m_2, \dots, m_r]$, where g is the genus of the underlying surface, $1 < m_1 \leq m_2 \leq \dots \leq m_r$ are integers, and m_i appears in the sequence $s_i > 0$ times if and only if σ takes the value m_i exactly s_i times. The *orbifold* *fundamental group* $\pi_1(M,\sigma)$ of (M,σ) is a F-group

$$\pi_1(M,\sigma) = F[g; m_1, m_2, \ldots, m_r] =$$

$$\langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g, e_1, \ldots, e_r | \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{r} e_j = 1, e_1^{m_1} = \ldots e_r^{m_r} = 1 \rangle. \tag{2.1}$$

Let $\psi : M \to N$ be a *regular* covering and σ be a signature defined on N. We say that ψ is σ*-compatible* if for each element $x \in V(N) \cup E(N) \cup F(N)$ the branch index of $b(x)$ of x is a divisor of $\sigma(x)$. Signature σ defined on N lifts along a σ-compatible regular covering $\psi : M \to N$ to a derived signature σ_ψ on M defined by the following rule: $\sigma_\psi(\tilde{x}) = \sigma(x)/b(x)$ for each $\tilde{x} \in \psi^{-1}(x)$ and each $x \in V(N) \cup E(N) \cup F(N)$. Let us remark that if $\sigma(x) = 1$ for each $x \in V(N) \cup E(N) \cup F(N)$ then σ-compatible covers over M are just smooth regular covers over M. Such a signature will be called *trivial*.

Let $M \to M/G$ be a regular covering with a covering transformation group G, let M be finite. Let the respective orbifold type of $N = M/G$ be $[g; m_1, m_2, \ldots, m_r]$. Then the Euler characteristic of the underlying surface of M is given by the Riemann-Hurwitz equation:

$$\chi = |G|\big(2 - 2g - \sum_{i=1}^{r}(1 - \frac{1}{m_i})\big).$$

A topological counterpart of a (combinatorial) map M with a signature σ can be established as follows. By an *orbifold* O we mean a surface S with a distinguished discrete set of points B assigned by integers $m_1, m_2, \ldots, m_i, \ldots$ such that $m_i \geq 2$, for $i = 1, 2, \ldots$. Elements of B will be called branch points. If S is a compact connected orientable surface of genus g then B is finite of cardinality $|B| = r$ and O is determined by its type $[g; m_1, m_2, \ldots, m_r]$. Hence we write $O = O[g; m_1, m_2, \ldots, m_r]$. The fundamental group $\pi_1(O)$ of O is an F-group defined by (2.1). A (topological) map on an orbifold O is a map on the underlying surface S_g of genus g satisfying the following properties:

(P1) if $x \in B$ then x is either an internal point of a face, or a vertex, or an end-point of a semiedge which is not a vertex,

(P2) each face contains at most one branch point,

(P3) the branch index of x lying at the free end of a semiedge is two.

A mapping $\psi : \tilde{O} \to O$ is *a covering* if it is a branched covering between underlying surfaces mapping the set of branch points $\tilde{B}$ of $\tilde{O}$ onto the set B of branch points of O and each $\tilde{x}_i \in \psi^{-1}(x_i)$ is mapped uniformly with the same branch index d dividing the prescribed index r_i of $x_i \in B$. The following result is a consequence of the well-known theorem of Koebe:

Theorem 2.1 (Koebe [41]) *Let O be a compact connected orbifold of type*

$[g; m_1, m_2, \ldots, m_r]$. Then there is a universal orbifold $\tilde{O}$ covering O satisfying the following conditions:

(a) if there is a regular covering $\varphi : O_1 \to O$ then there is a regular covering $\psi : \tilde{O} \to O_1$,

(b) the covering $\Phi : \tilde{O} \to O$ is regular with the covering transformation group isomorphic to $F(g; m_1, m_2, \ldots, m_r)$ and $\Phi = \psi \circ \varphi$.

Remark. The reader familiar with Koebe's theorem may ask where the 'bad orbifolds' of type $[0; r]$ and $[0; r, q]$ $r \neq q$, $\gcd(r, q) = 1$ disappeared. They are included in the statement only that they give rise to trivial universal covers. Note that $F[0; r] = F[0; r, q] = 1$ is a trivial group in this case. The underlying surface of the universal cover is either a sphere or a plane, depending on whether the respective F-group is finite or infinite. In general, the universal cover of the orbifold $O = O[0; r, q]$, $r \neq q$ is $O = O[0; r/d, q/d]$, where $d = \gcd(r, q)$.

It is easy to see a bridge between maps with signatures and orbifolds. Indeed, a finite map M with signature σ of orbifold type $[g; m_1, m_2, \ldots, m_r]$ determines an orbifold $O = O[g; m_1, m_2, \ldots, m_r]$ with signature $[g; m_1, m_2, \ldots, m_r]$ by taking the corresponding topological map and placing a branch point of index m_i inside the corresponding vertex, edge or face x with $\sigma(x) = m_i$, for each $i = 1, \ldots, r$. Moreover, σ-compatible covers over M are in correspondence with orbifolds covering O. Having the universal covering $\Phi : \tilde{O} \to O$ we can lift the map M to a map $\tilde{M}$ on $\tilde{O}$. The respective map $\tilde{M}$ will be called a *universal cover* with respect to (M, σ). As a consequence we have the following statement.

A homomorphism $\alpha : F[g; m_1, m_2, \ldots, m_r] \to H$ is called order-preserving if it preserves the orders $m_1, m_2, \ldots, m_r$ of generators $e_1, e_2, \ldots, e_r$.

Theorem 2.2 *Let N be a finite map with signature σ induced by a regular covering $\varphi : M \to N$ with a group of covering transformations A. Let $\Phi : \tilde{N} \to N$ be the universal covering with respect to (N, σ). Then a regular covering $\psi : \tilde{N} \to M$ such that $\Phi = \psi \circ \varphi$ induces an order-preserving group epimorphism $\psi^* : \pi_1(N, \sigma) \to A$. Moreover, the monodromy action of $Mon(M)$ on a fibre $\varphi^{-1}(x)$, $x \in N$, is uniquely determined by ψ^*.*

Proof. Let $M = (D, R, L)$ and $\tilde{N} = (\tilde{D}, \tilde{R}, \tilde{L})$. Fix a dart $x_0 \in N$ and fibres $\Phi^{-1}(x_0)$, $\varphi^{-1}(x_0)$. In what follows all the considered darts will be elements of these two fibres. We show that every covering transformation $\tilde{\tau}$ of Φ projects onto some covering transformation $\tau \in A$. Choose a dart $\tilde{x} \in \Phi^{-1}(x_0)$.

Let $\tilde{\tau}$ take $\tilde{x} \mapsto \tilde{y}$. Let $x = \psi(\tilde{x})$ and $y = \psi(\tilde{y})$. By regularity of the action of A there is a unique covering transformation $\tau \in A$ taking $x \mapsto y$. For any $\tilde{z} \in \Phi^{-1}(x_0)$ there exists $w(\tilde{R}, \tilde{L}) \in Mon(\tilde{N})$ such that $w(\tilde{R}, \tilde{L})\tilde{x} = \tilde{z}$ We show that $\psi\tilde{\tau} = \tau\psi$. We have

$$\psi\tilde{\tau}(\tilde{z}) = \psi\tilde{\tau}w(\tilde{R}, \tilde{L})(\tilde{x}) = \psi w(\tilde{R}, \tilde{L})\tilde{\tau}(\tilde{x}) = \psi w(\tilde{R}, \tilde{L})(\tilde{y}) =$$

$$= w(R, L)\psi(\tilde{y}) = w(R, L)\tau(x) = \tau w(R, L)(x) = \tau(z) = \tau\psi(\tilde{z})$$

Hence the mapping $\psi^* : \tilde{\tau} \mapsto \tau$ is a group homomorphism. Since for each $y \in \varphi^{-1}(x_0)$ there is a preimage $\tilde{y} \in \Phi^{-1}(x_0)$, it is an epimorphism.

