Mathematical Aspects of Mirror Symmetry

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Introduction

Mirror symmetry is the remarkable discovery in string theory that certain "mirror pairs" of Calabi–Yau manifolds apparently produce isomorphic physical theories—related by an isomorphism which reverses the sign of a certain quantum number—when used as backgrounds for string propagation. The sign reversal in the isomorphism has profound effects on the geometric interpretation of the pair of physical theories. It leads to startling predictions that certain geometric invariants of one Calabi–Yau manifold (essentially the numbers of rational curves of various degrees) should be related to a completely different set of geometric invariants of the mirror partner (period integrals of holomorphic forms). The period integrals are much easier to calculate than the numbers of rational curves, so this idea has been used to make very specific predictions about numbers of curves on certain Calabi–Yau manifolds; hundreds of these predictions have now been explicitly verified. Why either the pair of manifolds, or these different invariants, should have anything to do with each other is a great mathematical mystery.

The focus in these lectures will be on giving a precise mathematical description of two string-theoretic quantities which play a primary rôle in mirror symmetry: the so-called A-model and B-model correlation functions on a Calabi–Yau manifold. The first of these is related to the problem of counting rational curves while the second is related to period integrals and variations of Hodge structure. A natural mathematical consequence of mirror symmetry is the assertion that Calabi–Yau manifolds often come in pairs with the property that the A-model correlation function of the first manifold coincides with the B-model correlation function of the second, and *vice versa*. Our goal will be to formulate this statement as a precise mathematical conjecture. There are other recent mathematical expositions of mirror symmetry, by Voisin [**15**] and by Cox and Katz [**3**], which concentrate on other aspects of the subject; the reader may wish to consult those as well in order to obtain a complete picture.

I have only briefly touched on the physics which inspired mirror symmetry (in lectures one and eight), since there are a number of good places to read about some of the physics background: I recommend Witten's address at the International Congress in Berkeley [**100**], a book on "Differential Topology and Quantum Field Theory" by Nash [**14**], and the first chapter of Hübsch's "Calabi–Yau Manifolds: A Bestiary for Physicists" [**10**]. There are, in addition, three collections of papers related to string theory and mirror symmetry which contain some very accessible expository material: "Mathematical Aspects of String Theory" (from a 1986 conference at U.C. San Diego) [**16**], "Essays on Mirror Manifolds" (from a 1991 conference at MSRI) [**17**], and its successor volume "Mirror Symmetry II" [**7**]. I particularly recommend the paper by Greene and Plesser "An introduction to mirror manifolds" [**136**] and the paper by Witten "Mirror manifolds and topological field theory" [**152**], both in the MSRI volume.

This is a revised version of the lecture notes which I prepared in conjunction with my July, 1993 Park City lectures, and which I supplemented when delivering a similar lecture series in Trento during June, 1994. The field of mirror symmetry is a rapidly developing one, and in finalizing these notes for publication I have elected to let them remain as a "snapshot" of the field as it was in 1993 or 1994, making only minor modifications to the main text to accommodate subsequent developments. I have, however, added a postscript that sketches the progress which has been made in a number of different directions since then.

Acknowledgments

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Contents

LECTURE 1
Some Ideas From String Theory

1.1. String theory and quantum field theory

The origins of the startling calculations which have led to tremendous interest among mathematicians in the phenomenon of "mirror symmetry" lie in string theory. String theory is a proposed model of the physical world which idealizes its fundamental constituent particles as one-dimensional mathematical objects ("strings") rather than zero-dimensional objects ("points"). In theories such as general relativity, one has traditionally imagined a point as tracing out what is known as a "worldline" in spacetime; the corresponding notion in string theory is of a "worldsheet" which will describe the trace of a *string* in spacetime. We will consider here only "closed string theory" in which the string is a closed loop; the worldsheets are then (locally) closed surfaces: if we look at a portion of a worldsheet which represents the history of several interacting particles over a finite time interval, we will see a closed surface which has a boundary consisting of a finite number of closed loops.

An early version of string theory was proposed as a model for nuclear processes in the 1960's. Those early investigations revealed a somewhat disturbing property: in order to get a sensible physical theory, the spacetime M in which the string is propagating must have dimension twenty-six. Obviously, when we look around us, we do not see twenty-six dimensions. A later variant which incorporates supersymmetry[1] is sensible exactly when the spacetime has dimension ten—again a bit larger than the four-dimensional spacetime which we observe. Partly for this reason, but primarily because a better model for nuclear processes was found, the original research activity in string theory largely died out in the early 1970's.

String theory was subsequently revived in the 1980's when it was shown [**131**] that if the ten-dimensional string theory were used to model things at much smaller distance scales, an apparently consistent quantum theory of gravity could be produced. (In fact, gravity is predicted as an essential ingredient of this theory.) This "anomaly cancellation" result explained how certain potential inconsistencies in the quantum theory are avoided through an interaction between gravity and the other

[1] I shall not attempt to explain supersymmetry in these lectures.

forces present. Tremendous optimism and excitement pervaded this period, particularly since the new model contains a rich spectrum of elementary particles at low energies and exhibits many features one would expect of a "grand unified field theory" which could describe in a single theory all of the forces observed in nature. The "problem" of ten dimensions in this context can be resolved by assuming that the ten-dimensional spacetime is locally a product $M = M^{1,3} \times M^6$ of a macroscopic four-dimensional spacetime and a compact six-dimensional space whose size is on the order of the Planck length (10^{-33} cm). Because this is so small compared to macroscopic lengths, one wouldn't expect to observe the compact space directly, but its effect on four-dimensional physics could be detectable in various indirect ways.

The next step was even more remarkable for mathematicians—a group of string theorists [**122**] calculated that the compact six-dimensional space must have a Ricci-flat metric on it. (The physically relevant metric is actually a perturbation of this Ricci-flat one.) This is a very restrictive property—it implies, for example, that the six-manifold is a complex Kähler manifold of complex dimension three which has trivial canonical bundle; conversely, such Kähler manifolds always admit Ricci-flat metrics. (This had been conjectured by Calabi [**33**] in the late 1950's and proved by Yau [**103**] in the mid 1970's.) These manifolds have since been named "Calabi–Yau manifolds;" finding and studying them become problems in algebraic geometry, thanks to Yau's theorem.

The model being described here of a string propagating in a spacetime (with a specified metric) is generally regarded as a woefully inadequate description of the "true" string theory, a good formulation of which is as yet unknown. Indeed, if string theory is truly a theory of gravity as we observe it, then the theory should approximate general relativity when the distance scale approaches macroscopic levels. Since the metric on spacetime is part of general relativity, it should be a part of that "approximation" which is somehow to be deduced from a solution to the ultimate "string equations," rather than being something which is put in by hand in advance. Even the *topology* of spacetime should be dictated by the string theory. However, neither these "string equations" nor their exact solutions are known at present.

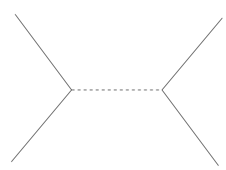
The Calabi–Yau manifolds and their connections with string theory have been studied intensively for more than a decade. In the earliest period, these manifolds were analyzed using standard mathematical techniques, and the results were applied in a string-theoretic context. However, at the same time, other advances were being made in string theory which suggested other ways of looking at certain aspects of the theory of Calabi–Yau manifolds. This eventually led to the discovery of a surprising new phenomenon known as "mirror symmetry," in which it was observed that different Calabi–Yau manifolds could lead to identical physical theories in a way that implied surprising connections between certain geometric features of the manifolds.

To explain this mirror symmetry observation in more detail, we must first describe a few aspects of quantum field theory and its relationship to string theory. In classical mechanics, the worldline representing a particle is required to minimize the "action" (which is the energy integrated with respect to time), or more precisely, to be a stationary path for the action functional. Due to this "stationary action principle," the location of the path in spacetime is completely determined by a knowledge of boundary conditions. Other physically measurable quantities associated to the particle (which are often represented as some kind of "internal variables") will also evolve from their boundary states in a completely predictable manner, again minimizing the action.

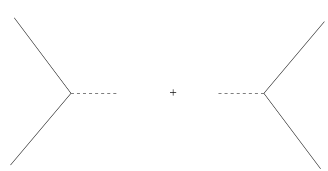
In quantum field theory, however, this changes. Only the probability of various possible outcomes can be predicted with certainty, and *all* trajectories—not just the action-minimizing ones—contribute to the measurement of this probability. The probability is calculated from an integral over the space of all possible paths with these initial and final states,[2] and the classical trajectory is recovered as the leading term in a stationary phase approximation to the path integral.

Relativistic quantum field theories are frequently studied by treating the theory as a small perturbation of a simple type of theory—called a free field theory—whose functional integrals are well-understood. For example, the path integral describing the interaction of two charged particles can be expanded in a perturbative series whose terms are described by "Feynman diagrams." The zeroth order term is the diagram

which represents two particles which do not interact at all, the leading perturbative correction is described by the diagram



which represents a transfer of momentum from one particle to the other via the emission and absorption of a third particle carrying the force, and higher order corrections involve diagrams with more complicated topologies—loops are allowed, for example. Such diagrams can be cut into simpler pieces, at the expense of performing an integral over all possible intermediate states. For example, the interacting Feynman diagram illustrated above can be decomposed into two more primitive pieces,

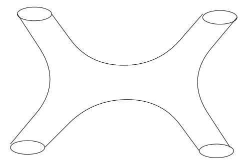


each of which represents a fundamental "interaction" vertex.

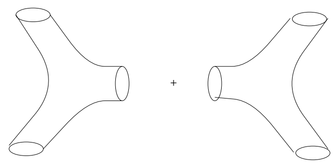
In string theory, the paths are replaced by surfaces: the interacting diagram might be represented as a sphere with four disks removed (or perhaps as something

[2] There are enormous mathematical difficulties in dealing with these "path integrals" or "functional integrals," and they do not in general have a rigorous mathematical formulation. Nevertheless, in the hands of skilled practitioners they can be used to make predictions which agree with laboratory experiments to a remarkable degree of precision.

with more complicated topology),



and this could also be decomposed into more primitive pieces



(called "pair of pants" surfaces). One of the advantages of string theory is that this fundamental piece, the "pair of pants" surface, is a smooth surface, in contrast to the interaction vertex which introduced a singularity into the worldline.

The methods of quantum field theory are applied to string theory in a rather interesting way. If we fix the spacetime and consider a string propagating through it, the location of the worldsheet can be viewed as a map from the worldsheet to the spacetime, and we can regard the coordinate functions on the spacetime as functions on the worldsheet. These spacetime coordinate functions are then treated as the "internal variables" of a two-dimensional quantum field theory—formulated on the worldsheet itself—which captures many of the important physical features of the string theory. (The functional integral in this theory involves an integration over all possible metrics on the worldsheet as well as all possible maps from the worldsheet to the spacetime.) The two-dimensional quantum field theories arising from string theory are of a particular type known as a *conformal field theory*; this means that a conformal change of metric on the worldsheet will act as an automorphism of the theory (typically acting linearly on various spaces of "internal fields" of the theory). The formulation in terms of conformal field theory has turned out to be a very fruitful viewpoint for the study of string theory.

1.2. Correlation functions and pseudo-holomorphic curves

The basic quantities which one needs to evaluate in any quantum field theory are the *correlation functions* which determine the probabilities for a specified final state, given an initial state. Specifying the initial and final states means not only specifying positions, but also the values of any "internal variables" which form a part of the theory. The possible initial or final states in a conformal field theory can be represented as operators $\mathcal{O}_P$ on some fixed Hilbert space $\mathcal{H}$, often referred to as "vertex operators."[3] (The label P indicates the position; we should in principle be specifying initial conditions on an entire boundary circle, but in fact it suffices to consider a limit—within the conformal class of the given metric—in which the circles have been shrunk to zero size and the vertex operators are located at points.) The conjugate transpose of an initial state is a final state, so we often don't distinguish

[3] Generally, in quantum field theories states are represented as elements of a Hilbert space $\mathcal{H}$ but in conformal field theories there is also an operator interpretation.

between those in our notation; with these conventions, the correlation function of a number of vertex operators is denoted by

$$\langle \mathcal{O}_{P_1} \mathcal{O}_{P_2} \ldots \mathcal{O}_{P_k} \rangle.$$

(Note that the correlation functions are complex-valued and do not directly calculate probabilities, but also include the phase of the quantum-mechanical wavefunction.)

If we fix the topology of the worldsheet we must in general integrate over the choice of metric on that worldsheet. A conformal change of metric leaves the correlation functions invariant, so we only need to integrate over the set of conformal classes of metrics, i.e., over the (finite-dimensional) moduli space $\mathcal{M}_{g,k}$ of k-punctured Riemann surfaces of genus g.

The two-dimensional quantum field theories which are related to mirror symmetry have a subset of their correlation functions whose values do not depend on the position of the points P_j on the worldsheet; these are called *topological* correlation functions. (We don't need to consider an integral over $\mathcal{M}_{g,k}$ in this case.) They will be the primary objects of interest for us. In fact, due to the possibility of decomposing the worldsheet into more primitive pieces, the main case to consider is the case of three vertex operators on surfaces of genus zero, i.e., we take Σ to be the "pair of pants" surface $\mathbb{CP}^1 - \{P_1, P_2, P_3\}$. To evaluate a correlation function

$$\langle \mathcal{O}_{P_1} \mathcal{O}_{P_2} \mathcal{O}_{P_3} \rangle,$$

however, we must still integrate over the infinite-dimensional space $\text{Maps}(\Sigma, M)$ of maps from Σ to the spacetime M.

To proceed further, we need to introduce the "action" functional on the space of maps. We fix Riemannian metrics[4] on both the worldsheet Σ and the spacetime M and define for any sufficiently smooth $\varphi \in \text{Maps}(\Sigma, M)$

$$\mathcal{S}[\varphi] = \int_\Sigma \|d\varphi\|^2 \, d\mu$$

using the metrics to define the norm. (In practice, we take M to be the compact six-dimensional manifold rather than the entire space.) The properties of the action functional are easier to analyze if we assume that M has some additional structure—the minimal structure needed is a symplectic form ω and an almost-complex structure J which is ω-tamed. (We will review the definitions of these in lecture three.) When these have been chosen, there is a "d-bar" operator $\overline{\partial}_J$ on maps, and an alternate formula for the action

$$\mathcal{S}[\varphi] = \int_\Sigma \|\overline{\partial}_J \varphi\|^2 \, d\mu + \int_\Sigma \varphi^*(\omega).$$

A lower bound for $\mathcal{S}[\varphi]$ in any homotopy class of maps is thus given by $\int_\Sigma \varphi^*(\omega)$; this bound will be achieved by the so-called pseudo-holomorphic maps—the ones for which $\overline{\partial}_J \varphi \equiv 0$. These have been extensively studied by Gromov [**51**] and others as a natural generalization of complex curves on Kähler manifolds.