By Theorem 2.1 N lifts to a map $\tilde{N} \to N$ on the universal orbifold with the group of covering transformations acting regularly on a fibre over a dart x. Moreover, this group is isomorphic to $\pi_1(N,\sigma)$. Thus ψ^* takes $\pi_1(N,\sigma)$ onto A. Furthermore, by the regularity we may label darts of $\Phi^{-1}(x_0)$ by elements of $\pi_1(N,\sigma)$ and darts of $\varphi^{-1}(x_0)$ by elements of A. If ψ^* is determined then the covering ψ is determined on $\Phi^{-1}(x_0)$, and consequently, the action of $Mon(M)$ on $\varphi^{-1}(x_0)$ is prescribed by the projection of the action of $Mon(\tilde{N})$ along ψ.

The assumption that the derived signature σ_φ is trivial forces the covering $\Phi : \tilde{N} \to M$ to be smooth. Take an element $g \in \pi_1(N,\sigma)$ of finite order n. Then there exists an associated word $w(\tilde{R},\tilde{L})$ taking a dart labelled by 1 onto a dart labelled by g. Then $w^j(\tilde{R},\tilde{L})$ takes $1 \mapsto g^j$, and in particular, $w^n(\tilde{R},\tilde{L})(1) = 1$. Thus it gives rise to a closed walk in $\tilde{N}$. The covering ψ takes $w^j(\tilde{R},\tilde{L}) \mapsto w^j(R,L)$. The respective walk in M is closed if and only if $(\psi^*(g))^j = 1$. Since ψ is smooth, $w^j(R,L)$ is not closed for $1 \le j < n$. Then $(\psi^*(g))^j \neq 1$ for $j = 1,\dots n-1$. Hence ψ^* is order preserving.

□

Reconstruction of M. With the above notation given $N = (\bar{D};\bar{R},\bar{L})$ on an orbifold $\bar{O}$ and an epimorphism $\psi^* : \pi_1(N,\sigma) \to A$ one may ask whether there is way to reconstruct the cover $M = (D;R,L)$ explicitly. To do this one can use the idea of ordinary voltage assignments used to describe regular covers of graphs [14] and modified in [30] to describe branched coverings of maps with branch points at vertices, faces and edges. Firstly we form a truncated map $T(N)$ which vertices are darts of N and arcs are of ordered pairs the form $x\bar{R}x$, $x\bar{R}^{-1}x$ and xLx. The dart reversing involution of $T(N)$ interchanges pairs $x\bar{R}x$, $(\bar{R}x)x$; and $x\bar{L}x$, $(\bar{L}x)x$, while the rotation cyclically permutes $(x\bar{L}x, xRx, xR^{-1}x)$, for any $x \in \bar{D}$. We choose a spanning tree T of $T(N)$ and define an ordinary voltage assignment ν in A on darts of T to be 1. We fix a vertex $x_0 \in \bar{D}$ of $T(N)$. If z is a dart of $T(N)$ not belonging to T it creates (together with some paths of T joining x_0 to the initial and terminal vertex of z) a closed walk based at x_0. This closed walk corresponds to some word $w(\bar{R},\bar{L})$ which lifts to $w(\tilde{R},\tilde{L})$ taking $\tilde{x}_0$ onto $\tilde{y} = w(\tilde{R},\tilde{L})x_0$. By regularity there is a unique element $h \in \pi_1(N,\sigma)$ such that $h(\tilde{x}_0) = \tilde{y}$. We set $\nu(z) = \psi^*(h)$. In this way the voltage assignment is defined at each dart of $T(N)$. We lift $T(N)$ using the definition of the derived graph and derived map (see [14, pages 162-170]) onto a truncation $T(M)$ of a map M. Then we contract the faces of $T(M)$ which correspond to vertices of M to points thus getting M. Taking different epimorphisms $\psi^* : \pi_1(N,\sigma) \to A$ we get all σ-compatible regular covers over N with the covering transformation group isomorphic to A.

3 A formula for counting maps of given genus

In this section we shall deal with the problem of enumeration of oriented unrooted maps of given genus γ. Recall, that a map is called *rooted* if it has one distinguished dart x_0 called a root. A morphism between rooted maps takes root onto root. A map is called *labelled* if all its darts are distinguished by some labelling. Since the automorphism group of a rooted map as well as that of a labelled map is trivial, each rooted map with n darts gives rise to $(n-1)!$ labelled maps. Moreover, if (M,x) and (M,y) are two rooted maps based on the same map with a dart set D then the number of isomorphism classes for of (M,x) and (M,y) is the same. Let us remark that there is a 1-1 correspondence between isomorphism classes of rooted (and labelled) maps defined in the category of oriented maps, and isomorphism classes of rooted (and labelled) maps in the category of maps on orientable surfaces as they are defined, for example in monograph [29, page 7].

To be more precise, let us fix the set of darts D and consider different maps based on D. We want to determine the number of isomorphism classes of (unrooted) maps based on n darts and with given genus γ. This number will be denoted by $NUM_\gamma(n)$. Denote by $\mathcal{M}=\mathcal{M}(n)$ the set of all (labelled) maps on D of given genus. The symmetric group S_n, $|D|=n$, acts on $\mathcal{M}$ by conjugation as follows: $M=(D;R,L)\mapsto M^\psi=(D;R^\psi,L^\psi)$. By definition ψ is a map isomorphism taking $M\mapsto M^\psi$. Then the number of orbits $\mathcal{M}/S_n=NUM_\gamma(n)$ and the number of orbits of the stabiliser S_{n-1} of a dart $x_0\in D$ is equal to the number of rooted maps: $NRM_\gamma(n)=\mathcal{M}/S_{n-1}$.

By Burnside's lemma [12, pages 494-495]

$$NUM_\gamma(n)=\sum_{\alpha\in S_n}\frac{|Fix_{\mathcal{M}}(\alpha)|}{n!},$$

where $Fix_{\mathcal{M}}(\alpha)$ is the set of maps on D fixed by the action of α. Since the set of darts is fixed, each such a map is determined by a pair of permutations (R,L) acting on D such that $\langle R,L\rangle$ is transitive and $L^2=1$. In what follows we shall concentrate on $Fix_{\mathcal{M}}(\alpha)$.

Hall's result, see [26, 27], implies:

if $Fix_{\mathcal{M}}(\alpha)\neq\emptyset$ then α is a regular permutation, that means α can be expressed as a product of m (disjoint) cycles of the same length ℓ, say $\alpha=C_1C_2\dots C_m$, where $\ell m=n$.

Thus we may reduce our investigation to regular permutations. Since all permutations with a prescribed cyclic structure are conjugate in S_n, the size of sets $Fix_{\mathcal{M}}(\alpha)$ depends only on the decomposition $n=\ell m$. Denote by $[\ell^m]$ the conjugacy class of regular permutations of order ℓ. By some well-known formula $|[\ell^m]|=\frac{n!}{m!\ell^m}$. Hence Burnside's formula transfers to

$$NUM_\gamma(n)=\sum_{\ell|n,n=\ell m}\frac{|Fix_{\mathcal{M}}[\ell^m]|}{\ell^m m!},$$

where $Fix_{\mathcal{M}}[\ell^m]$ is the set of maps in $\mathcal{M}$ fixed by some regular permutation α with cycle structure ℓ^m.