This action functional now appears in an integrand which is supposed to be integrated over the infinite-dimensional space of all maps. We will outline the standard manipulations which are made with these functional integrals in physics

[4] This is a "Euclidean" version of the theory, whose correlation functions are related by analytic continuation to those of the "Lorentzian" version in which the worldsheet metric has signature $(1, 1)$.

in order to express the correlation function as an infinite sum of finite-dimensional integrals. We will subsequently use the outcome of those manipulations to make mathematical definitions of the corresponding quantities in the form of a formal sum of these finite-dimensional integrals.

The topological correlation functions we are studying are to be evaluated by a functional integral of the form

$$\begin{aligned}\langle \mathcal{O}_{P_1} \dots \mathcal{O}_{P_k} \rangle &= \int \mathcal{D}\varphi \, \mathcal{O}_{P_1} \dots \mathcal{O}_{P_k} \, e^{-2\pi \, \mathcal{S}[\varphi]} \\ &= e^{-2\pi \int_\Sigma \varphi^*(\omega)} \int \mathcal{D}\varphi \, \mathcal{O}_{P_1} \dots \mathcal{O}_{P_k} \, e^{-2\pi \int_\Sigma \|\overline{\partial}_J \varphi\|^2 \, d\mu}.\end{aligned} \tag{1.1}$$

(We are suppressing the "fermionic" part of this functional integral, which is actually very important, but explaining it would take us too far afield.) The "topological" property of these correlation functions turns out to imply [**150, 152**] that if we introduce a parameter t into the exponent of the last functional integral in eq. (1.1) to produce

$$\int \mathcal{D}\varphi \, \mathcal{O}_{P_1} \dots \mathcal{O}_{P_k} \, e^{-2\pi t \int_\Sigma \|\overline{\partial}_J \varphi\|^2 \, d\mu},$$

then the resulting expression is independent of t and can be evaluated in a limit in which $t \to \infty$. In such a limit, the only contributions to the functional integral are the maps φ for which $\overline{\partial}_J \varphi \equiv 0$, i.e., the pseudo-holomorphic maps. (This trick for reducing to a finite-dimensional integral is known as the "method of stationary phase.") The space of pseudo-holomorphic maps in a given homotopy class is finite-dimensional, so we have reduced the evaluation of our correlation function to an infinite sum of finite-dimensional integrals, of the form

$$\langle \mathcal{O}_{P_1} \dots \mathcal{O}_{P_k} \rangle = \sum_{\text{homotopy classes}} e^{-2\pi \int_\Sigma \varphi^*(\omega)} \int_{\mathcal{M}} \mathcal{D}\varphi \, \mathcal{O}_{P_1} \dots \mathcal{O}_{P_k}, \tag{1.2}$$

where $\mathcal{M}$ denotes the moduli space of pseudo-holomorphic maps in a fixed homotopy class. It is these finite-dimensional integrals on which we shall eventually base our definitions. The convergence of the infinite sum will remain an issue in our approach, and will lead us to (in some cases) assign a provisional interpretation to this formula as being a formal power series only. From the physics one expects convergence whenever the volume of the corresponding metric is sufficiently large.

1.3. A glimpse of mirror symmetry

If the target space M for our maps is a Calabi–Yau manifold (equipped with a Ricci-flat metric), all of the vertex operators which participate in a given topological correlation functions must be of one of two distinct types. Correlation functions involving vertex operators of the first type are called *A-model correlation functions* while those involving vertex operators of the second type are known as *B-model correlation functions* [**152**]. (These are actually the correlation functions in two "topological field theories" [**150**] which are closely related to the original quantum field theories.) For each type, the vertex operators $\mathcal{O}_P$ in the quantum field theory or topological field theory have a geometric interpretation; we will treat the correlation functions as functions of these geometric objects.

The A-model correlation functions can be defined in a much broader context than Calabi–Yau manifolds: they can be defined for any semipositive symplectic manifold M (where semipositive roughly means that $-c_1(M)$ is nonnegative—we will give the precise definition in lecture three). The vertex operators $\mathcal{O}_{P_j}$ in the topological field theory correspond to harmonic differential forms α_j on M, and the correlation functions $\langle \alpha_1\, \alpha_2\, \alpha_3 \rangle$ take the form of an infinite series whose constant term—corresponding to homotopically trivial maps from Σ to M—is the familiar trilinear function $\int_M \alpha_1 \wedge \alpha_2 \wedge \alpha_3$.

To evaluate the non-constant terms we need an integral over the moduli space of pseudo-holomorphic two-spheres. In the Calabi–Yau case, those two-spheres are expected to be discrete (based on a formal dimension count), so there should be invariants which count the number of rational curves in a given homology class. (There are certain technical difficulties with this, as we shall see in lecture two.) More generally, the non-constant terms in the A-model correlation functions will be related to certain kinds of counting problems for pseudo-holomorphic curves on a semipositive symplectic manifold.

The B-model correlation functions, on the other hand, require a choice of non-vanishing holomorphic n-form Ω on M for their definition, so they are restricted to the Calabi–Yau case. The vertex operators in the topological correlation functions correspond to elements in the space $H^q(\Lambda^p T_M^{(1,0)})$, where we use $T^{(1,0)}$ to denote the holomorphic tangent bundle of an almost-complex manifold. (More precisely, we should use Dolbeault cohomology to describe $H^q(\Lambda^p T_M^{(1,0)})$, and take harmonic representatives to get the vertex operators in the topological field theory.) The "first term" in the correlation function is then defined as a composition of the standard map on cohomology groups

$$H^{q_1}(\Lambda^{p_1} T_M^{(1,0)}) \times H^{q_2}(\Lambda^{p_2} T_M^{(1,0)}) \times H^{q_3}(\Lambda^{p_3} T_M^{(1,0)}) \to H^n(\Lambda^n T_M^{(1,0)})$$

(for $p_1 + p_2 + p_3 = q_1 + q_2 + q_3 = n$) with some isomorphisms depending on the choice of $\Omega^{\otimes 2}$

$$H^n(\Lambda^n(T_M^{(1,0)})) \xrightarrow{\lrcorner\Omega} H^n(\mathcal{O}_M) \cong \big(H^0(K_M)\big)^* \xrightarrow{\otimes\Omega} \mathbb{C},$$

where the middle isomorphism is Serre duality. (This can be written as an integral over M, and so can be thought of as coming from integrating over the moduli space of homotopically trivial maps from Σ to M—this is why we identify it with the first term in an expansion like eq. (1.2).) Remarkably, all of the other terms in the expansion (1.2) of a B-model correlation function are known to vanish on physical grounds [**127, 152**], so we can calculate these correlation functions exactly using geometry, and even use the geometric version of the correlation function as a mathematical definition.

In brief, the idea of mirror symmetry is this. There could be pairs of complex manifolds M, W (each with trivial canonical bundle) which produce identical physics when used for string compactification, except that the rôles of the A-model and B-model correlation functions are reversed. In particular, this would imply the existence of isomorphisms

$$H^q(\Lambda^p(T_M^{(1,0)})^*) \cong H^q(\Lambda^p(T_W^{(1,0)}))$$

(and vice versa), as well as formulas relating the A-model correlation functions on M (which count the number of rational curves) to the B-model correlation functions on W (which are related to period integrals of Ω).

LECTURE 2
Counting Rational Curves

In this lecture we begin the discussion of the problem of counting rational curves on a complex threefold with trivial canonical bundle (a "Calabi–Yau threefold"). These curve-counting invariants will eventually be used to formulate a mathematical version of the A-model correlation functions. In the present lecture, we focus on the problems one encounters in formulating these invariants purely algebraically; we give a number of examples.

Consider the deformation theory of holomorphic maps from $\mathbb{CP}^1 \to M$, where M is a complex projective variety. If we are given such a map $\varphi : \mathbb{CP}^1 \to M$, then a first order variation of that map can be described by specifying in which direction (and at what rate) each point of the image moves. That is, we need to specify a holomorphic tangent vector of M for every point on $\mathbb{CP}^1$, or in other words, a section of $H^0(\mathbb{CP}^1, \varphi^*(T_M^{(1,0)}))$. As might be expected from other deformation problems, the obstruction group for these deformations is $H^1(\mathbb{CP}^1, \varphi^*(T_M^{(1,0)}))$. The moduli problem for such maps will be best-behaved if the obstruction group vanishes, that is, if $h^1(\mathbb{CP}^1, \varphi^*(T_M^{(1,0)})) = 0$. When that is true, the moduli space will be a smooth complex manifold of complex dimension $h^0(\mathbb{CP}^1, \varphi^*(T_M^{(1,0)}))$. More generally, the Euler–Poincaré characteristic

$$\chi(\varphi^*(T_M^{(1,0)})) = h^0(\mathbb{CP}^1, \varphi^*(T_M^{(1,0)})) - h^1(\mathbb{CP}^1, \varphi^*(T_M^{(1,0)}))$$

can be regarded as the "expected complex dimension" of the moduli space.

Although the Euler–Poincaré characteristic can be easily computed from the Riemann–Roch theorem for vector bundles, we shall make a more elementary calculation, based on a structure theorem for bundles on $\mathbb{CP}^1$.

Theorem 2.1 (Grothendieck)**.** *Every vector bundle $\mathcal{E}$ on $\mathbb{CP}^1$ can be written as a direct sum of line bundles:*

$$\mathcal{E} \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n).$$

Using a Grothendieck decomposition for $\varphi^*(T_M^{(1,0)})$, we can calculate the cohomology directly. For if

$$\varphi^*(T_M^{(1,0)}) \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$$

then using the fact the $h^0(\mathcal{O}(a)) = 1 + a$ we find

$$h^0(\varphi^*(T_M^{(1,0)})) = \sum_j \begin{cases} 1 + a_j & \text{if } a_j \geq -1 \\ 0 & \text{if } a_j < -1 \end{cases}$$

while since $H^1(\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_m)) \cong H^0(\mathcal{O}(-2 - a_1) \oplus \cdots \oplus \mathcal{O}(-2 - a_m))^*$ we have

$$h^1(\varphi^*(T_M^{(1,0)})) = \sum_j \begin{cases} -(1 + a_j) & \text{if } -2 - a_j \geq 0 \\ 0 & \text{if } -2 - a_j < 0 \end{cases}$$

since $1 + (-2 - a_j) = -(1 + a_j)$.

Taking the difference, we find

$$\chi(\varphi^*(T_M^{(1,0)})) = \sum_j (1 + a_j) = n + \sum_j a_j = \dim_{\mathbb{C}} M + \deg \varphi^*(-K_M).$$

Thus, the "expected dimension" is independent of the decomposition. The same result can be obtained from Riemann–Roch.

But our calculation shows more—to ensure vanishing of the obstruction group, we must have $a_j \geq -1$ for all j. In addition to this condition on the a_j's, they must also satisfy $\max\{a_j - 2\} \geq 0$, which is seen as follows. From the exact sequence

$$0 \to T_{\mathbb{CP}^1}^{(1,0)} \to \varphi^*(T_M^{(1,0)}) \to N_\varphi \to 0$$

(where N_φ denotes the normal bundle) and the fact that $T_{\mathbb{CP}^1}^{(1,0)} \cong \mathcal{O}(2)$, we see that there must be a nontrivial homomorphism

$$\mathcal{O}(2) \to \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_m),$$

which implies that $\max\{a_j - 2\} \geq 0$ as claimed. Without loss of generality, we may therefore assume that $a_1 \geq 2$.

In the case relevant to string theory ($K_M = 0$, $\dim_{\mathbb{C}} M = 3$) we then find that in order to have vanishing obstruction group we need

$$0 = a_1 + a_2 + a_3 \geq 2 - 1 - 1 = 0$$

and so $a_1 = 2$, $a_2 = a_3 = -1$. In this case, the moduli space of holomorphic maps will be smooth of dimension three; if we mod out by the automorphism group $\mathrm{PGL}(2, \mathbb{C})$, the moduli space of unparameterized maps will be smooth of dimension 0. The points in that space are what we would like to "count." We discuss some examples, drawn largely from [**59**], to which we refer the reader for more details.

Example 2.2. *Lines on the Fermat quintic threefold.* All of the lines on the Fermat quintic threefold

$$\{x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0\} \subset \mathbb{CP}^4$$

can be described as follows.[1]

First type (375 lines): The line described by $x_0 + x_1 = x_2 + x_3 = x_4 = 0$, and others whose equations are obtained from these by permutations and multiplication by fifth roots of unity.

[1] We use the Fermat quintic because it is an easily-described nonsingular hypersurface, and because it will be related to a mirror symmetry construction later on, *not* because Wiles announced a proof of Fermat's Last Theorem while the 1993 Park City Institute was underway!

Second type (50 one-parameter families of lines): The lines described parametrically by

$$(u, v) \mapsto (u, -u, av, bv, cv)$$

for fixed constants a, b, c satisfying $a^5 + b^5 + c^5 = 0$, and others whose parameterizations are obtained from these by permutations and multiplication by fifth roots of unity.

So we see that the lines are not always finite in number, even for smooth hypersurfaces. (One might have suspected such a "universal finiteness for smooth hypersurfaces" based on experience with cubic surfaces—every smooth cubic surface in $\mathbb{CP}^3$ has precisely twenty-seven lines.)

Example 2.3. *Lines on the general quintic threefold.* However, if we deform from the Fermat quintic threefold to a general one, it is possible to show that the number of lines is finite. The generic number of lines can then be computed as follows. Start from the Grassmannian $\mathrm{Gr}(\mathbb{CP}^1, \mathbb{CP}^4)$ of lines in $\mathbb{CP}^4$. Consider the universal bundle U whose fiber at a line L is the two-dimensional subspace $U_L \subset \mathbb{C}^5$ such that $\mathbb{P}(U_L) = L$. We define a bundle $\mathcal{B} = \mathrm{Sym}^5(U^*)$ whose fibers describe the quintic forms on the lines L. Then every quintic threefold M determines a section $s_M \in \Gamma(\mathcal{B})$: the equation of M is restricted to L to give a homogeneous quintic there. Clearly, the lines contained in M are precisely those whose corresponding points in the Grassmannian are zeros of the section s_M.

The Grassmannian $\mathrm{Gr}(\mathbb{CP}^1, \mathbb{CP}^4)$ has complex dimension six, and the bundle $\mathcal{B}$ has rank six; when things are generic, the section s_M will have finitely many zeros, which can be counted by calculating

$$\#\{L \mid s_M(L) = 0\} = c_6(\mathcal{B}) = 2875.$$

Example 2.2 (bis). Katz [**59**] has found a way to assign multiplicities to each of the isolated lines, and one-parameter families of lines, on the Fermat quintic threefold. His multiplicity assignment for each of the 375 isolated lines is "5," and that for each of the 50 one-parameter families is "20." Thus, the total count is

$$5 \cdot 375 + 20 \cdot 50 = 2875.$$

Katz's methods of assigning multiplicities are not yet completely general,[2] but they do hold out the hope that a "count" of rational curves might be made even in cases when the actual number of curves is not finite.