Since $Fix_{\mathcal{M}}(\alpha)=\{(D;R,L)\in\mathcal{M}|R^\alpha=R,L^\alpha=L\}$, $\langle\alpha\rangle$ is a cyclic group of map automorphisms for each $M=(D;R,L)\in Fix_{\mathcal{M}}(\alpha)$. Take the quotient $N=M/\langle\alpha\rangle=(\bar{D},\bar{R},\bar{L})$. The covering $\varphi:M\mapsto N$ determines a signature σ on N assigning to vertices, faces and edges their branch index with respect to φ. Denote by O the respective orbifold associated with (N,σ). By Theorem 2.2 there is a covering $\psi:\tilde{N}\to M$ which induces an order preserving epimorphism $\psi^*:\pi_1(N,\sigma)\to Z_\ell$. The map N is a labelled map on orbifold O which darts are assigned by $C_1, C_2, \dots C_m$. Since for given $N = (\bar{R}, \bar{L})$ every monodromy action on the cycle C_1 is determined by an epimorphism from the orbifold fundamental group into the cyclic group $Z_\ell \cong \langle \alpha \rangle$ we have $Epi_0(\pi_1(N, \sigma), Z_\ell) = Epi_0(\pi_1(O), Z_\ell)$ possibilities to reconstruct the action of $Mon(M)$ on C_1, here O denotes the orbifold associated with (N, σ). Now in each cycle $C_i = \{v_{i,1}, v_{i,2}, \dots, v_{i,\ell}\}$, $i \neq 1$ we choose one dart. We have ℓ^{m-1} such choices. In this way the labelling of arcs of M is determined by the following rule: $v_{i,x} \in C_i$ $(i \neq 1)$ has the second coordinate $x = j$ if and only if a monodromy transformation τ taking $v_{1,1} \mapsto v_{1,j}$ maps $v_{i,1} \mapsto v_{i,x}$. Thus the permutations (R, L) are completely determined by the action of $\langle \bar{R}, \bar{L} \rangle$ and by the action of the set-wise stabiliser of C_1.

Denote by $Orb(S_\gamma/Z_\ell)$ the set of all orbifolds arising as cyclic quotients by some action of Z_ℓ from a surface of genus γ and by $NLM_O(m)$ the number of labelled quotient maps for a given orbifold type O which lift onto maps on a surface of genus γ, having $n = \ell m$ darts.

We have proved that

$$NUM_\gamma(n) = \sum_{\ell | n, n = \ell m} \frac{|Fix_{\mathcal{M}}[\ell^m]|}{\ell^m m!} =$$

$$\sum_{\ell | n, n = \ell m} \sum_{O \in Orb(S_\gamma/Z_\ell)} \frac{Epi_0(\pi_1(O), Z_\ell)\ell^{m-1} NLM_O(m)}{\ell^m m!}$$

Finally, since $NLM_O(m) = (m-1)! NRM_O(m)$ we get the following theorem.

Theorem 3.1 *With the above notation the following enumeration formula holds:*

$$NUM_\gamma(n) = \frac{1}{n} \sum_{\ell | n, n = \ell m} \sum_{O \in Orb(S_\gamma/Z_\ell)} Epi_0(\pi_1(O), Z_\ell) NRM_O(m).$$

Remark 1. The above theorem establishes a general counting principle which allows to reduce the problem of enumeration of maps of given genus γ sharing certain map property $\mathcal{P}$ onto a problem to enumerate rooted maps on associated cyclic orbifolds which lifts to maps of genus γ sharing the property $\mathcal{P}$. In this paper we are interseted in enumeration of ordinary maps of genus γ, so $\mathcal{P}$ means here: no semiedges in M. Generally, by a map property we mean a property preserved by isomorphisms of unrooted maps. Checking the proof of Theorem 3.1 one can see that its proof is independent on the choice of $\mathcal{P}$, hence one can apply this counting principle for more restricted families of maps such as one-face maps, loopless maps, non-separable maps, e.t.c. It remains, however, to solve the problem to determine the numbers $Epi_0(\pi_1(O), Z_\ell)$ and $NRM_O(\mathcal{P}, m)$, where $NRM_O(\mathcal{P}, m)$ denotes the number of rooted maps on a cyclic orbifold S_γ/Z_ℓ which lift to maps with $m\ell$ darts sharing property $\mathcal{P}$. In what follows we shall deal with both problems.

Remark 2. As noted by V. Liskovets (personal communication) using results of the following sections one can prove that the above formula derived in Theorem 3.1 agrees with the general reductive formula derived in [26, Theorem 2.8] (see [27] as well).

4 Number of epimorphisms from an F-group onto a cyclic group

As one can see in Theorem 3.1 to derive an explicit formula for the number of unrooted maps with given genus and given number of edges one needs to deal with the numbers $Epi_0(\Gamma, \mathsf{Z}_\ell)$ of order preserving epimorphisms from an F-group Γ onto a cyclic group Z_ℓ. The aim of this section is to calculate these numbers.

Denote by $\mathrm{Hom}_0(\Gamma,\, \mathsf{Z}_\ell)$ the set of order preserving homomorphisms from the group Γ into Z_ℓ. Let

$$\Gamma = F[g; m_1, \ldots, m_r] =< \mathsf{a}_1, \mathsf{b}_1, \ldots, \mathsf{a}_g, \mathsf{b}_g, \mathsf{x}_1, \ldots, \mathsf{x}_r\colon \prod_{i=1}^{g}[\mathsf{a}_i,\, \mathsf{b}_i]\prod_{j=1}^{r}\mathsf{x}_j = 1,\ \mathsf{x}_1^{m_1} = 1, \ldots, \mathsf{x}_r^{m_r} = 1>$$

be an $F-$group of signature $(g; m_1, \ldots, m_r)$.

Following the arguments used by G. Jones in [18] we obtain

$$Epi_0(\Gamma,\, \mathsf{Z}_\ell) = \sum_{d|\ell}\mu\left(\frac{\ell}{d}\right)|\mathrm{Hom}_0(\Gamma,\, \mathsf{Z}_d)|,$$

where $\mu\left(\frac{\ell}{d}\right)$ denotes the Möbius function. Set $m = lcm(m_1, \ldots, m_r)$ to be the least common multiple of $m_1, m_2, \ldots, m_r$. We note that if $r = 0$ then the group $F[g, \emptyset] = F[g, 1]$. So, we set $m = 1$ for $r = 0$. Since $\mathrm{Hom}_0(\Gamma,\, \mathsf{Z}_d)$ is empty if at least one of $m_1, \ldots, m_r$ is not a divisor of d we have also

$$Epi_0(\Gamma,\, \mathsf{Z}_\ell) = \sum_{m|d|\ell}\mu\left(\frac{\ell}{d}\right)|\mathrm{Hom}_0(\Gamma,\, \mathsf{Z}_d)|. \tag{4.1}$$

We suppose that the numbers $m_1, \ldots, m_r$ are divisors of d. Identify the group Z_d with additive group of residues $\{1, \ldots, d\} \bmod d$. Since, the group Z_d is abelian, there is one-to-one correspondence between order preserving epimorphisms from $\mathrm{Hom}_0(\Gamma,\, \mathsf{Z}_d)$ and the elements of the set

$$\{(a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_r) \in \mathsf{Z}_d^{2g+r} : x_1 + \ldots + x_r = 0 \bmod d,\ (x_1,\, d) = d_1, \ldots, (x_r,\, d) = d_r\},$$

where $(x,\, d)$ is the greatest common divisor of x and d (well defined in the group Z_d). Set $d_1 = \frac{d}{m_1}, \ldots, d_r = \frac{d}{m_r}$.

Hence

$$|\mathrm{Hom}_0(\Gamma,\, \mathsf{Z}_d)| = d^{2g} \cdot E_d(m_1, \ldots, m_r), \tag{4.2}$$

where $E_d(m_1, \ldots, m_r)$ is the number of solutions of the equation $x_1 + \ldots + x_r = 0 \bmod d$, $(x_1,\, d) = d_1, \ldots, (x_r,\, d) = d_r$.

Denote by $\mu(n)$, $\phi(n)$ and $\Phi(x, n)$ the Möbius, Euler and von Sterneck functions, respectively. The relationship between them is given by the formula

$$\Phi(x, n) = \frac{\phi(n)}{\phi(\frac{n}{(x,n)})}\,\mu\left(\frac{n}{(x, n)}\right),$$

where (x,n) is the greatest common divisor of x and n. It was shown by O. Hölder that $\Phi(x,n)$ coincides with the Ramanujan sum $\sum_{\substack{1\le k\le n\\(k,\,n)=1}} \exp(\frac{2\,ikx}{n})$. For the proof, see Apolstol [1, p.164] and [31].

Lemma 4.1 *Let* $m_1,\ldots,m_r$ *be divisors of* d *and* $d_1=\dfrac{d}{m_1},\ldots,d_r=\dfrac{d}{m_r}$. *Then the number* $E=E_d(m_1,\ldots,m_r)$ *of solutions* $(x_1,x_2,\ldots,x_r)$, $x_j\in Z_d$ *for* $j=1,2,\ldots,r$, *of the system of the equations*

$$x_1+\ldots+x_r=0 \bmod d,\ (x_1,\ d)=d_1,\ldots,(x_r,\ d)=d_r$$

is given by the formula

$$E=\frac{1}{d}\sum_{k=1}^{d}\Phi(k,\ m_1)\cdot\Phi(k,\ m_2)\ldots\Phi(k,\ m_r).$$

Proof. Consider the polynomial

$$P(z)=\sum_{\substack{1\le x_1,\ldots,x_r\le d\\(x_1,\,d)=d_1,\ldots,(x_r,\,d)=d_r}} z^{x_1+\ldots+x_r}.$$

Then the number of solutions E coincide with the sum of the coefficients of $P(z)$ whose exponents are divisible by d. Hence

$$E=\frac{1}{d}\sum_{k=1}^{d}P(\varepsilon^k),\quad\text{where }\ \varepsilon=e^{\dfrac{2\pi\,i}{d}}.$$

We have

$$P(\varepsilon^k)=\sum_{\substack{1\le x_1\le d\\(x_1,\,d)=d_1}}\ \sum_{\substack{1\le x_2\le d\\(x_2,\,d)=d_2}}\cdots\sum_{\substack{1\le x_r\le d\\(x_r,\,d)=d_r}}(\varepsilon^k)^{x_1+\ldots+x_r}$$

$$=\sum_{\substack{1\le x_1\le d\\(x_1,\,d)=d_1}}\varepsilon^{k\,x_1}\sum_{\substack{1\le x_2\le d\\(x_2,\,d)=d_2}}\varepsilon^{k\,x_2}\ldots\sum_{\substack{1\le x_r\le d\\(x_r,\,d)=d_r}}\varepsilon^{k\,x_r}$$

$$=\sum_{\substack{1\le x_1\le d\\(x_1,\,d)=d_1}}e^{\dfrac{2\pi\,ik\,x_1}{d}}\sum_{\substack{1\le x_2\le d\\(x_2,\,d)=d_2}}e^{\dfrac{2\pi\,ik\,x_2}{d}}\ldots\sum_{\substack{1\le x_r\le d\\(x_r,\,d)=d_r}}e^{\dfrac{2\pi\,ik\,x_r}{d}}$$

$$=\sum_{\substack{1\le y_1\le m_1\\(y_1,\,m_1)=1}}e^{\dfrac{2\pi\,ik\,y_1}{m_1}}\sum_{\substack{1\le y_2\le m_2\\(y_2,\,m_2)=1}}e^{\dfrac{2\pi\,ik\,y_2}{m_2}}\ldots\sum_{\substack{1\le y_r\le m_r\\(y_r,\,m_r)=1}}e^{\dfrac{2\pi\,ik\,y_r}{m_r}}$$

$$=\Phi(k,\ m_1)\cdot\Phi(k,\ m_2)\ldots\Phi(k,\ m_r).$$

Hence

$$E = \frac{1}{d} \sum_{k=1}^{d} \Phi(k,\, m_1) \cdot \Phi(k,\, m_2) \ldots \Phi(k,\, m_r).$$

□

As was observed by V. Liskovets (personal communication) $E_d(m_1, m_2, \ldots, m_r) = E_m(m_1, m_2, \ldots, m_r)$ for any d, $m|d$. Thus the function $E_d(m_1, m_2, \ldots, m_r)$ does not depend on d and we set

$$E(m_1, m_2, \ldots, m_r) = \frac{1}{m} \sum_{k=1}^{m} \Phi(k,\, m_1) \cdot \Phi(k,\, m_2) \ldots \Phi(k,\, m_r), \tag{4.3}$$

where $m = lcm(m_1, m_2, \ldots, m_r)$. Recall that the Jordan multiplicative function $\phi_k(n)$ of order k can be defined as (for more information see [13, p.199],[20, 33])

$$\phi_k(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d^k.$$

From the above arguments we obtain the following proposition.

Proposition 4.2 *Let* $\Gamma = F[g; m_1, \ldots, m_r]$ *be an* $F-$*group of signature* $(g; m_1, \ldots, m_r)$. *Denote by* $m = lcm\,(m_1, \ldots, m_r)$ *the least common multiple of* $m_1, \ldots, m_r$ *and let* $m|\ell$. *Then the number of order-preserving epimorphisms of the group* Γ *onto a cyclic group* Z_ℓ *is given by the formula*

$$Epi_0(\Gamma, Z_\ell) = m^{2g} \phi_{2g}(\ell/m) E(m_1, m_2, \ldots, m_r),$$

where

$$E(m_1, m_2, \ldots, m_r) = \frac{1}{m} \sum_{k=1}^{m} \Phi(k,\, m_1) \cdot \Phi(k,\, m_2) \ldots \Phi(k,\, m_r),$$

$\phi_{2g}(\ell)$ *is the Jordan multiplicative function of order* $2g$, *and* $\Phi(k,\, m_j)$ *is the von Sterneck function.*

In particular, if $\Gamma = F[g; \emptyset] = F[g; 1]$ *is a surface group of genus* g *we have*

$$Epi_0(\Gamma, Z_\ell) = \phi_{2g}(\ell).$$

Proof. By (4.1) and (4.2)

$$Epi_0(\Gamma,\, Z_\ell) = \sum_{m|d|\ell} \mu\left(\frac{\ell}{d}\right) |\mathrm{Hom}_0(\Gamma,\, Z_d)| = \sum_{m|d|\ell} \mu\left(\frac{\ell}{d}\right) d^{2\,g} \cdot E_d(m_1, \ldots, m_r).$$

By Lemma 4.1 and the following note the function

$$E_d(m_1, \ldots, m_r) = \frac{1}{m} \sum_{k=1}^{m} \Phi(k,\, m_1) \cdot \Phi(k,\, m_2) \ldots \Phi(k,\, m_r).$$

Hence

$$Epi_0(\Gamma,\, \mathsf{Z}_\ell) = \sum_{m|d|\ell} \mu\left(\frac{\ell}{d}\right) d^{2\,g} \cdot E(m_1, \ldots, m_r).$$

Inserting $d = d_1 m$ and $\ell = \ell_1 m$ we get

$$Epi_0(\Gamma,\, \mathsf{Z}_\ell) = m^{2g} \sum_{d_1|\ell_1} \mu\left(\frac{\ell_1 m}{d_1 m}\right) d_1^{2\,g} \cdot E(m_1, \ldots, m_r) = m^{2g}\phi_{2g}(\ell/m) E(m_1, m_2, \ldots, m_r).$$

□

Let us note that the condition $m|\ell$ in the above proposition gives no principal restriction, since $Epi_0(\Gamma, \mathsf{Z}_\ell) = 0$ by the definition provided m does not divide ℓ. An orbifold $O = O[g; m_1, \ldots, m_r]$ will be called γ-*admissible* if it can be represented in the form $O = S_\gamma/\mathsf{Z}_\ell$, where S_γ is an orientable surface of genus γ surface and Z_ℓ is a cyclic group of automorphisms of S_γ. By the Köbe's theorem there is an orbifold $O = S_\gamma/\mathsf{Z}_\ell$ with signature $[g; m_1, m_2, \ldots, m_r]$ if and only if there exists ℓ such that the number $Epi_0(\pi_1(O), Z_\ell) \neq 0$ and the numbers γ, g, $m_1, \ldots, m_r$ and ℓ are related by the Riemann-Hurwitz equation $2 - 2\gamma = \ell(2 - 2g - \sum_{i=1}^{r}(1 - 1/m_i))$. Although the condition $Epi_0(\pi_1(O), Z_\ell) \neq 0$ can be checked using Proposition 4.2 for practical use it is more convenient to employ the following result by Harvey [17], see [9, 6] as well. The Wiman theorem [10, page 131] makes us sure that $1 \leq \ell \leq 4\gamma + 2$ for $\gamma > 1$.