Example 2.4. *Conics on the general quintic threefold.* We can make a similar calculation for conics on the general quintic threefold. The key observation is that every conic spans a $\mathbb{CP}^2$, so the starting point for describing them is the Grassmannian $\mathrm{Gr}(\mathbb{CP}^2, \mathbb{CP}^4)$. We need the bundle over the Grassmannian whose fiber is the set of conics in the $\mathbb{CP}^2$ in question: this is described by $\mathbb{P}(\mathrm{Sym}^2(U^*))$, where U is the universal subbundle as before.

The space $\mathbb{P}(\mathrm{Sym}^2(U^*))$ contains degenerate conics (pairs of lines, and double lines) as well as smooth conics. However, if M is sufficiently general, then the actual locus of conics which lie in M will be finite in number, and contain only smooth conics.

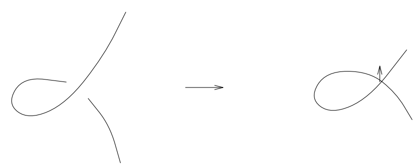
[2] See the "Postscript: Recent Developments" section for the current status.

The vector bundle which will get a section s_M for every quintic M is the bundle $\mathcal{B} := \mathrm{Sym}^5(U^*)/(\mathrm{Sym}^3(U^*) \oplus \mathcal{O}_{\mathbb{P}}(-1))$. This describes the effect of restricting the quintic equation to the conic: one gets a quintic equation on the $\mathbb{CP}^2$, but must mod out by those quintics which can be written as the product of a cubic (the $\mathrm{Sym}^3(U^*)$ factor) and the given conic.

We have $\dim_{\mathbb{C}} \mathbb{P}(\mathrm{Sym}^2(U^*)) = \operatorname{rank} \mathcal{B} = 11$, so the computation is made by calculating:

$$\#\{C \mid s_M(C) = 0\} = c_{11}(\mathcal{B}) = 609250.$$

Example 2.5. *Twisted cubics on the general quintic threefold.* The problem gets more difficult for twisted cubics. Again, we can look at the linear span (a $\mathbb{CP}^3$) and begin by considering a Grassmannian $\mathrm{Gr}(\mathbb{CP}^3, \mathbb{CP}^4)$. But this time we must use a bundle $\mathcal{H} \to \mathrm{Gr}(\mathbb{CP}^3, \mathbb{CP}^4)$ whose fibers are isomorphic to the Hilbert scheme of twisted cubics in $\mathbb{CP}^3$. That scheme contains limits which are quite complicated. (For example, there is a limit which is a nodal plane curve with an embedded point at the node which points out of the plane:



see Hartshorne [**9**], pp. 259–260.) Although the bundle $\mathcal{B}$ and the section s_M can be defined and understood at points representing smooth twisted cubics, their extension to the locus of degenerate cubics is by no means easy.

Ellingsrud and Strømme [**40**] have, however, carried this out, and they find that the number of twisted cubics on the general quintic threefold is 317206375.

Clemens [**37**] has conjectured that the general quintic threefold will have only a finite number of rational curves of each degree, and that all of them will satisfy $\varphi^*(T_M^{(1,0)}) = \mathcal{O}(2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. This has been verified up through degree nine by Katz [**58**], Johnsen–Kleiman [**56**] and Nijsse [**80**], and the prospects are good for degrees as high as twenty-four [**57**]. However, as we have seen, making the calculation of the number becomes very difficult past degree two. In fact, for degree greater than three, effective techniques for calculating this number are not presently known.

We now turn to another example which demonstrates that we cannot always expect finiteness, even for the generic deformation of a given threefold with trivial canonical bundle.

Example 2.6. *Rational curves on double solids.* We let M be the double cover of $\mathbb{CP}^3$, branched along a general surface S of degree eight in $\mathbb{CP}^3$; the double cover map is denoted by $\pi : M \to \mathbb{CP}^3$. We let $\pi^*(H)$ be the pullback of a hyperplane H from $\mathbb{CP}^3$; the *degree* of a rational curve C will mean $\pi^*(H) \cdot C$.

To find "lines" on M, that is, curves L with $\pi^*(H) \cdot L = 1$ we consider their images $\pi(L)$. Since $\pi^*(H)$ meets L in a single point P, H meets $\pi(L)$ in the single point $\pi(P)$. Thus, $\pi(L)$ must itself be a line. But its inverse image on M will necessarily have two components: $\pi^{-1}(\pi(L)) = L + L'$. In order to have this splitting into two components, the line $\pi(L)$ must be tangent to S at every point of intersection with S, i.e., it must be four-times tangent to S. Now the Grassmannian

$\mathrm{Gr}(\mathbb{CP}^1, \mathbb{CP}^3)$ has dimension four, and it is one condition to be tangent to a surface, so the dimension of the set of four-tangent lines is nonnegative, and can be expected to be equal to zero. (In fact, it turns out to equal zero as expected, when S is general.) The number of such lines in the Grassmannian can be calculated with the Schubert calculus; it turns out to be 14752. The corresponding count of lines on M is 29504.

Finding "conics" on M is a different story, as has been observed by Katz and by Kollár. Given a curve C with $\pi^*(H) \cdot L = 2$, there are two possibilities for $\pi(C)$: it could be a line, or it could be a conic. In the latter case, the conic $\pi(C)$ must be eight-times tangent to S. But in the former case, in order to have an irreducible double cover with a rational normalization, the line $\pi(C)$ must be three-times tangent to S. By our previous dimension count, there is at least a one-parameter family of such lines for any choice of S.

So we won't always have a finite number of things to "count," even if we perturb to a general member of a particular family. And there is an additional difficulty if we wish to count maps from $\mathbb{CP}^1$ to M when multiple covers are allowed, as the next example shows.

Example 2.7. *Multiple covers.* Suppose that $\varphi : \mathbb{CP}^1 \to M$ is generically one-to-one, but that we consider a map $\varphi' := u \circ \varphi$, where $u : \mathbb{CP}^1 \to \mathbb{CP}^1$ is a covering of degree μ. Even if $\varphi^*(T_M^{(1,0)}) = \mathcal{O}(2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, we will get a bad splitting of the pullback via the new map:

$$\varphi'^*(T_M^{(1,0)}) = \mathcal{O}(2\mu) \oplus \mathcal{O}(-\mu) \oplus \mathcal{O}(-\mu).$$

Furthermore, the dimension of the moduli space can be calculated: the moduli space of maps $u : \mathbb{CP}^1 \to \mathbb{CP}^1$ of degree μ has dimension $2\mu+1$. So we see that the dimension of the space of maps will go up and up.

To handle cases such as multiple covers, "virtual" numbers of curves must be introduced; Katz's approach to this is to use excess intersection theory [**6**]. However, this introduction of "virtual" numbers leads to another complication, as our final example shows.

Example 2.8. *Negative numbers of curves* (see [**120**], section 8). There are cases in which the "virtual" number of curves is negative. In general, when the parameter space B for a family of curves is smooth of dimension b, the virtual number of curves should be the top Chern class of the holomorphic cotangent bundle $c_b((T_B^{(1,0)})^*)$. If M is a complex threefold with $K_M = 0$ which contains $\mathbb{CP}^2$ as a submanifold (which can arise from resolving a $\mathbb{Z}/3\mathbb{Z}$-quotient singularity, for example), then the lines on $\mathbb{CP}^2$ are parameterized by $\mathbb{CP}^2$ and have virtual number $c_2((T_{\mathbb{CP}^2}^{(1,0)})^*) = 3$, but the conics on $\mathbb{CP}^2$, being parameterized by $\mathbb{CP}^5$, have virtual number $c_5((T_{\mathbb{CP}^5}^{(1,0)})^*) = -6$. This negative value actually agrees with the predictions of mirror symmetry as shown in [**120**].

LECTURE 3
Gromov–Witten Invariants

3.1. Counting curves via symplectic geometry

The difficulties we encountered in trying to count rational curves on a Calabi–Yau threefold can be avoided by enlarging the category we are considering, and using Gromov's theory of pseudo-holomorphic spheres in symplectic manifolds [**51**]. This approach has the advantage that the almost-complex structure can be slightly perturbed to make the number of such spheres finite, and the finite number so obtained is independent of the choice of small perturbation.

Let (M, ω) be a compact *symplectic manifold* of dimension $2n$. This means that M is a compact oriented differentiable manifold of (real) dimension $2n$ and ω is a closed real two-form on M which is nondegenerate in the sense that its n^{th} exterior power $\omega^{\wedge n}$ is nonzero at every point.

An *almost complex structure* on a manifold M is a map $J : T_M \to T_M$ whose square is -1. If we complexify the tangent spaces, we get $T_{M,p} \otimes \mathbb{C} = T^{(1,0)}_{M,p} \oplus T^{(0,1)}_{M,p}$, the decomposition into $+i$ and $-i$ eigenspaces for J. If these subspaces are closed under Lie bracket, we say that the almost-complex structure is *integrable*; in this case, these subspaces give M the structure of a complex manifold.

If (M, ω) is a symplectic manifold, an almost-complex structure J on M is said to be ω*-tamed* if $\omega(\xi, J\xi) > 0$ for all nonzero $\xi \in T_pM$. If we have fixed an (ω-tamed) almost-complex structure J on M, and φ is a differentiable map from S^2 to M, we define

$$\overline{\partial}_J \varphi = \frac{1}{2}(d\varphi + J\, d\varphi\, J_0),$$

where J_0 is the standard almost-complex structure on S^2.

The main example we have in mind is this: M is a compact Kähler manifold, ω is the Kähler form, and J is an ω-tamed perturbation of the original complex structure on M.

Definition 3.1 (McDuff [**71**])**.** (M, ω) is *semipositive* if there is no map $\varphi : S^2 \to M$ satisfying

$$\int_{S^2} \varphi^*(\omega) > 0, \quad \text{and} \quad 3 - n \le \int_{S^2} \varphi^*(-K_M) < 0,$$

where we are writing $-K_M$ as in algebraic geometry to indicate the first Chern class $c_1(M)$, which may be represented as a two-form.

Examples. Here are three ways of producing semipositive symplectic manifolds.

1. If $K_M = 0$ (the Calabi–Yau case) then (M, ω) is semipositive for any ω.
2. If M is a complex projective manifold with $|-K_M|$ ample (a "Fano variety"), then we can take $\omega = -K_M$ to produce a semipositive (M, ω).
3. If $n \leq 3$ then M is automatically semipositive.

Because it is sometimes difficult to check whether a homology class η is represented as the image of a map $\varphi : S^2 \to M$ we also introduce a variant of this property.

Definition 3.2. (M, ω) is *strongly semipositive* if there is no class $\eta \in H_2(M, \mathbb{Z})$ satisfying

$$\omega \cdot \eta > 0, \quad \text{and} \quad 3 - n \leq (-K_M) \cdot \eta < 0.$$

All three of our examples satisfy this stronger property as well.

Fix a homology class $\eta \in H_2(M, \mathbb{Z})$. As we saw in example 2.7, there are technical problems caused by "multiple-covered" maps—maps whose degree onto the image is greater than one. Let us call a map *simple* if its degree onto its image is one. We let $\mathrm{Maps}^*_\eta(S^2, M) \subset \mathrm{Maps}_\eta(S^2, M)$ denote the subset of simple maps with fundamental class η. We also let $\mathrm{Maps}^*_\eta(S^2, M)_{(p)}$ be the set of simple differentiable maps $S^2 \to M$ with fundamental class η whose derivative lies in L_p. Using an appropriate Sobolev norm, $\mathrm{Maps}^*_\eta(S^2, M)_{(p)}$ can be given the structure of a Banach manifold. We can then regard $\overline{\partial}_J$ as a section of the bundle $\mathcal{W} \to \mathrm{Maps}^*_\eta(S^2, M)_{(p)}$ whose fibers are

$$\mathcal{W}_\varphi := H^0_{(p)}(S^2, \mathcal{A}^{(0,1)}_{S^2} \otimes \varphi^*(T^{(1,0)}_M)),$$

where the subscript (p) denotes L_p-cohomology, and $\mathcal{A}^{(0,1)}_{S^2}$ denotes the sheaf of $(0,1)$-forms on S^2 (with respect to the complex structure J_0).

The key technical properties we need are summarized in the following two theorems.

Theorem 3.3 (McDuff [**70**]). *If J is generic, then*

$$\widetilde{\mathcal{M}}^*_{(\eta,J)} := \{\varphi \in \mathrm{Maps}^*_\eta(S^2, M)_{(p)} \mid \overline{\partial}_J \varphi = 0\}$$

is a smooth manifold of dimension

$$\dim_{\mathbb{R}} \widetilde{\mathcal{M}}^*_{(\eta,J)} = 2\,\chi(\varphi^*(T^{(1,0)}_M)).$$

(The dimension is calculated using the Atiyah–Singer index theorem, which yields the same result as the Riemann–Roch theorem did in algebraic geometry.)

(This theorem would have failed if we had allowed multiple-covered maps to be included.)

The next theorem is due to Gromov [**51**], based on some techniques of Sacks–Uhlenbeck [**88**] and with further improvements by several authors [**84, 102, 104**]. (We refer the reader to those papers for a more precise statement.)

Theorem 3.4. *$\widetilde{\mathcal{M}}^*_{(\eta,J)}$ can be compactified by using limits of graphs of maps; this compactification has good properties.*

In the case relevant to string theory in which M is a projective manifold with $K_M = 0$ of complex dimension three, we find that for generic J, the (real) dimension of $\widetilde{\mathcal{M}}^*_{(\eta,J)}$ is six, and the dimension of

$$\mathcal{M}^*_{(\eta,J)} := \widetilde{\mathcal{M}}^*_{(\eta,J)} / \operatorname{PGL}(2,\mathbb{C})$$

is zero. The space $\mathcal{M}^*_{(\eta,J)}$ itself is already compact in this case; the number of points in that space is our desired invariant. (These points may need to be counted with multiplicity, or with signs.) This invariant counts the number of rational curves (of fixed topological type) on M with respect to its original complex structure, if that number is finite; it can be used as a substitute for that count in the general case.[1]

To describe the invariants in situations more general than complex threefolds with trivial canonical bundle, we must introduce the oriented bordism group $\Omega_*(M)$. The elements of $\Omega_k(M)$ are equivalence classes of pairs (B^k, F) consisting of a compact oriented differentiable manifold B of dimension k (but not necessarily connected), together with a differentiable map $F : B^k \to M$. We say that $(B^k, F) \sim 0$ if there exists an *oriented bordism* (C^{k+1}, H): i.e., a differentiable manifold C of dimension $k+1$ and a differentiable map $H : C^{k+1} \to M$ with $\partial C^{k+1} = B^k$ and $H|_{B^k} = F$. We add elements of $\Omega_k(M)$ by means of disjoint union: $(B_1^k, F_1) + (B_2^k, F_2) = (B_1^k \cup B_2^k, F_1 \cup F_2)$; the additive inverse is given by reversing orientation.