Theorem 4.3 [17] *Let $O = O[g; m_1, \ldots, m_r]$ be an orbifold. Then O is γ-admissible if and only if there exists an integer ℓ such that following conditions are satisfied*

1. *$m = lcm(m_1, m_2, \ldots, m_r)$ divides ℓ and $m = \ell$ if $g = 0$,*
2. *$2 - 2\gamma = \ell(2 - 2g - \sum_{i=1}^{r}(1 - 1/m_i))$ (Riemann-Hurwitz equation),*
3. *$lcm(m_1, \ldots, m_{i-1}, m_i, m_{i+1}, \ldots, m_r) = lcm(m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_r)$ for each $i = 1, 2, \ldots, r$,*
4. *if $m = lcm(m_1, m_2, \ldots, m_r)$ is even then the number of m_j divisible by the maximal power of 2 dividing m is even,*
5. *if $\gamma \geq 2$ then $r \neq 1$ and $r \geq 3$ for $g = 0$, if $\gamma = 1$ then $r \in \{0, 3, 4\}$, if $\gamma = 0$ then $r = 2$.*

If $\gamma > 1$ the integer ℓ is bounded by $1 \leq \ell \leq 4\gamma + 2$.

Using Theorem 4.3, see [9, 6, 21] as well, we derive the following lists of γ-admissible orbifolds, for $\gamma = 0, 1, 2, 3$. Employing Proposition 4.2 the numbers $Epi_O(\pi_1(O), Z_\ell)$ are calculated for each orbifold in the list .

Corollary 4.4 *0-admissible orbifolds are $O = O[0; \ell^2]$, with $Epi_0(\pi_1(O), Z_\ell) = \phi(\ell)$ for any positive integer ℓ.*

Corollary 4.5 *Let $O = O[g; m_1, m_2, \ldots, m_r] = S_1/Z_\ell$ be a 1-admissible orbifold. Then one of the following cases happens:*

$O = O[1; \emptyset]$, *with* $Epi_0(\pi_1(O), Z_\ell) = \sum_{k|\ell} \mu(\ell/k)k^2 = \phi_2(\ell)$ *for any* ℓ,

$\ell = 2$ *and* $O = O[0; 2^4]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 1$,

$\ell = 3$ *and* $O = O[0; 3^3]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 2$,

$\ell = 4$ *and* $O = O[0; 4^2, 2]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 2$,

$\ell = 6$ *and* $O = O[0; 6, 3, 2]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 2$.

Corollary 4.6 *Let $O = O[g; m_1, m_2, \ldots, m_r] = S_2/Z_\ell$ be a 2-admissible orbifold. Then one of the following statements holds:*

$\ell = 1$ *and* $O = O[2; \emptyset]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 1$,

$\ell = 2$ *and* $O = O[1; 2^2]$ *or* $O[0; 2^6]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 4, 1$, *respectively,*

$\ell = 3$ *and* $O = O[0; 3^4]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 6$,

$\ell = 4$ *and* $O = O[0; 2^2, 4^2]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 2$,

$\ell = 5$ *and* $O = O[0; 5^3]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 12$,

$\ell = 6$ *and* $O = O[0; 2^2, 3^2]$ *or* $O = O[0; 3, 6^2]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 2, 2$, *resp.,*

$\ell = 8$ *and* $O = O[0; 2, 8^2]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 4$,

$\ell = 10$ *and* $O = O[0; 2, 5, 10]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 4$.

Corollary 4.7 *Let $O = O[g; m_1, m_2, \ldots, m_r] = S_3/Z_\ell$ be a 3-admissible orbifold. Then one of the following statements holds:*

$\ell = 1$ *and* $O = O[3; \emptyset]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 1$,

$\ell = 2$ *and* $O = O[2; \emptyset], O[1; 2^4]$ *or* $O[0; 2^8]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 15, 4, 1$, *resp.,*

$\ell = 3$ *and* $O = O[1; 3^2]$ *or* $O[0; 3^5]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 18, 10, resp.$,

$\ell = 4$ *and* $O = O[1; 2^2], O[0; 2^3, 4^2]$ *or* $O[0; 4^4]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 12, 2, 8$, *resp.,*

$\ell = 6$ *and* $O = O[0; 2, 3^2, 6]$ *or* $O[0; 2^2, 6^2]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 2, 2$, *resp.,*

$\ell = 7$ *and* $O = O[0; 7^3]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 30$,

$\ell = 8$ *and* $O = O[0; 4, 8^2]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 8$,

$\ell = 9$ *and* $O = O[0; 3, 9^2]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 12$,

$\ell = 12$ *and* $O = O[0; 2, 12^2]$ *or* $O[0; 3, 4, 12]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 4, 4$, *resp.,*

$\ell = 14$ *and* $O = O[0; 2, 7, 14]$, *with* $Epi_0(\pi_1(O), Z_\ell) = 6$.

5 Numbers of rooted maps on cyclic orbifolds

Notation. Let M be a rooted map on an orbifold O such that $M = \tilde{M}/Z_\ell = (D; R, L)$ is a quotient of an ordinary finite map $\tilde{M}$ on a surface S. Thus $O = S/Z_\ell$. It follows that each branch index is a divisor of ℓ and we can write $O = O[g; 2^{q_2}, \dots, \ell^{q_\ell}]$, where $q_i \geq 0$ denotes the number of branch points of index i, for $i = 2, \dots, \ell$. In order to shorten the length of expressions given orbifold $O = O[g; 2^{q_2}, \dots, \ell^{q_\ell}]$ we denote the number of rooted maps with m darts sitting on O such that each semiedge is endowed with a branch point of index two by $\nu_O(m) = \nu_{[g;\, 2^{q_2}, \dots, \ell^{q_\ell}]}(m) = NRM_O(m)$. Also we use the convention $\nu_g(m) = \nu_{[g;\, \emptyset]}(m)$ denoting the number of rooted maps with m darts on a surface of genus g. Let us remark that in this case m is necessarily even and $\nu_g(m) = \mathcal{N}_g(m/2)$, where $\mathcal{N}_g(m/2)$ denotes the (Tutte's) number of rooted maps with $m/2$ edges on a surface of genus g.

Let us denote by v, f, m and s the number of vertices, faces, darts and semiedges of a map M on an orbifold O, respectively. Since we are primarily interested in enumeration of maps without semiedges we assume that a free-end of each semiedge is incident with a branch point of index two. Hence $0 \leq s \leq q_2$. Moreover, by Euler-Poincare formula $v - \frac{m-s}{2} + f = 2 - 2g$. By $Cor(M)$ we denote an ordinary rooted map on S_g which arises from M by using the following rules:

(1) delete all semiedges of M,

(2) if the root of M occupies a semiedge x in M, we choose a root of $Cor(M)$ to be the first dart following x in the local rotation of M sharing an edge of degree 2,

(3) if $Cor(M)$ is a map without darts we consider it as a unique rooted map.

Given integers $x_1, x_2, \dots, x_q$ and $y \geq x_1 + x_2 + \cdots + x_q$ we denote by

$$\binom{y}{x_1, x_2, \dots, x_q} = \frac{y!}{x_1! x_2! \dots x_q! (y - \sum_{j=1}^{q} x_j)!},$$

the multinomial coefficient. Note that the meaning of the symbol consistently extends also to the case of non-negative y satisfying $y < x_1 + x_2 + \cdots + x_q$. In this case the multinomial coefficient takes value 0.

Reconstruction of M from $Cor(M)$. Let us start from the map $Cor(M)$ which is an ordinary rooted map with $\frac{m-s}{2}$ edges. How many different rooted maps M on the orbifold O comes from a fixed ordinary rooted map $Cor(M)$? We split the discussion into three subcases.

Case 1. Number of distributions of branch points which are not attached to semiedges.

We have to find the number of divisions of the set $V(M) \cup F(M)$ of cardinality $v + f = e + 2 - 2g$ into disjoint subsets of cardinalities $q_1, q_2 - s, \dots, q_\ell$. This is exactly the number

$$\binom{\frac{m-s}{2} + 2 - 2g}{q_2 - s, q_3, \dots, q_\ell}$$

(see for instance [12, page 62])

Case 2. Number of distributions of semiedges if the root of M is not located at a semiedge.

The family of semiedges of $M = (D, R, L)$ splits into families S_i, $i = 1, 2, \dots,$ defined by the following rule: a semiedge determined by a unique dart x belongs to S_i if and only if x belongs to a sequence of darts $x_0, x_1, \dots, x_i, x_{i+1}$ satisfying

(i) $x_j = R(x_{j-1})$ for $j = 1, \dots, i+1$,

(ii) for the initial and terminal darts we have $L(x_0) \neq x_0$ and $L(x_{i+1}) \neq x_{i+1}$, or $R = (x_1, x_2, \dots, x_i)$,

(iii) for the internal darts $x_j = L(x_j)$, $j = 1, 2, \dots, i$.

Set $c_i = \frac{|S_i|}{i}$. Clearly, c_i is the number of sequences of darts satisfying the above conditions (i), (ii) and (iii).

We have $s = \sum ic_i \leq m - s$, because a position of such a sequence in M is uniquely determined by choosing its initial dart x_0, which is a dart of $Cor(M)$, as well. Note that $c_j = 0$ if $j > s$. Given partition $s = c_1 + 2c_2 + \dots sc_s$ we have

$$\binom{m-s}{c_1, c_2, \dots, c_s}$$

choices to distribute the respective sequences of semiedges in $Cor(M)$. Denote by $Par(s)$ the set of partitions of s. In what follows we write a partition of s in the exponential form as $1^{c_1}2^{c_2}\dots s^{c_s}$.

It follows that the total number of distributions of semiedges is

$$\sum_{Par(s)} \binom{m-s}{c_1, c_2, \dots, c_s},$$

where the sum runs through all non-negative solutions $(c_1, c_2, \dots, c_s)$ of the equation $x_1 + 2x_2 + \dots + sx_s = s$.

In fact it makes sense to consider only partitions satisfying $c_1 + c_2 + \dots + c_s \leq m - s$ but in view of the remark after the definition of the multinomial coefficient the expression is correct even if we do not write this condition in the subscript of the sum.

Case 3. The root of M lies at a semiedge.

By the definition of $Cor(M)$ the position of the root of M is determined by the position of the root of $Cor(M)$ up to its position in the internal part of a sequence $x_0, x_1, \dots, x_i, x_{i+1}$ satisfying (i), (ii) and (iii). We use one semiedge z_0 for the root. The remaining $s - 1$ semiedges have to be distributed in $m - (s-1) = m - s + 1$ places which are given by darts of $Cor(M)$ and by z_0. Similar arguments as in Case 2 apply. We get

$$\sum_{Par(s-1)} \binom{m-s+1}{c_1, c_2, \dots, c_{s-1}}$$

distributions in this case.

Depending on the parity of m the number s of semiedges takes either even, or odd values only, since we are assuming that $\frac{m-s}{2}$ is the number of edges of $Cor(M)$ which is an integer. Denote by $p(m,s)$ the parity function taking value 0 if the numbers have different parity, and 1 otherwise.

Given integers n and s denote

$$\beta(n,s)=\sum_{Par(s)}\binom{n}{c_1,c_2,\ldots,c_s}.$$

We set $\beta(n,-1)=0$ and $\beta(n,0)=1$ as well.

Summarizing all the above calculations we finally get:

$$\nu_O(m)=\sum_{s=0}^{q_2}p(m,s)\big(\beta(m-s,s)+\beta(m-s+1,s-1)\big)\binom{\frac{m-s}{2}+2-2g}{q_2-s,q_3,\ldots,q_\ell}\mathcal{N}_g\left(\frac{m-s}{2}\right)$$

The following lemma significantly simplifies the computation of $\beta(n,s)$.

Lemma 5.1

$$\beta(n,\, s)=\binom{n+s-1}{s}.$$

Proof. By the multinomial formula [12, page 123] we have

$$(1+x+x^2+x^3+\ldots)^n=\sum_{s=0}^{\infty}\sum_{c_1+2c_2\ldots+sc_s=s}\binom{n}{c_1,\ldots,c_s}x^{c_1}x^{2c_2}\ldots x^{sc_s}=\sum_{s=0}^{\infty}\beta(n,\, s)x^s.$$

On the other hand,

$$(1+x+x^2+x^3+\ldots)^n=\frac{1}{(1-x)^n}=\binom{n-1}{0}+\binom{n}{1}x+\ldots+\binom{n+s-1}{s}x^s+\ldots.$$

Comparing the coefficients at x^s we get the result. □

Since $\beta(m-s,\, s)+\beta(m-s+1,\, s-1)=\binom{m-1}{s}+\binom{m-1}{s-1}=\binom{m}{s}$ we have proved the following statement.

Proposition 5.2 *Let $O=O[g;2^{q_2},\ldots,\ell^{q_\ell}]$ be an orbifold, $q_i\geq 0$ for $i=2,\ldots,\ell$. Then the number of rooted maps $\nu_O(m)$ with m darts on the orbifold O is*

$$\nu_O(m)=\sum_{s=0}^{q_2}\binom{m}{s}\binom{\frac{m-s}{2}+2-2g}{q_2-s,q_3,\ldots,q_\ell}\mathcal{N}_g((m-s)/2), \qquad (5.1)$$

with a convention that $\mathcal{N}_g(n)=0$ if n is not an integer.

6 Counting unrooted maps on the sphere

In this section we apply the above results to calculate the number of unrooted maps with given number of edges on the sphere. These numbers were derived by Liskovets in [23] and [25].

First we deal with the numbers $\nu_O(m)$ where O is one of the spherical orbifolds $O = O[0;\ell^2]$.

If $\ell > 2$ then the number s of semiedges of a rooted map M which are lifted to a spherical map with $m\ell$ darts is equal to 0. By Proposition 5.2 we have

$$\nu_{[0;\ell^2]}(m) = \binom{\frac{m}{2}+2}{2}\mathcal{N}_0(m/2), \quad \ell > 2 \text{ and } m \text{ even}$$

and

$$\nu_{[0;\ell^2]}(m) = 0, \quad \ell > 2 \text{ and } m \text{ odd}.$$

Let $\ell = 2$ then $O = O[0;2^2]$ and the number of semiedges is $s = 0$ or $s = 2$ for m even and it is $s = 1$ for m odd.

By Proposition 5.2

$$\nu_{[0;2^2]}(m) = \binom{\frac{m}{2}+2}{2}\mathcal{N}_0(m/2) + \binom{m}{2}\mathcal{N}_0(m/2-1), \quad \text{if } m \text{ even}$$

and

$$\nu_{[0;2^2]}(m) = m\left(\frac{m-1}{2}+2\right)\mathcal{N}_0((m-1)/2), \quad \text{if } m \text{ odd}.$$

Now we are ready to apply our formula to express the number of ordinary unrooted maps on the sphere with e edges in terms of the Tutte numbers $\mathcal{N}_0(e)$ denoting the number of rooted ordinary maps with e edges on the sphere.