The oriented bordism group $\Omega_*(M)$ is a module over the Thom bordism ring Ω (consisting of oriented manifolds modulo oriented bordisms, with no maps to target spaces) via

$$N^j \cdot (B^k, F) = (N^j \times B^k, G)$$

where $G(x,y) = F(y)$.

Theorem 3.5 (Thom [**92**], Conner–Floyd [**2**])**.**

1. *$\Omega_*(M) \otimes \mathbb{Q} \cong H_*(M, \mathbb{Q}) \otimes \Omega$.*
2. *If $H_*(M, \mathbb{Z})$ is torsion-free, then $\Omega_*(M) \cong H_*(M, \mathbb{Z}) \otimes \Omega$.*

To describe our basic invariants, we choose three classes α_1, α_2, α_3 in $\Omega_*(M)$ represented by elements $(B_1^{k_1}, F_1)$, $(B_2^{k_2}, F_2)$, $(B_3^{k_3}, F_3)$, and let $Z_j = \operatorname{Image}(F_j)$. We call the invariants defined below the *Gromov–Witten invariants*, since it was Witten [**150**] who pointed out how Gromov's study of $\widetilde{\mathcal{M}}^*_{(\eta,J)}$ could be used in principle to describe invariants relevant in topological quantum field theory. The detailed construction of these invariants was recently carried out by Ruan [**86**]. There are two cases to consider, with one being more technically challenging than the other.[2]

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[1] It has not yet been verified that Katz's method of assigning multiplicities to positive-dimensional components in the algebro-geometric context produces the same results as this method from symplectic geometry. Because of the need to include signs in certain circumstances, this invariant can even accommodate the "negative virtual numbers" which occurred in example 2.8.

[2] To simplify the exposition, we have altered Ruan's description of the second case, ignored the necessity of passing to the inhomogeneous $\overline{\partial}$ equation (introduced already by Gromov [**51**]), and built into our definition the so-called "multiple cover formula" expected from the physics [**118, 110**]. (This latter step is now justified thanks to a theorem of Voisin [**98**]; there is also a related result of Manin [**69**].) We are also abusing notation somewhat by using Φ_η in both cases, since the second case is actually related to Ruan's $\widetilde{\Phi}_\eta$ invariant.

Construction (Ruan). *Let $\eta \in H_2(M, \mathbb{Z})$, let $\alpha_j = (B^{k_j}, F_j)$ be a bordism class, and let $Z_j = \mathrm{Image}(F_j)$, for $j = 1, 2, 3$. Suppose that $\sum_{j=1}^{3}(2n - k_j) = 2n - 2K_M \cdot \eta$, where η is the class of the image of φ, and suppose that the almost-complex structure J is generic.*

(a) *If $-K_M \cdot \eta > 0$, then*

$$\{\varphi \in \widetilde{\mathcal{M}}^*_{(\eta,J)} \mid \varphi(0) \in Z_1, \varphi(1) \in Z_2, \varphi(\infty) \in Z_3\}$$

is a finite set. Let $\Phi_\eta(\alpha_1, \alpha_2, \alpha_3)$ denote the signed number of points in this set, with signs assigned according to orientations at the specified points of intersection.

(b) *If $-K_M \cdot \eta = 0$, then there exists an integer $\Phi_\eta(\alpha_1, \alpha_2, \alpha_3)$ which agrees with*

$$\#\{\text{generically injective } \varphi \in \widetilde{\mathcal{M}}^*_{(\eta,J)} \mid \varphi(0) \in Z_1, \varphi(1) \in Z_2, \varphi(\infty) \in Z_3\}$$

(counted with signs) whenever the latter makes sense. (The signs are all positive if the almost-complex structure is integrable.)

These invariants $\Phi_\eta(\alpha_1, \alpha_2, \alpha_3)$ depend only on the bordism classes α_1, α_2, α_3, and do not change under small variation of J.

3.2. Simple properties of Gromov–Witten invariants

In spite of the fact that we needed to pass to bordism to ensure that the Gromov–Witten invariants are well-defined, their dependence on bordism-related phenomena is minimal. In fact, Ruan checks that the invariants are trivial with respect to the Ω-module structure on $\Omega_*(M)$, and so it follows from the theorem of Thom and Conner–Floyd that we get a well-defined $\mathbb{Q}$-valued invariant on rational homology $H_*(M, \mathbb{Q})$. If M has no torsion in homology, we even get an integer-valued invariant on integral homology.

The Gromov–Witten invariants $\Phi_\eta(\alpha_1, \alpha_2, \alpha_3)$ will vanish if α_1 corresponds to a class of real codimension zero or one. This is easy to see—if there are any elements in the set

$$\{\varphi \in \widetilde{\mathcal{M}}^*_{(\eta,J)} \mid \varphi(0) \in Z_1, \varphi(1) \in Z_2, \varphi(\infty) \in Z_3\}$$

then the intersection of the image of φ with the image Z_1 of F_1 has real dimension two or one. By varying the location of $\varphi(0)$, we will produce a two- or one-parameter family of maps. This contradicts the set being finite; thus, the set must be empty and the invariant vanishes.

Note what happens to the Gromov–Witten invariants in the case of interest to string theory ($\dim_{\mathbb{C}} M = 3$, $K_M = 0$): the only relevant invariants are those with $k_1 = k_2 = k_3 = 4$. (This is because $k_j \leq 4$ to get a nonzero invariant, so that $6 = \sum(6 - k_j) \geq \sum_{j=1}^{3} 2 = 6$, which implies that each k_j is 4.) The possible location of 0 under a generically injective map is easy to spot: the image curve $\varphi(S^2)$ is some rational curve on M, and meets the four-manifold Z_1 in precisely $\#(Z_1 \cap \eta) = \alpha_1 \cdot \eta$ points; we can choose any of these for the image of 0. Similar remarks about the images of 1 and ∞ lead to the calculation:

$$\Phi_\eta(Z_1, Z_2, Z_3) = (\alpha_1 \cdot \eta)(\alpha_1 \cdot \eta)(\alpha_1 \cdot \eta)\,\#(\mathcal{M}^*_{(\eta,J)}).$$

3.3. The A-model correlation functions

Although we have defined the Gromov–Witten invariants for oriented bordism classes, we will now use them in cohomology instead. As previously remarked, thanks to the triviality of the invariants under the Ω-module structure, if we tensor with $\mathbb{Q}$ we can move the invariants to homology (and then by Poincaré duality, to cohomology). This is at the expense of possibly allowing them to become $\mathbb{Q}$-valued on integral classes. One hopes that they will remain integer valued on integer classes, but this has not yet been established. Therefore, we will give a presentation using $\mathbb{Q}$-coefficients, but the reader should bear in mind that most of the formulas are expected to be valid with integer coefficients if one uses integer cohomology classes.

In brief, the bordism class of $\alpha = (B^k, F)$ gives rise to a homology class $[Z] \in H_k(M, \mathbb{Z})$ (using the image Z of F to represent the class), and by duality to a cohomology class $\zeta = [Z]^\vee \in H^{2n-k}(M, \mathbb{Q})$. (Our retreat to $\mathbb{Q}$-coefficients will be in part because we do not know that every integer cohomology class can be so represented.) We extend the definition of Gromov–Witten invariants to cohomology by defining

$$\Phi_\eta(\zeta_1, \zeta_2, \zeta_3) := \Phi_\eta(\alpha_1, \alpha_2, \alpha_3)$$

when $\alpha_j = (B_j^{k_j}, F_j)$ and $\zeta_j = [\mathrm{Image}(F_j)]^\vee$; then extend by linearity to all of $H^*(M, \mathbb{Q})$.

Our "A-model correlation functions" are then built from the Gromov–Witten invariants, following a calculation from the physics literature [**145, 126, 118, 110**]. There is a certain danger in using the *outcome* of a physics calculation as a *definition*—later, the physicists may become interested in a slightly different problem, whose outcome is radically different from the original one, and we mathematicians will find that our definitions are inadequate.

Nevertheless, we will go ahead and define the A-model correlation functions. These are trilinear functions on the cohomology $H^*(M, \mathbb{Q})$ defined by:

$$\begin{aligned}\langle \zeta_1\,\zeta_2\,\zeta_3 \rangle := (\zeta_1 \cup \zeta_2 \cup \zeta_3)|_{[M]} \quad &+ \sum_{\substack{\eta \in H_2(M,\mathbb{Z}),\\ -K_M\cdot\eta>0}} \Phi_\eta(\zeta_1, \zeta_2, \zeta_3)\, q^\eta \\ &+ \sum_{\substack{0\neq\eta \in H_2(M,\mathbb{Z}),\\ -K_M\cdot\eta=0}} \Phi_\eta(\zeta_1, \zeta_2, \zeta_3) \sum_{m=1}^{\infty} q^{m\eta}\end{aligned} \tag{3.6}$$

It is sometimes convenient to formally sum the geometric series in the final term, and write $q^\eta/(1-q^\eta)$ in place of $\sum_{m=1}^\infty q^{m\eta}$, in which case eq. (3.6) becomes

$$\begin{aligned}\langle \zeta_1\,\zeta_2\,\zeta_3 \rangle := (\zeta_1 \cup \zeta_2 \cup \zeta_3)|_{[M]} \quad &+ \sum_{\substack{\eta \in H_2(M,\mathbb{Z}),\\ -K_M\cdot\eta>0}} \Phi_\eta(\zeta_1, \zeta_2, \zeta_3)\, q^\eta \\ &+ \sum_{\substack{0\neq\eta \in H_2(M,\mathbb{Z}),\\ -K_M\cdot\eta=0}} \Phi_\eta(\zeta_1, \zeta_2, \zeta_3)\, \frac{q^\eta}{1-q^\eta}\end{aligned} \tag{3.7}$$

The terms with $K_M \cdot \eta = 0$ have been separated out because they are where the multiple-covered maps cause the greatest difficulty. Heuristically, the coefficients in these functions (as we have defined them) are expected to count the simple maps only.

The symbol q^η which appears in these formulas has not yet been defined. In fact, there are two natural interpretations of eq. (3.6), one algebraic and one geometric, and we consider them in turn in the next two lectures.

LECTURE 4
The Quantum Cohomology Ring

4.1. Coefficient rings

There are several possible ways to interpret the "A-model correlation functions" defined by eq. (3.6). In this lecture, we will focus on the algebraic interpretation, in which the symbol q^η can be regarded as an element of a group ring or semigroup ring.[1] Recall that for any commutative semigroup $\mathcal{S}$ and any commutative ring R, the *semigroup ring of $\mathcal{S}$ with coefficients in R* is the ring

$$R[q;\mathcal{S}] := \left\{ \sum_{\eta\in\mathcal{S}} a_\eta q^\eta \mid a_\eta \in R \text{ and } \{\eta \mid a_\eta \neq 0\} \text{ is finite} \right\}.$$

The symbol q serves as a placeholder, translating the semigroup operation (usually written additively) into a multiplicative structure on a set of monomials. If $\mathcal{S}$ is a group, this coincides with the usual "group ring" construction.

In the case of a Fano variety, the sum in eq. (3.6) defining the A-model correlation function is finite, and we can regard it as taking values in the rational group ring[2] $\mathbb{Q}[q;H_2(M,\mathbb{Z})]$. To be more concrete, if we assume for simplicity that $H_2(M,\mathbb{Z})$ has no torsion, and choose a basis $e_1, \ldots, e_r$ of $H_2(M,\mathbb{Z})$, then writing $\eta = \sum a^j e_j$ we may associate to η the rational monomial $q^\eta \in \mathbb{Q}(q_1,\ldots,q_r)$ defined by

$$\log q^\eta = \sum a^j \log q_j .$$

(One can also write this multiplicatively:

$$q^\eta = \prod (q_j)^{(a^j)}$$

but then great care is required in distinguishing exponents from superscripts.)

If we choose our basis so that the coefficients a^j are nonnegative for all classes η which have nonvanishing Gromov–Witten invariants $\Phi_\eta(\zeta_1,\zeta_2,\zeta_3)$ for some ζ_1, ζ_2,

[1] I am grateful to A. Givental for pointing out the relevance of group rings.

[2] If we knew that the Gromov–Witten invariants were integers, we could use the integral group ring $\mathbb{Z}[q;H_2(M,\mathbb{Z})]$. But when we passed from bordism to cohomology we lost control of the integer structure.

ζ_3, then each q^η occurring in eq. (3.6) is a *regular* monomial, i.e., q^η belongs to the polynomial ring $\mathbb{Q}[q_1, \ldots, q_r]$, and we can calculate eq. (3.6) in that ring.

In the Calabi–Yau case in which $K_M = 0$, the sum in eq. (3.6) is not finite and we must work harder. The simplest interpretation would be to simply allow infinite sums $\sum a_\eta q^\eta$ as formal expressions. However, in order to construct quantum cohomology (which we shall do in the next section) we need the values of the correlation function to lie in a *ring*. In the definition of semigroup rings one restricts to finite sums in order to ensure that the partial sums which occur in the expansion of a product will be finite. That finiteness can still be guaranteed for products of infinite sums if the semigroup satisfies a special property, given below.

We say that a semigroup $\mathcal{S}$ has the *finite partition property* if for every $\eta \in \mathcal{S}$ there are only finitely many pairs $(\eta_1, \eta_2) \in \mathcal{S} \times \mathcal{S}$ such that $\eta = \eta_1 + \eta_2$. For such semigroups, any expression of the form

$$\sum_{\substack{(\eta_1,\eta_2) \text{ s.t.} \\ \eta_1+\eta_2=\eta}} a_{\eta_1} a_{\eta_2}$$

(for fixed η) will be finite. Thus, infinite sums can be multiplied. So if $\mathcal{S}$ is a semigroup with the finite partition property, we define the *formal semigroup ring of $\mathcal{S}$ with coefficients in R* to be

$$R[[q;\mathcal{S}]] := \{\sum_{\eta\in\mathcal{S}} a_\eta q^\eta\},$$

with the product defined by

$$(\sum_{\eta_1\in\mathcal{S}} a_{\eta_1} q^{\eta_1}) \cdot (\sum_{\eta_2\in\mathcal{S}} a_{\eta_2} q^{\eta_2}) = \sum_{\eta\in\mathcal{S}} (\sum_{\substack{(\eta_1,\eta_2) \text{ s.t.} \\ \eta_1+\eta_2=\eta}} a_{\eta_1} a_{\eta_2}) q^\eta.$$

The semigroup $H_2(M, \mathbb{Z})$ of interest to us is actually a *group* with a nontrivial free abelian part, and so does not satisfy the finite partition property. However, in eq. (3.6) we are only required to sum over classes which can be realized by pseudo-holomorphic curves—these generate a smaller semigroup. If we are using an integrable almost-complex structure J on M for which M is a Kähler manifold, this smaller semigroup is the *integral Mori semigroup* defined (in the case $h^{2,0} = 0$, for simplicity) as

$$\overline{\mathrm{NE}}(M, \mathbb{Z}) := \{\eta \in H_2(M, \mathbb{Z}) \mid (\omega, \eta) \geq 0 \ \forall\ \omega \in \overline{\mathcal{K}}_J\},$$

where $\mathcal{K}_J$ is the Kähler cone and $\overline{\mathcal{K}}_J$ is its closure. The Mori semigroup has the finite partition property (the free part lies in a strongly convex cone, and the torsion part is finite), so we can form the formal semigroup ring $R[[q; \overline{\mathrm{NE}}(M, \mathbb{Z})]]$. Presumably, by using the symplectic version of the Kähler cone, we would find a similar property for the analogous semigroup in the almost-complex case and could form a similar ring in that case.

There is an important variant which we will have occasion to consider. Let $\mathrm{Aut}_J(M)$ be the image in $\mathrm{Aut}\, H_2(M, \mathbb{Z})$ of the group of diffeomorphisms of M compatible with the almost-complex structure J. This group acts on the pseudo-holomorphic curves and so permutes the Gromov–Witten invariants. The values of the A-model correlation function are preserved by the group action, and can be regarded as lying in the ring of invariants

$$R[[q; \overline{\mathrm{NE}}(M, \mathbb{Z})]]^{\mathrm{Aut}_J(M)}.$$

As in the Fano variety case, if we choose an appropriate basis (and assume $H_2(M,\mathbb{Z})$ is torsion-free) then we can regard the correlation function defined in eq. (3.6) as taking values in a formal power series ring $\mathbb{Q}[[q_1,\dots,q_r]]$. Note that if we set all q_j's to 0, we simply recover the topological trilinear function $(\zeta_1\cup\zeta_2\cup\zeta_3)|_{[M]}$. But although the formal series in eq. (3.6) is expected by the physicists to converge near $q_j = 0$, no convergence properties of the series (as we have defined it) are known at present.

There is an alternative to using the semigroup rings: we could instead use the Novikov rings [**82**] which have played a rôle elsewhere in symplectic geometry [**55**]. For each Kähler class ω, the *Novikov ring* Λ_ω consists of all formal power series

$$\sum_{\eta\in H_2(M,\mathbb{Z})} a_\eta q^\eta$$

such that the set

$$\{\eta \mid a_\eta \neq 0 \text{ and } (\omega,\eta) < c\}$$

is finite for all $c \in \mathbb{R}$. (If it is necessary to specify the ring R in which the coefficients a_η take their values, the notation $\Lambda(\omega, R)$ is used.) The product of two elements of Λ_ω is well-defined, and also belongs to Λ_ω. In the case $H_2(M,\mathbb{Z}) = \mathbb{Z}^r$, Λ_ω is the ring of *generalized Laurent series*

$$\{\sum a_{\vec{k}} q^{\vec{k}} \mid \text{ there are only finitely many terms with } \omega\cdot\vec{k} < c \text{ for any } c \in \mathbb{R}\}.$$

4.2. A new algebra structure

The correlation functions defined in the previous lecture can be used to describe a new algebra structure on the cohomology of M, in the following way. Let R be an integral domain (usually we use $R = \mathbb{Z}$ or $R = \mathbb{Q}$), and choose a coefficient ring $\mathcal{R}$ from among

1. the group ring $R[q; H_2(M,\mathbb{Z})]$ (in the case of a Fano variety),
2. the formal semigroup ring with coefficients in R for the Mori semigroup $R[[q; \overline{\mathrm{NE}}(M,\mathbb{Z})]]$ (when this is well-defined, such as in the case of a Kähler manifold),
3. the subring $R[[q; \overline{\mathrm{NE}}(M,\mathbb{Z})]]^{\mathrm{Aut}_J(M)}$ of $\mathrm{Aut}_J(M)$-invariants, or
4. one of the Novikov rings $\Lambda(\omega, R)$.

We introduce a binary operation $\zeta_1 \star \zeta_2$ on $H^*(M,\mathcal{R})$ defined by the requirement

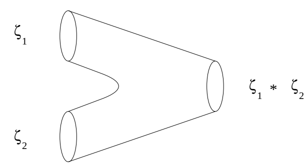
$$((\zeta_1\star\zeta_2)\cup\zeta_3)|_{[M]} = \langle\zeta_1\,\zeta_2\,\zeta_3\rangle.$$

(This is well-defined since the cup product is a perfect pairing.) The class $\mathbb{1} := [M]^\vee \in H^0(M)$ which is dual to the fundamental class $[M] \in H_{2n}(M)$ has the property that the Gromov–Witten invariants $\Phi_\eta(\mathbb{1},\zeta_2,\zeta_3)$ all vanish, hence

$$\langle\mathbb{1}\,\zeta_1\,\zeta_2\rangle = (\zeta_2\cup\zeta_3)|_{[M]};$$

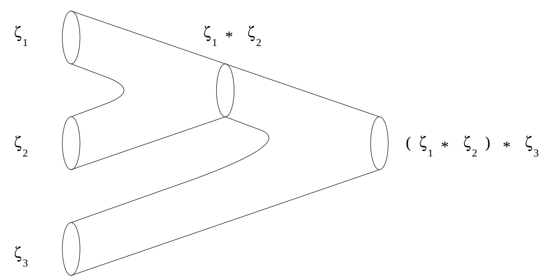
it follows that $\mathbb{1}$ serves as the identity element for the binary operation $\star$.

This interpretation of the correlation function as a binary operation also comes from physics [**141, 151**]. Let us return to the picture we had of the “pair of pants” surface

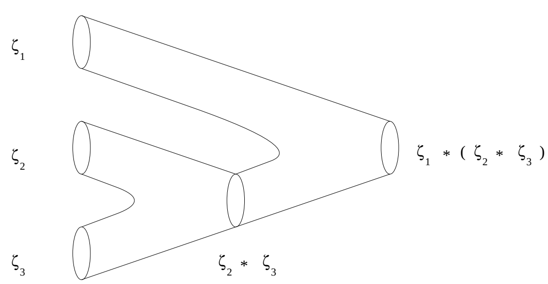


as describing a possible evolution between an initial state with two "incoming" vertex operators ζ_1, ζ_2 on the left and a final state with one "outgoing" vertex operator $\zeta_1 \star \zeta_2$ on the right.

This point of view leads to the remarkable expectation that the binary operation should be associative! A heuristic argument for this runs as follows: the product $(\zeta_1 \star \zeta_2) \star \zeta_3$ is evaluated by means of the surface



(with an outgoing vertex operator of one piece attached to an incoming vertex operator of the other) while the product $\zeta_1 \star (\zeta_2 \star \zeta_3)$ is evaluated by means of the surface



which is a deformation of the first one. So long as the values of the resulting quadrilinear function do not depend on the location of the four points in $\mathbb{CP}^1$ used in defining it, these two products will agree. In fact, as we pointed out in the introduction, the correlation functions we are studying are expected from the physics to be precisely of this "topological" nature which makes them independent of the location of the points [**150**].

This associativity property of the binary operation $\star$ can be rewritten as a set of relations which must be satisfied among the Gromov–Witten invariants themselves. This turns out to be a very deep property, which had not been proved at the time these lectures were delivered (although proofs were given not too long thereafter [**87, 67, 12**]). We have formulated the Gromov–Witten invariants and the binary operation at this level of generality primarily because this associativity property is such an interesting one. However, as we will see in more detail below, for the case of primary interest in mirror symmetry—that of Calabi–Yau threefolds—the associativity is automatic, and there is nothing to prove. (Associativity *does* say something interesting for Calabi–Yau manifolds of higher dimension.)

The $\mathcal{R}$-module $H^*(M, \mathcal{R})$ equipped with the binary operation $\star$ is called the *quantum cohomology ring* of M, or the *quantum cohomology algebra* when we wish to emphasize the $\mathcal{R}$-module structure.

We can give a more geometric description of the new binary operation, by turning each Gromov–Witten invariant itself into a kind of binary operation. Here is a heuristic description of what this construction should look like.

We want a cohomology class $Q_\eta(\zeta_1, \zeta_2)$ with the property that

$$(Q_\eta(\zeta_1, \zeta_2) \cup \zeta_3)|_{[M]} = \Phi_\eta(\zeta_1, \zeta_2, \zeta_3).$$

Consider the set of pseudo-holomorphic curves which satisfy the conditions imposed by ζ_1 and ζ_2 only:

$$\mathcal{M}_\eta(\zeta_1, \zeta_2) := \{\varphi \in \widetilde{\mathcal{M}}^*_{(\eta,J)} \mid \varphi(0) \in Z_1, \varphi(1) \in Z_2\},$$

where $\zeta_j = [Z_j]^\vee$. To count the maps contributing to $\Phi_\eta(\zeta_1, \zeta_2, \zeta_3)$, we must look for all maps in $\mathcal{M}_\eta(\zeta_1, \zeta_2)$ which also send ∞ into Z_3. What subset of M has the property that its intersection with Z_3 is in one-to-one correspondence with such maps? It is the subset consisting of *all possible* points $\varphi(\infty)$ which might be mapping to Z_3. In other words, we can write $Q_\eta(\zeta_1, \zeta_2) = [T_\eta(Z_1, Z_2)]^\vee$, where $T_\eta(Z_1, Z_2)$ is the cycle defined by

$$\begin{aligned} T_\eta(Z_1, Z_2) &:= \{P \in M \mid P = \varphi(\infty) \text{ for some } \varphi \in \mathcal{M}_\eta(\zeta_1, \zeta_2)\} \\ &= \bigcup_{\varphi \in \mathcal{M}_\eta(\zeta_1, \zeta_2)} \operatorname{Image}(\varphi). \end{aligned}$$

Then $T_\eta(Z_1, Z_2) \cap Z_3$ will correspond to the maps counted by $\Phi_\eta(\zeta_1, \zeta_2, \zeta_3)$, where $\zeta_3 = [Z_3]^\vee$. Note that for this heuristic description to work, we need the set $T_\eta(Z_1, Z_2)$ to be of the expected dimension. A better formal definition of $Q_\eta(\zeta_1, \zeta_2)$ would be the pushforward under evaluation at ∞ of the pullback of $\mathcal{M}_\eta(\zeta_1, \zeta_2)$ to the universal family of maps.

Expressed in these terms, then, the binary operation can be written:

$$\begin{aligned} \zeta_1 \star \zeta_2 := \zeta_1 \cup \zeta_2 \quad &+ \sum_{\substack{\eta \in H_2(M,\mathbb{Z}), \\ -K_M \cdot \eta > 0}} q^\eta \, Q_\eta(\zeta_1, \zeta_2) \\ &+ \sum_{\substack{0 \neq \eta \in H_2(M,\mathbb{Z}), \\ -K_M \cdot \eta = 0}} \frac{q^\eta}{1 - q^\eta} \, Q_\eta(\zeta_1, \zeta_2) \end{aligned} \tag{4.1}$$

Recall that the Gromov–Witten invariant $\Phi_\eta(\zeta_1, \zeta_2, \zeta_3)$ with $\zeta_j \in H^{\ell_j}(M, \mathbb{Q})$ is zero unless

$$\ell_1 + \ell_2 + \ell_3 = 2n + 2(-K_M \cdot \eta), \quad \text{and} \quad \ell_j \geq 2.$$

It follows that if the cycle $Q_\eta(\zeta_1, \zeta_2)$ is nonzero, we have

$$Q_\eta(\zeta_1, \zeta_2) \in H^{2n-\ell_3}(M, \mathbb{Q}) = H^{\ell_1+\ell_2-2(-K_M\cdot\eta)}(M, \mathbb{Q}).$$

Thus, if $K_M = 0$, then the binary operation $\star$ preserves the grading on cohomology, while if $-K_M \cdot \eta > 0$ the grading is shifted down by $2(-K_M \cdot \eta)$. But note that in any case, the $\mathbb{Z}/2\mathbb{Z}$-grading on cohomology is preserved.

Note also that $\ell_j \leq 2n$ implies $\ell_1 + \ell_2 + \ell_3 \leq 6n$ and hence $-K_M \cdot \eta \leq 2n$.

Exercise. Show that the semipositivity condition $3 - n < -K_M \cdot \eta$ implies that the grading cannot shift up, it can only shift down.

Example 4.2. (cf. [**138, 149**]) We now compute an example of the quantum cohomology ring. Let $M = \mathbb{CP}^n$ (with ω induced from the Fubini–Study metric, which will ensure semipositivity). The formal semigroup ring in this case can be written as $\mathcal{R} = \mathbb{Q}[[q]]$ (or we could use $\mathcal{R} = \mathbb{Q}[q]$ since we know the sums are finite, this being a Fano variety).

If C is any complex curve on M, then $-K_M \cdot C = d(n+1)$, where d is the degree of the curve. Since $-K_M \cdot C \le 2n$, we must have $d = 1$. So only lines (and constant maps) will contribute to our correlation function.

Now the predicted real dimension of the space of maps $\mathbb{CP}^1 \to M$ whose image L has degree one is

$$2n + 2(-K_M \cdot L) = 2n + 2(n+1) = 4n + 2$$

while the actual dimension is

$$\dim_{\mathbb{R}} \mathrm{PGL}(2, \mathbb{C}) + \dim_{\mathbb{R}} \mathrm{Gr}(\mathbb{CP}^1, \mathbb{CP}^n) = 6 + 2 \cdot 2(n-1) = 4n + 2$$

so we should be able to use the given complex structure to compute the invariants.

The Gromov–Witten invariants are evaluated as follows. A basis for $H^*(M, \mathbb{Q})$ is given by the classes $\zeta^k \in H^{2k}(M, \mathbb{Q})$ where ζ is the class of a hyperplane. We choose k_1, k_2, k_3, satisfying

$$2k_1 + 2k_2 + 2k_3 = 4n + 2$$

and find that there is a *unique* line in $\mathbb{CP}^m$ meeting three fixed linear spaces of codimensions k_1, k_2 and k_3. And there is a unique map sending 0, 1, ∞ to the intersection points with the three linear spaces. Thus,

$$\Phi_L(\zeta^{k_1}, \zeta^{k_2}, \zeta^{k_3}) = 1.$$

Expressed in terms of the binary operation, we find that

$$\zeta^{k_1} \star \zeta^{k_2} = \begin{cases} \zeta^{k_1+k_2} & \text{if } k_1 + k_2 \le n \\ \zeta^{k_1+k_2-n-1}\, q & \text{if } k_1 + k_2 \ge n+1 \end{cases}.$$

It follows that the quantum cohomology ring can be described as:

$$\mathcal{R}[\zeta]/(\zeta^{\star(n+1)} - q).$$

Example 4.3. If we consider the case relevant to string theory ($\dim_{\mathbb{C}}(M) = 3$, $K_M = 0$), we find that the only products which differ from the cup product are products $\zeta_1 \star \zeta_2$, with $\zeta_1, \zeta_2 \in H^2(M)$, and these are given by

$$\zeta_1 \star \zeta_2 := \zeta_1 \cup \zeta_2 \quad + \sum_{0 \ne \eta \in H_2(M,\mathbb{Z})} \Big(\zeta_1(\eta) \cdot \zeta_2(\eta) \cdot \#(\mathcal{M}^*_{(\eta,J)}) \Big) \, \frac{q^\eta}{1 - q^\eta} \, \eta$$

Here, $\#(\mathcal{M}^*_{(\eta,J)})$ denotes the number of curves in class η (counted with appropriate multiplicity).