We distinguish two cases.

Case 1. The number of edges e is even. Note that $n = 2e$ and $n \equiv 0 \bmod 4$.
We have

$$\Theta_0(e) = NUM(n) = \frac{1}{n}\sum_{\substack{\ell|n\\ n=\ell m}} \sum_{O\in Orb(S_0/Z_\ell)} Epi_0(\pi_1(O), Z_\ell)\nu_O(m)$$

Writing the terms for $\ell = 1$ and $\ell = 2$ separately and using the fact that given $\ell > 1$ there is only one 0-admissible orbifold, namely $O[0;\ell^2]$, with $Epi_0(\pi_1(O), Z_\ell) = \phi(\ell)$ (see Theorem 4.4), we get

$$\Theta_0(e) = \frac{1}{n}\left(\nu_0(n) + \binom{\frac{n}{4}+2}{2}\nu_0(n/2) + \binom{\frac{n}{2}}{2}\nu_0(n/2-2) + \sum_{\substack{\ell|n,\ell>2\\ n=\ell m, m \text{ even}}} \phi(\ell)\binom{\frac{m}{2}+2}{2}\nu_0(m)\right).$$

Using $e = 2n$, $\nu_0(n) = \mathcal{N}_0\left(\frac{n}{2}\right) = \mathcal{N}_0(e)$ and $\nu_0(m) = \mathcal{N}_0\left(\frac{m}{2}\right)$ we rewrite it as follows:

$$\Theta_0(e) = \frac{1}{2e}\left(\mathcal{N}_0(e) + \binom{e}{2}\mathcal{N}_0(e/2-1) + \sum_{\substack{\ell|e\\ \ell\geq 2}}\phi(\ell)\binom{\frac{e}{\ell}+2}{2}\mathcal{N}_0(e/\ell)\right).$$

Setting $d = \frac{e}{\ell}$ we have

$$\Theta_0(e) = \frac{1}{2e}\left(\mathcal{N}_0(e) + \binom{e}{2}\mathcal{N}_0(e/2-1) + \sum_{\substack{d|e\\ d<e}}\phi(e/d)\binom{d+2}{2}\mathcal{N}_0(d)\right), \tag{6.1}$$

where e is an even number.

Assume now that $n \equiv 2 \bmod 4$. Then extracting the first two terms from the sum and inserting $n = 2e$ we get

$$\Theta_0(e) = \frac{1}{2e}\left(\mathcal{N}_0(e) + e\left(\frac{e-1}{2}+2\right)\mathcal{N}_0((e-1)/2) + \sum_{\substack{\ell|n=2e,\ell>2\\ n=\ell m, m \text{ even}}}\phi(\ell)\binom{\frac{m}{2}+2}{2}\nu_0(m)\right).$$

All the conditions in the sum are satisfied if and only if $m = 2d$ for some $d|e$. Hence we have

$$\Theta_0(e) = \frac{1}{2e}\left(\mathcal{N}_0(e) + e\left(\frac{e-1}{2}+2\right)\mathcal{N}_0((e-1)/2) + \sum_{\substack{d|e\\ d<e}}\phi(e/d)\binom{d+2}{2}\mathcal{N}_0(d)\right) \tag{6.2}$$

for e odd. Hence we have proved the following result of Liskovets [23]. Recall that $\mathcal{N}_0(e)$ denotes the number of rooted planar maps with e edges and is given by $\mathcal{N}_0(e) = \dfrac{2(2e)!3^e}{e!(e+2)!}$ (Tutte [34]).

Theorem 6.1 [23] *The number of spherical unrooted maps with e edges is given by (6.1) if e is even, and (6.2) if e is odd.*

7 Counting unrooted maps on surfaces of genus 1, 2 and 3

The aim of this section is to derive a more explicit formula for counting unrooted maps on torus. The list of 1-admissible orbifolds and the respective numbers $Epi_o(\pi_1(O), Z_\ell)$ were derived in Theorem 4.5. Rooted toroidal maps were enumerated in [1]. It was proved that

$$\mathcal{N}_1(e) = \sum_{k=0}^{e-2} 2^{e-3-k}(3^{e-1} - 3^e)\binom{e+k}{k}.$$

Following Theorem 1.1 and taking into account Corollary 4.5 we have

$$\begin{aligned}\Theta_1(e) = NUM_1(n) = \frac{1}{n}\big(&\nu_{[0;2^4]}(n/2) + 2\nu_{[0;3^3]}(n/3) + 2\nu_{[0;2,4^2]}(n/4) + \\ &2\nu_{[0;2,3,6]}(n/6) + \sum_{\substack{\ell \mid n \\ n=\ell m}} \sum_{k \mid \ell} \mu(\ell/k) k^2 \nu_1(n/\ell)\big). \qquad (7.1)\end{aligned}$$

Since $\nu_1(n/\ell) = \mathcal{N}_1(e/\ell)$ for $e = n/2$, it remains to calculate the numbers of rooted maps on orbifolds $O[0;2^4]$, $O[0;3^3]$, $0[2;4^2]$ and $O[0;2,3,6]$.

By Proposition 5.2 we have

$$\nu_{[0;3^3]}(m) = \binom{\frac{m}{2}+2}{3}\mathcal{N}_0(m/2), \qquad (7.2)$$

for m even, and it is 0 for m odd.

For the orbifold $O = O[0;2,4^2]$ we have

$$\nu_{[0;2,4^2]}(m) = \binom{\frac{m}{2}+2}{1,2}\mathcal{N}_0(m/2), \quad m \text{ even}, \qquad (7.3)$$

and

$$\nu_{[0;2,4^2]}(m) = m\binom{\frac{m-1}{2}+2}{2}\mathcal{N}_0((m-1)/2), \quad m \text{ odd}. \qquad (7.4)$$

For the orbifold $O = O[0;2,3,6]$ we get

$$\nu_{[0;2,3,6]}(m) = \binom{\frac{m}{2}+2}{1,1,1}\mathcal{N}_0(m/2), \quad m \text{ even}, \qquad (7.5)$$

and

$$\nu_{[0;2,3,6]}(m) = m\binom{\frac{m-1}{2}+2}{1,1}\mathcal{N}_0((m-1)/2), \quad m \text{ odd}, \qquad (7.6)$$

And finally, by Proposition 5.2 we get

$$\nu_{[0;2^4]}(m) = \binom{\frac{m}{2}+2}{4}\mathcal{N}_0(m/2) + \binom{m}{2}\binom{\frac{m-2}{2}+2}{2}\mathcal{N}_0((m-2)/2) + \binom{m}{4}\mathcal{N}_0((m-4)/2) \qquad (7.7)$$

for m even.

For m odd Proposition 5.2 implies

$$\nu_{[0;2^4]}(m) = m\binom{\frac{m-1}{2}+2}{3}\mathcal{N}_0((m-1)/2) + \binom{m}{3}\left(\frac{m-3}{2}+2\right)\mathcal{N}_0((m-3)/2) \qquad (7.8)$$

for odd m.

Now we are ready to formulate the statement establishing the number of unrooted toroidal maps with given number of edges.