Remark. Note that the associativity of the binary operation $\star$ is automatically satisfied by threefolds with trivial canonical bundle, since only one of the products being associated can be different from the cup product.

Example 4.4. Let $\lambda \in H^2(M)$ be represented by L, a submanifold of real codimension two. If we define

$$\mathcal{M}_\eta(\zeta) := \{\varphi \in \widetilde{\mathcal{M}}^*_{(\eta,J)} \mid \varphi(1) \in \zeta\},$$

and

$$\Gamma_\eta(\zeta) := [\{P \in M \mid P = \varphi(\infty) \text{ for some } \varphi \in \mathcal{M}_\eta(\zeta)\}]^\vee,$$

then we can expect that

$$Q_\eta(\lambda, \zeta) = \lambda(\eta) \cdot \Gamma_\eta(\zeta).$$

This is because the image of each φ should meet L in precisely $\lambda(\eta)$ points, any of which may be chosen as $\varphi(0)$.

The binary operation can then be written:

$$\begin{aligned}\lambda \star \zeta := \lambda \cup \zeta \quad &+ \sum_{\substack{\eta \in H_2(M,\mathbb{Z}),\\ -K_M \cdot \eta > 0}} \lambda(\eta)\, q^\eta\, \Gamma_\eta(\zeta) \\ &+ \sum_{\substack{0 \neq \eta \in H_2(M,\mathbb{Z}),\\ -K_M \cdot \eta = 0}} \lambda(\eta)\, \frac{q^\eta}{1 - q^\eta}\, \Gamma_\eta(\zeta)\end{aligned}$$

We regard Γ_η as a map on cohomology, and call it the *Gromov–Witten map.*

4.3. Algebraic properties of the correlation functions

Let K be the field of fractions of our coefficient ring $\mathcal{R}$; tensoring the quantum cohomology ring with K makes it into a K-algebra. This quantum cohomology algebra carries some additional structure which makes it into what is known as a *Frobenius algebra.*[3] By definition this is a K-algebra A with a multiplicative identity element $\mathbb{1}$, such that there exists a linear functional $\varepsilon : A \to K$ for which the induced bilinear pairing $(x, y) \mapsto \varepsilon(x \star y)$ is nondegenerate. There does not seem to be a standard name for such a functional; we call it an *expectation function* (cf. [**143**]). If an expectation function exists at all, then most linear functionals on A can serve as expectation functions. If A is $\mathbb{Z}$-graded, we call ε a *graded expectation function* when $\ker(\varepsilon)$ is a graded subalgebra of A (and we call A a *graded Frobenius algebra* when such a function exists). There is much less freedom to choose graded expectation functions.

The cohomology of a compact manifold M has the structure of a graded Frobenius algebra, with multiplication given by cup product, $\mathbb{1}$ given by the standard generator of $H^0(M)$, and a graded expectation function given by "evaluation on the fundamental class." The quantum cohomology algebra is a deformation of this algebra, with the expectation function given by

$$\varepsilon(\zeta) = \langle \zeta\, \mathbb{1}\, \mathbb{1} \rangle,$$

which again can be interpreted as evaluation on the fundamental class. The induced bilinear pairing

$$(\zeta_1, \zeta_1) \mapsto \varepsilon(\zeta_1 \star \zeta_2) = \langle \zeta_1\, \zeta_2\, \mathbb{1} \rangle$$

[3] We follow standard mathematical usage [**4, 11**] and do not require a Frobenius algebra to be commutative; our definition therefore differs slightly from that in [**129**]. However, we will primarily be interested in the even part $H^{ev}(M)$ of the cohomology of M, on which the quantum product will in fact be commutative.

coincides with the usual cup product pairing. Note that the correlation function is also determined by ε and $\star$, via

$$\langle\zeta_1\,\zeta_2\,\zeta_3\rangle = \varepsilon(\zeta_1 \star \zeta_2 \star \zeta_3).$$

That is, rather than specifying the correlation function first and using it to determine the quantum product, we can simply work with the quantum product and the expectation function.

For most symplectic manifolds, the Frobenius algebra structure on quantum cohomology is not graded; however, in the Calabi–Yau case we get the structure of a graded Frobenius algebra.

Generally, given any associative K-algebra A with multiplicative identity, and any linear functional φ on A, the kernel of the bilinear form $(x, y) \mapsto \varphi(x * y)$ is an ideal $\mathcal{J}_\varphi$, and the quotient ring $A/\mathcal{J}_\varphi$ is a Frobenius algebra with expectation function induced by φ. If A is itself a Frobenius algebra with an expectation function ε, then by a theorem of Nakayama [**79**] (see [**11**] for a modern discussion), φ takes the form $\varphi(x) = \varepsilon(\alpha * x)$ for some fixed element $\alpha \in A$, and $\mathcal{J}_\varphi$ coincides with the annihilator of α.

Although the correlation functions determine the ring structure, the opposite does not hold in general—there can be many expectation functions on a given algebra. However, if A is a graded Frobenius algebra of finite length as a K-module and all elements of A have nonnegative degree, then the graded expectation functions on A are in one-to-one correspondence with degree 0 elements of A which are not zero-divisors. (This is because they must all be of the form $\varphi(x) = \varepsilon(\alpha * x)$ for some α which is not a zero-divisor, but every element of degree > 0 must be a zero-divisor.) In particular, in the case of the quantum cohomology algebra of a Calabi–Yau manifold M (equipped with a symplectic structure), we have a graded Frobenius algebra of finite length in which the degree 0 elements are just the one-dimensional vector space $H^0(M)$. This means that the graded expectation function is unique up to multiplication by an element of K, and that the ring structure determines the correlation functions up to this overall factor. (It is not hard to see in the Calabi–Yau case that the graded expectation function is nonzero precisely on the top degree piece $H^{2n}(M)$, where $n = \dim_{\mathbb{C}} M$, and that $H^{2n}(M)$ must also be one-dimensional.) We will see this structure again when we study the B-model correlation functions in lecture six.

LECTURE 5
Moduli Spaces of σ-Models

5.1. Calabi–Yau manifolds and nonlinear σ-models

In this lecture, we wish to give a more geometric interpretation to the A-model correlation functions as defined by eq. (3.6). This geometric interpretation is motivated in part by a study of the moduli spaces of the conformal field theories associated to Calabi–Yau manifolds, so we begin with a description of those moduli spaces.

Let M be a Kähler manifold with $K_M = 0$. Underlying M is a differentiable manifold X of real dimension $2n$. We can regard M as consisting of X together with a chosen integrable almost-complex structure J and a Kähler metric g_{ij}, such that $K_M = 0$. (The complex manifold specified by J will then be denoted X_J.) If ω denotes the Kähler form of the metric, then by a theorem of Calabi [**33**] there is at most one Ricci-flat metric whose Kähler form is cohomologous to ω; by a theorem of Yau [**103**] such a Ricci-flat metric always exists. The global holonomy of such a metric is necessarily contained in $\mathrm{SU}(n)$. (The metric being Kähler implies that its holonomy is contained in $\mathrm{U}(n) \subset SO(2n)$; the Ricci-flatness further restricts the holonomy to $\mathrm{SU}(n)$, and also implies that the canonical bundle is trivial. See [**25**] for an account of these holonomy conditions.)

We use the term *Calabi–Yau manifold* to mean a compact connected orientable manifold X of dimension $2n$ which admits Riemannian metrics whose (global) holonomy is contained in $\mathrm{SU}(n)$. You should be aware that there are some places in the literature (including papers of mine [**73**]) where "Calabi–Yau manifold" is used in the more restrictive sense of a Riemannian manifold with holonomy precisely $\mathrm{SU}(n)$. These alternate definitions will often also insist that a complex structure has been chosen on X.

Given a Calabi–Yau manifold X (in our sense) and a metric on it whose holonomy lies in $\mathrm{SU}(n)$, there always exist complex structures on X for which the given metric is Kähler. If $h^{2,0} = 0$, then there are only a finite number of such complex structures. (If the universal cover is a written as a product of indecomposable pieces, one may apply conjugation on the various factors to obtain other complex structures.) When $h^{2,0} > 0$, however, the complex structures depend on parameters. There are some very interesting cases with $h^{2,0} > 0$, including the famous K3 surfaces, but lack of time in these lectures forces us to assume—with regret—that $h^{2,0} = 0$ henceforth.

The physical model discussed in lecture one which considers maps from surfaces to a six-dimensional target space is a special case of a class of physical theories called "nonlinear σ-models." One regards these as quantum field theories on the surfaces themselves, with various vertex operators and correlation functions derived from the space of maps from the surface to the target. The target should be a fixed Riemannian manifold, usually assumed to be compact.

When the Riemannian metric on the target is (a particular perturbation of) one which has holonomy in $\mathrm{SU}(n)$, the resulting "nonlinear σ-model" is believed to be invariant under conformal transformations of the surface. It thus is a type of "conformal field theory"—an even broader class of physical models which have a rich literature devoted to their study (see [**130**] for an introduction and further references). Conformal field theories typically depend on finitely many parameters, and in the case of a nonlinear σ-model those parameters have a direct geometric interpretation. In the Lagrangian formulation of the theory, one must specify the metric g_{ij} on the target X together with an auxiliary harmonic two-form B on X called the "B-field." (To simplify matters, we take our metrics to have holonomy in $\mathrm{SU}(n)$, even though the true metrics of interest in physics will be perturbations of those; we also assume that $H_2(X, \mathbb{Z})$ has no torsion.[1]) The data consisting of the pair (g_{ij}, B) accounts for all local parameters in the conformal field theory moduli space, so we get at least a good local description of moduli if we specify such a pair. More details about these moduli spaces can be found in [**75**].

Two pairs (g_{ij}, B) and (g'_{ij}, B') will determine isomorphic conformal field theories if there is a diffeomorphism $\varphi : X \to X$ such that $\varphi^*(g'_{ij}) = g_{ij}$, and $\varphi^*(B') - B \in H^2_{\mathrm{DR}}(X, \mathbb{Z})$. (We use the notation $H^k_{\mathrm{DR}}(X, \mathbb{Z})$ to denote the image of integral cohomology in de Rham cohomology.) This second condition arises because the appearance of B in the Lagrangian is always in the form $\int_\Sigma B$, and the Lagrangian is exponentiated (with an appropriate factor of $2\pi i$) in every physically observable quantity.

We call the set of all isomorphism classes of such pairs the *semiclassical nonlinear σ-model moduli space*, or simply the *σ-model moduli space* (for short). This may differ from the actual *conformal field theory moduli space* for three reasons.

1. It may happen that the physical theory does not exist for all values of g_{ij} and B. Most of the study of these theories uses perturbative methods, valid near a limit of "large volume" of the metric, but it may be that the theory breaks down when the volume (either of X, or of images of holomorphic maps into X) becomes too small.
2. On the other hand, there may be a sort of analytic continuation of the theory beyond the region where the σ-model description is valid. (This was shown to occur in [**106, 153**].) It was only claimed above that the specification of (g_{ij}, B) gave good *local* parameters for the moduli.
3. Furthermore, there could be subtle isomorphisms between conformal field theories which do not show up in the σ-model interpretation. This is known to happen in the K3 surface case [**113**], for example (which we have no time

[1] The correct description of the moduli space will be slightly different if torsion is included—see section 5.3 below.

to discuss here)—mirror symmetry provides a new identification of conformal field theories.

We will ignore these phenomena for the present, and concentrate on the "σ-model moduli space" which parameterizes pairs (g_{ij}, B) modulo equivalence.

To study this moduli space using the tools of algebraic geometry, we must choose a complex structure on X. In fact, if we consider the set of triples (g_{ij}, B, J) modulo equivalence, with J being an integrable almost-complex structure for which the metric g_{ij} is a Ricci-flat Kähler metric, then the map from the set of equivalence classes of triples to that of pairs is a finite map. (It is a map of degree two if the holonomy is precisely $SU(n)$.)

On the other hand, we can map the set of triples (g_{ij}, B, J) to the moduli space $\mathcal{M}_{\mathrm{cx}}$ of complex structures on X. That moduli space is quite well-behaved, both locally and globally. The local structure is given by the theorem of Bogomolov–Tian–Todorov [**29, 93, 95**], which says that all first-order deformations are unobstructed. (I recommend Bob Friedman's paper [**43**] for a very readable account of this theorem.) Thus, the moduli space $\mathcal{M}_{\mathrm{cx}}$ will be smooth, and the tangent space at $[J]$ can be canonically identified with $H^1(T^{(1,0)}_{X_J})$. Globally, $\mathcal{M}_{\mathrm{cx}}$ is known to be a quasi-projective variety (if one specifies a "polarization") by a theorem of Viehweg [**96**]. We will study the moduli space $\mathcal{M}_{\mathrm{cx}}$ in more detail (using variations of Hodge structure) in the next section.

The fibers of the map

$$\{(g_{ij}, B, J)\}/\!\sim \;\to\; \mathcal{M}_{\mathrm{cx}} \tag{5.1}$$

(from the set of equivalence classes of triples to the moduli space) are spaces of the form $\mathcal{D}/\Gamma$, with

$$\begin{aligned} \mathcal{D} &= H^2(X,\mathbb{R}) + i\,\mathcal{K}_J \\ \Gamma &= H^2_{\mathrm{DR}}(X,\mathbb{Z}) \rtimes \mathrm{Aut}_J(X). \end{aligned}$$

One hopes that the map (5.1) is some kind of fiber bundle (at least generically); this would require that both the family of Kähler cones and the family of automorphism groups are generically locally constant. This has been shown for the Kähler cones in the case of complex dimension three by Wilson [**99**].

The tangent spaces to the fibers of the map (5.1) can be canonically identified with $H^1((T^{(1,0)}_{X_J})^*)$. Mirror symmetry predicts that X should have a mirror partner Y, such that the moduli spaces of conformal field theories on X and Y should be isomorphic, but with a reversal of rôles of $H^1(T^{(1,0)}_{X_J})$ and $H^1((T^{(1,0)}_{X_J})^*)$. That is, under the isomorphism between the conformal field theory moduli spaces, the part of the tangent space corresponding to $H^1((T^{(1,0)}_{X_J})^*)$ on X should map to the part corresponding to $H^1(T^{(1,0)}_{Y_{J'}})$ on Y, and vice versa.

In particular, the rôles of base and fiber in (5.1) should be reversed. This is at first sight a rather peculiar statement, since the base and the fiber do not look much alike: the base $\mathcal{M}_{\mathrm{cx}}$ is a quasi-projective variety, whereas the fiber $\mathcal{D}/\Gamma$ looks much more like a Zariski open subset of a bounded domain—a typical model for the space is $(\Delta^*)^r$, where Δ^* is the punctured disk.

This is in fact one of the indicators that the conformal field theory moduli space must be analytically continued beyond the realm of σ-models, as suggested in point 2 above. We will see further evidence of this at the end of lecture seven.

5.2. Geometric interpretation of the A-model correlation functions

We turn now to a geometric interpretation of the A-model correlation functions, which in the case of Calabi–Yau manifolds will turn out to be closely related to the spaces $\mathcal{D}/\Gamma$ described above.

In the previous lecture, the symbols q^η were treated purely formally, which allowed us to discuss some algebraic aspects of quantum cohomology. Now, however, we would like to make the new product more geometric by giving specific values to the q^η's, thereby making the quantum cohomology ring into a deformation of the usual cohomology ring. Turning algebraic parameters into geometric data is a familiar task for algebraic geometers; however here, we only have formal parameters. We will describe a natural parameter space as a formal completion of a certain geometric space—if some day someone proves that the series (3.6) and (4.1) are convergent power series, then the true parameter space will be a neighborhood (in the classical topology) of the completion point within the geometric space which we will construct.

Let $\mathcal{R} = \mathbb{Q}[[q; \overline{\mathrm{NE}}(X_J, \mathbb{Z})]]$ be the formal semigroup ring of the integral Mori semigroup. If $\overline{\mathrm{NE}}(X_J, \mathbb{Z})$ is finitely generated, then we can take as the geometric space $\operatorname{Spec} \mathbb{C}[q; \overline{\mathrm{NE}}(X_J, \mathbb{Z})]$ (the spectrum of the semigroup ring), and as its completion the formal scheme $\operatorname{Spf} \mathcal{R}_{\mathbb{C}}$, where $\mathcal{R}_{\mathbb{C}}$ denotes $\mathcal{R} \otimes_{\mathbb{Q}} \mathbb{C}$ and Spf is the formal spectrum. More generally, if the ring of $\operatorname{Aut}_J(X)$-invariants $\mathcal{R}^{\operatorname{Aut}_J(X)}$ is the formal completion of a ring of finite type over $\mathbb{Q}$, we take our completed parameter space to be $\operatorname{Spf}(\mathcal{R}_{\mathbb{C}}^{\operatorname{Aut}_J(X)})$.

In the finitely generated case, this geometric space $\operatorname{Spec} \mathbb{C}[q; \overline{\mathrm{NE}}(X_J, \mathbb{Z})]$ is in a natural way an affine toric variety, and as such admits a rather concrete description: the geometric points are in one-to-one correspondence with the set of semigroup homomorphisms $\operatorname{Hom}_{\mathrm{sg}}(\overline{\mathrm{NE}}(X_J, \mathbb{Z}), \mathbb{C})$, where $\mathbb{C}$ is given the structure of a *multiplicative* semigroup. Any geometric point ξ in the parameter space—regarded as a semigroup homomorphism—specifies compatible values $\xi(q^\eta)$ for the symbols q^η. An important open problem is to decide for which ξ the series expressions (3.6) for the correlation functions converge. If convergent, the correlation functions would become actual $\mathbb{C}$-valued functions on a parameter space (as expected by the physicists), which would be an open subset of $\operatorname{Spec} \mathbb{C}[q; \overline{\mathrm{NE}}(X_J, \mathbb{Z})]$ in the classical topology.

To make this even more concrete, consider the case in which the Mori semigroup is freely generated by elements $e_1, \ldots, e_r$ which also serve as a basis of the lattice $H_2(X, \mathbb{Z})$. In this case, we can define $q_j := q^{e_j}$, and write the ring $\mathcal{R}$ as a formal power series ring $\mathcal{R} = \mathbb{Q}[[q_1, \ldots, q_r]]$. The geometric space $\operatorname{Spec} \mathbb{C}[q_1, \ldots, q_r]$ can then be identified as $\mathbb{C}^r$ with coordinates $q_1, \ldots q_r$. One natural candidate for the open set on which the correlation functions might converge is

$$\{(q_1, \ldots, q_r) \in \mathbb{C}^r \mid 0 \le |q_j| < 1\}.$$

More generally, still assuming that $H_2(X, \mathbb{Z})$ is torsion-free, suppose we choose a basis $e_1, \ldots, e_r$ whose span as a semigroup *contains* $\overline{\mathrm{NE}}(X_J, \mathbb{Z})$. Then the corresponding formal power series ring $\mathbb{Q}[[q_1, \ldots, q_r]]$ contains our coefficient ring $\mathcal{R}$. If we let σ denote the open real cone generated by the dual basis $e^1, \ldots, e^r$, then that formal power series ring can be more canonically described as the formal semigroup ring $\mathcal{R}_\sigma := \mathbb{Q}[[q; \check{\sigma} \cap H_2(X, \mathbb{Z})]]$. The same cone σ can be used to give a canonical description of the open set specified by $0 < |q_j| < 1$ in the form

$$(H^2(X,\mathbb{R}) + i\sigma)/H^2(X,\mathbb{Z}).$$

(To see this, write a general element of $H^2(X,\mathbb{C})$ modulo $H^2(X,\mathbb{Z})$ in the form

$$\frac{1}{2\pi i}\sum(\log q_j)e^j,$$

and note that the condition $0 < |q_j| < 1$ is equivalent to $\mathrm{Im}(\frac{1}{2\pi i}\log q_j) > 0$.) The Mori semigroup $\overline{\mathrm{NE}}(X_J,\mathbb{Z})$ will be contained in the semigroup spanned by $\{e_j\}$ precisely when the cone σ is contained in the Kähler cone of X_J.

We will treat such a choice of cone σ as specifying a coordinate chart on the geometric space we are trying to construct. For any such cone, we define

$$\mathcal{D}_\sigma = H^2(X,\mathbb{R}) + i\,\sigma \subset H^2(X,\mathbb{C})$$

In terms of local coordinates, as pointed out above we have

$$\mathcal{D}_\sigma/H^2(X,\mathbb{Z}) = \{(q_1,\ldots,q_r) \mid 0 < |q_j| < 1\}.$$

The open subset of our desired geometric space will be a partial compactification of this, defined by

$$(\mathcal{D}_\sigma/H^2(X,\mathbb{Z}))^- = \{(q_1,\ldots,q_r) \mid 0 \le |q_j| < 1\}.$$

We call the origin $0 \in (\mathcal{D}_\sigma/H^2(X,\mathbb{Z}))^-$ the *distinguished limit point* in this space.

It is hoped that the expressions for the A-model correlation functions, or for the binary operation $\zeta_1 \star \zeta_2$, will converge in a neighborhood of the distinguished limit point 0 in $(\mathcal{D}_\sigma/H^2(X,\mathbb{Z}))^-$. The different possible choices of σ will correspond to operations—such as blowing up the boundary—which change the compactification without changing the underlying space.

Intrinsically, we can describe $\mathcal{R}_\sigma \otimes \mathbb{C}$ as the formal completion of the local ring of $(\mathcal{D}_\sigma/H^2(X,\mathbb{Z}))^-$ at its distinguished limit point 0.

The geometric space which is emerging from this discussion is very closely related to the space $\mathcal{D}/\Gamma$ which formed part of the nonlinear σ-model moduli space in the case of a Calabi–Yau manifold with $h^{2,0} = 0$. In fact, if $\mathcal{K}_J$ is the Kähler cone of such a Calabi–Yau manifold which can be partitioned into cones σ_α which are spanned by various bases of $H^2(X,\mathbb{Z})$, then $\mathcal{D}/H^2(X,\mathbb{Z})$ is the interior of the closure of the union of the sets $\mathcal{D}_{\sigma_\alpha}/H^2(X,\mathbb{Z})$. Ideally, one could make such a partition in an $\mathrm{Aut}_J(X)$-equivariant way. This would be guaranteed by the following conjecture.

The Cone Conjecture. *Let X be a Calabi–Yau manifold on which a complex structure J has been chosen, and suppose that $h^{2,0}(X) = 0$. Let $\mathcal{K}_J$ be the Kähler cone of X, let $(\mathcal{K}_J)_+$ be the convex hull of $\overline{\mathcal{K}}_J \cap H^2(X,\mathbb{Q})$, and let $\mathrm{Aut}_J(X)$ be the group of holomorphic automorphisms of X. Then there exists a rational polyhedral cone $\Pi \subset (\mathcal{K}_J)_+$ such that $\mathrm{Aut}_J(X).\Pi = (\mathcal{K}_J)_+$.*

A nontrivial case of this conjecture—Calabi–Yau threefolds which are fiber products of generic rational elliptic surfaces with section (as studied by Schoen [**90**])—has been checked by Grassi and the author [**49**]. There are some other pieces of supporting evidence in examples worked out by Borcea [**30**] and Oguiso [**83**].

When this conjecture holds, there is a partial compactification of $\mathcal{D}/\Gamma$ constructed in [**74**] by gluing together the spaces $(\mathcal{D}_{\sigma_\alpha}/H^2(X,\mathbb{Z}))^-$ for an $\mathrm{Aut}_J(X)$-equivariant partitioning of $\mathcal{K}_J$, and modding out by $\mathrm{Aut}_J(X)$. This produces a "semi-toric" partial compactification of the type introduced by Looijenga [**68**]. Because it is covered by explicit coordinate charts, this is a convenient type of compactification for making comparisons of correlation functions.

There is also a "minimal" semi-tori compactification determined from the same data, which partially compactifies $\mathcal{D}/\Gamma$ more directly, adding several new strata but only a single stratum of maximal codimension (the analogue of the "distinguished limit points"). When the cone conjecture holds, the ring of invariants $\mathcal{R}^{\mathrm{Aut}_J(X)}$ is the formal completion of a ring of finite type over $\mathbb{Q}$, and the completion of the local ring of the minimal semi-toric compactification at its distinguished point P coincides with $\mathrm{Spf}(\mathcal{R}_{\mathbb{C}}^{\mathrm{Aut}_J(X)})$. On such a compactification, we will expect

$$\lim_{Q\to P}\langle\zeta_1\,\zeta_2\,\zeta_3\rangle_Q = (\zeta_1\cup\zeta_2\cup\zeta_3)|_{[X]} \tag{5.2}$$

(the "$q_j = 0$ values" in coordinate charts). Such a point is called a "semiclassical limit" in the physics literature [**109**].

5.3. The rôle of torsion in the moduli space

Up to this point, we have not considered the effects of possible torsion in $H_2(X,\mathbb{Z})$ and in fact we have explicitly assumed at several points that there was no torsion. If torsion is present, we can define the formal semigroup ring $\mathcal{R} = \mathbb{Q}[[q;\overline{\mathrm{NE}}(X_J,\mathbb{Z})]]$ as before, and it will have a torsion part $\mathcal{R}_{\mathrm{torsion}}$ whose spectrum is a finite set of geometric points. This can be identified with the set of connected components of our parameter space. It can also be seen in the following description of the σ-model moduli space.

The complete description of the σ-model moduli space (with the torsion included) considers the quantity $e^{2\pi i(B+i\omega)}$ to lie in $\mathrm{Hom}(H_2(X,\mathbb{Z}),\mathbb{C}^*)$. This can be thought of concretely as having a torsion part, together with a free part which lies in the space

$$\mathrm{Hom}(H_2(X,\mathbb{Z})/\mathrm{torsion},\mathbb{C}^*) \cong H^2(X,\mathbb{C}^*) \cong H^2(X,\mathbb{C})/H^2_{\mathrm{DR}}(X,\mathbb{Z})$$

where (as in section 5.1) $H^2_{\mathrm{DR}}(X,\mathbb{Z})$ is the image of $H^2(X,\mathbb{Z})$ in de Rham cohomology, isomorphic to $H^2(X,\mathbb{Z})/\mathrm{torsion}$. A representative of the free part can be written as $B_{\mathrm{free}} + i\omega \in H^2(X,\mathbb{C})$, where ω is the Kähler form and B_{free} is the real two-form which appeared in section 5.1. The torsion part of $e^{2\pi i(B+i\omega)} \in \mathrm{Hom}(H_2(X,\mathbb{Z}),\mathbb{C}^*)$ can be identified with the torsion part of our coefficient ring $\mathcal{R}_{\mathrm{torsion}}$ from the algebraic interpretation. One way to interpret this "B-field with torsion included" is to regard it as an element of $H^2(X,\mathbb{R}/\mathbb{Z})$.

LECTURE 6
Variations of Hodge Structure

6.1. The B-model correlation functions

Our goal in this lecture is to describe the B-model correlation functions and how they are related to variations of Hodge structure. We work with Calabi–Yau manifolds on which complex structures have been chosen. That is, we let W be a complex manifold with $K_W = 0$. The assumption of trivial canonical bundle is needed in order to define the B-model correlation functions.

Let us define

$$H^{-p,q}(W) := H^q(\Lambda^p(T_W^{(1,0)})),$$

and consider all of these groups together:

$$H^{-*}(W) := \bigoplus_{p,q} H^{-p,q}(W).$$

There is a natural ring structure on $H^{-*}(W)$ which can be thought of as a sheaf cohomology version of the cup product pairing:

$$H^q(\Lambda^p(T_W^{(1,0)})) \otimes H^{q'}(\Lambda^{p'}(T_W^{(1,0)})) \to H^{q+q'}(\Lambda^{p+p'}(T_W^{(1,0)})).$$

Note that since these are sheaf cohomology groups, this ring structure is not "topological" in nature; in fact, it depends heavily on the choice of complex structure on W.

Recall that in the case of the A-model correlation functions on a symplectic manifold M, the expectation function which determined the Frobenius algebra structure was a very familiar object, given by evaluating a cohomology class on the fundamental class of M (which determines a canonical map $H^{n,n}(M) \to \mathbb{C}$). By contrast, the ring structure on quantum cohomology was unusual. In this new "B-model" case, however, the ring structure is straightforward but the expectation function is more elusive. To define it, we must choose a nonvanishing global section $\Omega^{\otimes 2}$ of $(K_W)^{\otimes 2}$. This is then used in two steps to specify the expectation function:

$$H^{-n,n}(W) = H^n(\Lambda^n(T_W^{(1,0)})) \xrightarrow{\lrcorner\Omega} H^n(\mathcal{O}_W) \cong \left(H^0(K_W)\right)^* \xrightarrow{\otimes\Omega} \mathbb{C},$$

where the middle isomorphism is Serre duality.