Theorem 7.1 *The number of oriented unrooted toroidal maps with e edges is*

$$\frac{1}{2e}\big(\alpha(e) + \sum_{\ell|e} \phi_2(\ell)\mathcal{N}_1(e/\ell)\big),$$

where

$$\alpha(e) = \nu_{[0;2^4]}(e) + 2\nu_{[0;3^3]}(2e/3) + 2\nu_{[0;2,4^2]}(e/2) + 2\nu_{[0;2,3,6]}(e/3), \quad \textit{if } e \equiv 0 \textit{ mod } 12,$$

$$\alpha(e) = \nu_{[0;2^4]}(e), \quad \textit{if } e \equiv \pm1, \pm5 \textit{ mod } 12,$$

$$\alpha(e) = \nu_{[0;2^4]}(e) + 2\nu_{[0;2,4^2]}(e/2), \quad \textit{if } e \equiv \pm2 \textit{ mod } 12,$$

$$\alpha(e) = \nu_{[0;2^4]}(e) + 2\nu_{[0;3^3]}(2e/3) + 2\nu_{[0;2,3,6]}(e/3), \quad \textit{if } e \equiv \pm3 \textit{ mod } 12,$$

$$\alpha(e) = \nu_{[0;2^4]}(e) + 2\nu_{[0;2,4^2]}(e/2), \quad \textit{if } e \equiv \pm4 \textit{ mod } 12,$$

$$\alpha(e) = \nu_{[0;2^4]}(e) + 2\nu_{[0;3^3]}(2e/3) + 2\nu_{[0;2,4^2]}(e/2) + 2\nu_{[0;2,3,6]}(e/3), \quad \textit{if } e \equiv 6 \textit{ mod } 12.$$

Let us remark that $\phi_2(\ell)$ denotes the Jordan function of order 2 and the other functions used in the statement are defined by (7.2)-(7.8).

The following list containing the numbers of rooted and oriented unrooted maps of genus 1 up to 30 edges follows.

No. edges, No. rooted maps on torus, No. unrooted maps on torus:

02, 1, 1
03, 20, 6
04, 307, 46
05, 4280, 452
06, 56914, 4852
07, 736568, 52972
08, 9370183, 587047
09, 117822512, 6550808
10, 1469283166, 73483256
11, 18210135416, 827801468
12, 224636864830, 9360123740
13, 2760899996816, 106189359544
14, 33833099832484, 1208328304864
15, 413610917006000, 13787042250528
16, 5046403030066927, 157700137398689
17, 61468359153954656, 1807893066408464
18, 747672504476150374, 20768681225892328
19, 9083423595292949240, 239037464947999900
20, 110239596847544663002, 2755989928117365244
21, 1336700736225591436496, 31826208029615881656
22, 16195256987701502444284, 368074022535205870382
23, 196082659434035163992720, 4262666509741017440552
24, 2372588693872584957422422, 49428931123444048643388
25, 28692390789135657427179680, 573847815786545413529104
26, 346814241363774726576771244, 6669504641624799675973078
27, 4190197092308320889669166128, 77596242450201993985513136
28, 50605520500653135912761192668 903670008940406050891508432
29, 610946861846663952302648987552 10533566583563768540393559344
30, 7373356726039234245335035186504 122889278767322703855171530872

Let us remark that the initial values confirm the available data for $e \leq 6$ obtained by Walsh [39] (the sequence M4253 in [32]). The statements establishing $\Theta_\gamma(e)$ for genus two and genus three surfaces follow.

Theorem 7.2 *The number of oriented unrooted maps on genus two surface with e edges is given by the formula*

$$\frac{1}{2e}\big(\mathcal{N}_2(e) + 4\nu_{[1;2^2]}(e) + \nu_{[0;2^6]}(e) + 6\nu_{[0;3^4]}(2e/3) + 2\nu_{[0;2^2,4^2]}(e/2) + 12\nu_{[0;5^3]}(2e/5) +$$
$$2\nu_{[0;2^2,3^2]}(e/3) + 2\nu_{[0;3,6^2]}(e/3) + 4\nu_{[0;2,8^2]}(e/4) + 4\nu_{[0;2,5,10]}(e/5)\big),$$

where $\nu_O(m)$ is defined in (5.1) and $\mathcal{N}_g(e)$ is the number of rooted maps of genus g.

Theorem 7.3 *The number of oriented unrooted maps on genus three surface with e edges is given by the formula*

$$\tfrac{1}{2e}\big(\mathcal{N}_3(e) + 15\mathcal{N}_2(e/2) + 4\nu_{[1;2^4]}(e) + \nu_{[0;2^8]}(e) + 18\nu_{[1;3^2]}(2e/3) + 10\nu_{[0;3^5]}(2e/3)$$

$$+12\nu_{[1;2^2]}(e/2) + 2\nu_{[0;2^3,4^2]}(e/2) + 8\nu_{[0;4^4]}(e/2)$$

$$+2\nu_{[0;2,3^2,6]}(e/3) + 2\nu_{[0;2^2,6^2]}(e/3) + 30\nu_{[0;7^3]}(2e/7) + 8\nu_{[0;4,8^2]}(e/4) + 12\nu_{[0;3,9^2]}(2e/9)$$

$$+4\nu_{[0;2,12^2]}(e/6) + 4\nu_{[0;3,4,12]}(e/6) + 6\nu_{[0;2,7,14]}(e/7)\big),$$

where $\nu_O(m)$ is defined in (5.1) and $\mathcal{N}_g(e)$ is the number of rooted maps of genus g.

The numbers of rooted and unrooted maps on genus two surface up to 30 edges:

No. of edges, No. rooted maps of genus 2, No. of unrooted maps of genus 2

04, 21, 4
05, 966, 106
06, 27954, 2382
07, 650076, 46680
08, 13271982, 830848
09, 248371380, 13804864
10, 4366441128, 218353000
11, 73231116024, 3328822880
12, 1183803697278, 49325772812
13, 18579191525700, 714586880940
14, 284601154513452, 10164338225482
15, 4272100949982600, 142403410942816
16, 63034617139799916, 1969831979334086
17, 916440476048146056, 26954132420126920
18, 13154166812674577412, 365393525753591368
19, 186700695099591735024, 4913176199287631232
20, 2623742783421329300190, 65593569635906036912
21, 36548087103760045010148, 870192550284377429780
22, 505099724454854883618924, 11479539192932030062066
23, 6931067091334952379275496, 150675371553731499821264
24, 94498867785495807431128548, 1968726412209522334197356
25, 1280884669005154962723094680, 25617693380147483835449016
26, 17269149245085316894987194432, 332099023944121243161761560
27, 231687461653506761485020818832, 4290508549139665515691123744
28, 3094389154894054750463387898444, 55256949194539206365604601052
29, 41156529959321075124439691833704, 709595344126234852207569048760
30, 545290525617230994007326084007416, 9088175426953885980802745018758

The numbers of rooted and unrooted maps on genus three surface up to 30 edges:

No. of edges, No. of rooted maps, No of unrooted maps

06, 1485, 131
07, 113256, 8158
08, 5008230, 313611
09, 167808024, 9326858
10, 4721384790, 236095958
11, 117593590752, 5345316004
12, 2675326679856, 111472798586
13, 56740864304592, 2182345314816
14, 1137757854901806, 40634231364914
15, 21789659909226960, 726322104184848
16, 401602392805341924, 12550075287918360
17, 7165100439281414160, 210738250570954064
18, 124314235272290304540, 3453173212810875280
19, 2105172926498512761984, 55399287587418128520
20, 34899691847703927826500, 872492296405529104608
21, 567797719808735191344672, 13518993329700676078500
22, 9084445205688065541367710, 206464663769623968602698
23, 143182713522809088357084720, 3112667685295345475820652
24, 2226449757923955373340520612, 46384369956820665320587902
25, 34199303698053326789771187600, 683986073961364663577206704
26, 519494783678325912052481379156, 9990284301507510446092217236
27, 7811251314435936176791882965696, 144652802119189104865404688680
28, 116359017952555222287628015931 5184, 2077839606295596379211506191640
29, 1718465311469518829323877355423840, 29628712266715926913818949155968
30, 25178356967150456246664822271180140, 419639282785841282782195528667536

The above tables were computed using MATHEMATICA, Ver. 4. The input numbers of rooted maps come from [4] for genus 1, and from [5] for genus 2 and 3.

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