Using this expectation function and the "sheaf cup product" binary operation, we define the B-model correlation functions (in the standard way from the Frobenius algebra structure):

$$\langle \beta_1\, \beta_2\, \beta_3 \rangle = ((\beta_1 \cup \beta_2 \cup \beta_3) \lrcorner\, \Omega) \otimes \Omega.$$

(Once again we have a definition which is inspired by the outcome of a calculation in the physics literature [**147**].) This gives a map

$$H^{-p,q}(W) \times H^{-p',q'}(W) \times H^{-(n-p-p'),n-q-q'}(W) \to \mathbb{C}.$$

Note that as in the A-model case, we actually have a graded Frobenius algebra of finite length, so the expectation function is uniquely defined up to a scalar multiple (which can be absorbed in the choice of $\Omega^{\otimes 2}$.)

In order to relate this correlation function to a more familiar mathematical object, we can proceed as follows: first use the two Ω's to transform two of the arguments, and then use the cup product:

$$\langle \beta_1\, \beta_2\, \beta_3 \rangle = ((\beta_1 \lrcorner\, \Omega) \cup \beta_2 \cup (\beta_3 \lrcorner\, \Omega)).$$

This variant of the correlation function can be regarded as a map

$$H^{n-p,q}(W) \times H^{-p',q'}(W) \times H^{p+p',n-q-q'}(W) \to \mathbb{C},$$

or, if we treat it as a modified "binary operation," as a map

$$H^{n-p,q}(W) \times H^{-p',q'}(W) \to H^{n-p-p',q+q'}(W).$$

This version of the "binary operation" expresses the cohomology $H^*(W)$ as a module over the ring $H^{-*}(W)$. As we shall see, this variant has the pleasant property that it can be directly interpreted in terms of variations of Hodge structure and the differential of the period map. Of course, the original version of the correlation function can be recovered from this, once we have specified $\Omega^{\otimes 2}$.

6.2. Variations of Hodge structure

We now briefly review the theory of variations of Hodge structure, in order to explain the mathematical origin of the B-model correlation functions. Variations of Hodge structure were introduced as a tool for measuring how the complex structure on a differentiable manifold can vary. Good general references for this are Griffiths et al. [**8**], and Schmid [**89**].

There are two primary ways one can view deformations of complex structure. In the first viewpoint, we fix a compact differentiable manifold Y, and consider various integrable almost-complex structures J on Y. Then the set of such, modulo diffeomorphism, is known to be a finite-dimensional space.

In the second viewpoint, we consider proper holomorphic maps $\pi : \mathcal{W} \to S$ with $W_s = \pi^{-1}(s)$ diffeomorphic to Y. Each fiber W_s has an induced structure of a complex manifold. If S is contractable, then π can be trivialized in the C^∞ category, and we can regard π as specifying a family of complex structures. One wants to represent the functor

$$S \mapsto \{\pi : \mathcal{W} \to S\}/(\text{isomorphism}),$$

by maps to a moduli space which has a "universal family." This is generally too much to hope for, but there are often "coarse moduli spaces" whose points are in one-to-one correspondence with the possible complex structures. (The appendices in [**13**] provide good background for moduli problems in general.)

We will study complex structures on Y by studying the Hodge decomposition induced on cohomology by each choice of complex structure. In general, if W_s is a Kähler manifold there is a *Hodge decomposition* of the cohomology:

$$H^k(W_s, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(W_s). \tag{6.1}$$

Now in a family over a contractable base, the bundle of $H^k(W_s, \mathbb{C})$'s may be canonically trivialized. Over more general bases S (assumed to be connected), it is convenient to consider $R^k\pi_*\mathbb{C}_{\mathcal{W}}$, which is simply the sheaf whose local sections are topologically constant families of cohomology classes. This sheaf has the structure of a *local system*: it can be characterized by its fiber $H^k(W_s, \mathbb{C})$ at a particular point $s \in S$ together with a representation of the fundamental group

$$\rho : \pi_1(S, s) \to \operatorname{Aut}(H^k(W_s, \mathbb{C}))$$

which specifies what happens when the locally constant sections are followed around loops. There is useful dictionary [**5**] between local systems and pairs $(\mathcal{H}, \nabla)$ consisting of a holomorphic vector bundle $\mathcal{H}$ on S and a flat holomorphic connection

$$\nabla : \mathcal{H} \to (T_S^{(1,0)})^* \otimes \mathcal{H}.$$

The way the dictionary works is this: given a local system $\mathbb{H}$, define $\mathcal{H} = \mathcal{O}_S \otimes \mathbb{H}$, and $\nabla(\sum \varphi_j h^j) = \sum d\varphi_j \otimes h^j$ for $\{h^j\}$ a local basis of sections of $\mathbb{H}$. Conversely, given $(\mathcal{H}, \nabla)$, define $\Gamma(U, \mathbb{H}) = \{h \in \Gamma(U, \mathcal{H}) \mid \nabla(h) = 0\}$ for every open set U.

In the case of the cohomology local system $R^k\pi_*\mathbb{C}_{\mathcal{W}}$, the associated connection ∇ on $\mathcal{H}^k$ is called the *Gauss–Manin connection*. An explicit version of this Gauss–Manin connection goes like this: if we choose a local basis $\alpha^1, \ldots, \alpha^r$ for the space of sections $\Gamma(U, R^k\pi_*\mathbb{C}_{\mathcal{W}})$, then any $\beta(s) \in \Gamma(U, \mathcal{H}^k)$ can be written $\beta(s) = \sum f_j(s)\alpha^j$ for some coefficient functions $f_j \in \Gamma(U, \mathcal{O}_S)$. Then

$$\nabla(\beta) = \sum df_j \otimes \alpha^j \in \Gamma(U, (T_S^{(1,0)})^* \otimes \mathcal{H}^k).$$

This can be given an interpretation in terms of classical "period integrals" as follows. The basis $\alpha^1, \ldots, \alpha^r$ is dual to some basis $\gamma_1, \ldots, \gamma_r \in H_k(W_{s_0}, \mathbb{C})$. Then the coefficient functions are the period integrals $f_j(s) = \int_{\gamma_j} \beta(s)$. (We use integration to denote the pairing between homology and cohomology.)

The great advantage of expressing everything in terms of the Gauss–Manin connection is that the Gauss–Manin connection can be computed algebraically, without knowing the topological cycles in advance.

Although the sheaf $R^k\pi_*\mathbb{C}_{\mathcal{W}}$ of cohomology groups can be locally trivialized over the base S, the Hodge decomposition (6.1) will vary as we vary the complex structure. The properties of this variation are more conveniently expressed using the *Hodge filtration*:

$$F^p(W_s) := \bigoplus_{p' \geq p} H^{p',k-p'}(W_s) \subset H^k(W, \mathbb{C})$$

rather than the Hodge groups $H^{p,q}(W_s)$ directly. The spaces $F^p(W_s)$ in the Hodge filtration vary holomorphically with parameters, fitting together to form a holomorphic subbundle $\mathcal{F}^p \subset \mathcal{H}^k$. One might also try to construct a bundle of $H^{p,q}$'s by the simple procedure

$$\mathcal{H}^{p,q}_{C^\infty} := \bigcup_{s \in S} H^{p,q}(W_s) \subset \mathcal{H}^k.$$

As the notation indicates, this defines a C^∞ bundle, but it is not in general holomorphic. There is a holomorphic bundle $\mathcal{H}^{p,k-p}$ defined by the exact sequence

$$0 \to \mathcal{F}^{p+1} \to \mathcal{F}^p \to \mathcal{H}^{p,k-p} \to 0, \tag{6.2}$$

but this exact sequence *has no canonical splitting*, and $\mathcal{H}^k$ cannot in general be written as a direct sum of these holomorphic $\mathcal{H}^{p,k-p}$ bundles.

The key property satisfied by the Hodge bundles is known as *Griffiths transversality*: when we differentiate with respect to parameters by using the Gauss–Manin connection, the Hodge filtration only shifts by one, i.e.,

$$\nabla(\mathcal{F}^p) \subset (T_W^{(1,0)})^* \otimes \mathcal{F}^{p-1}.$$

To study the totality of complex structures on W, we can map the moduli space, or any parameter space S for a family, to the classifying space for Hodge structures. Each Hodge structure on a fixed vector space H^k determines a point in a flag variety

$$\mathrm{Flags}_{(f_j)} := \{\{0\} \subset F^k \subset \cdots \subset F^0 = H^k \mid \dim F^j = f_j\},$$

with the f_j's specifying the dimensions of the spaces making up the filtration. The group $\mathrm{GL}(f_0, \mathbb{C})$ acts transitively on such flags, and if we fix a reference flag $F_0^\bullet$, then the flag variety can be described as $\mathrm{GL}(f_0, \mathbb{C}) / \mathrm{Stab}(F_0^\bullet)$. (The stabilizer $\mathrm{Stab}(F_0^\bullet)$ is the group of block lower triangular matrices.) There are some additional conditions which should be imposed to get a good Hodge structure (cf. [**8, 89**]); these restrict us to an open subset $\mathcal{U}$ of a subvariety[1] of the flag variety on which a discrete group Γ acts, and the desired classifying space for Hodge structures is $\mathcal{U}/\Gamma$. The classifying map $S \to \mathcal{U}/\Gamma$ for a family is often referred to as the *period map*.

The tangent space to the flag variety can be described as

$$\bigoplus_j \mathrm{Hom}(F^j/F^{j+1}, H^k/F^j).$$

So another way of stating Griffiths transversality is to say that the differential of the period map $S \to \mathcal{U}/\Gamma$ sends $T_S^{(1,0)}$ to the subspace

$$\bigoplus_j \mathrm{Hom}(F^j/F^{j+1}, F^{j-1}/F^j) = \bigoplus_j \mathrm{Hom}(H^{j,k-j}(W_s), H^{j-1,k+1}(W_s))$$

of the tangent space.

The differential of the map $S \to \mathrm{Flags}_{(f_j)}$ factors through a map $T_S^{(1,0)} \to H^1(T_W^{(1,0)})$ which describes the first-order deformations represented by S at $[W]$. The map which then induces the differential is the map

$$H^1(T_W^{(1,0)}) \to \bigoplus_j \mathrm{Hom}(H^{j,k-j}(W), H^{j-1,k+1}(W)) \tag{6.3}$$

given by sheaf cup product.

The success of this approach to studying the moduli of complex structures derives from the *local Torelli theorem* for Calabi–Yau manifolds, which states that the map (6.3) is injective. This means that at least locally, the moduli space can be accurately described by using variations of Hodge structure. However, that same

[1] We must pass to a subvariety to restrict to the so-called *polarized* Hodge structures—see [**8**] or [**38**] for an explanation of this.

map can now be given a new interpretation, as a B-model correlation function. That is, *the B-model correlation function*

$$H^1(T_W^{(1,0)}) \times H^{j,k-j}(W) \to H^{j-1,k+1}(W)$$

coincides with the differential of the period map!

We now restrict our attention to the middle-dimensional cohomology $H^n(W, \mathbb{C})$. Stated in terms of the Gauss–Manin connection, we find the following "bundle version" of our correlation function [**73**]: given a vector field θ on the moduli space and sections $\alpha \in \mathcal{F}^j$, $\beta \in \mathcal{F}^{j-1}$, the correlation function is

$$\langle \theta\, \alpha\, \beta \rangle = \int_W \nabla_\theta(\alpha) \wedge \beta$$

(where $\nabla_\theta = \theta \lrcorner \nabla$ denotes the directional derivative in direction θ).

However, as used in physics the correlation function is a specific function rather than a map between bundles. To find this interpretation, we will need to choose specific sections of these bundles on which to evaluate the map. It is this issue to which we now turn.

6.3. Splitting the Hodge filtration

Our method for specifying sections of the Hodge bundles will be given in terms of a choice of splitting for the Hodge filtration on the middle-dimensional cohomology $H^n(W, \mathbb{C})$, i.e., a set of splittings of the exact sequences (6.2) (but defined only locally in the parameter space). We determine such a splitting by means of a filtration on *homology*, which we think of as specifying "which periods to calculate."

Let $\mathbb{S}_\bullet$ be a filtration of the homology local system $\mathrm{Hom}(R^n\pi_*\mathbb{C}_\mathcal{W}, \mathbb{C}_S)$ by sub-local systems, and let

$$\mathbb{S}^\ell := \mathrm{Ann}(\mathbb{S}_{\ell-1}) := \{\alpha \in \mathcal{H}^n \mid \int_\gamma \alpha = 0 \ \forall\, \gamma \in \mathbb{S}_{\ell-1}\}.$$

be the associated filtration of annihilators of $\mathbb{S}_\bullet$ in cohomology. We say that $\mathbb{S}_\bullet$ is a *splitting filtration for* $\mathcal{F}^\bullet$ if $(\mathcal{H}^n)_s \cong (\mathcal{F}^p)_s \oplus (\mathbb{S}^{n-p+1})_s$ for every $s \in S$ and for every $0 \le p \le n$. (In this case, $\mathbb{S}^\bullet$ and $\mathcal{F}^\bullet$ are called *opposite filtrations of weight n* [**38**].)

One way of producing examples of splitting filtrations is as follows: fix a point $s \in S$, and consider the conjugate of the Hodge filtration at s_0, namely, $\overline{F^q}_{s_0}$. The "opposite" property for these filtrations is easy to check: by definition

$$(\mathcal{F}^p)_s = H^{n,0}(W_s) \oplus \cdots \oplus H^{p,n-p}(W_s)$$

and so

$$\begin{aligned}(\overline{\mathcal{F}^{n-p+1}})_s &= \overline{H^{n,0}(W_s) \oplus \cdots \oplus H^{n-p+1,p-1}(W_s)} \\ &= H^{0,n}(W_s) \oplus \cdots \oplus H^{p-1,n-p+1}(W_s),\end{aligned}$$

where we have used the fact that $\overline{H^{p,q}(W_s)} = H^{q,p}(W_s)$. The Gauss–Manin connection can be used to extend this from a filtration at one point to a filtration of the local system. Although this filtration only coincides with the conjugate of the Hodge filtration at one point in the parameter space, it remains opposite to the Hodge filtration at all points nearby.

Given a splitting filtration $\mathbb{S}_\bullet$, we define

$$\mathcal{H}_{\mathbb{S}}^{p,q} := \mathcal{F}^p \cap \operatorname{Ann}(\mathbb{S}_{q-1}),$$

on any open set on which $\mathbb{S}_\bullet$ is single-valued. Then

$$\mathcal{H} = \bigoplus_{p=0}^{n} \mathcal{H}_{\mathbb{S}}^{p,q} \quad \text{and} \quad \mathcal{F}^p = \bigoplus_{p' \geq p} \mathcal{H}_{\mathbb{S}}^{p',n-p'}.$$

(This is the promised splitting of the Hodge filtration.) More concretely, this space can be described in terms of conditions on the periods as follows. The sections of $\mathcal{H}_{\mathbb{S}}^{p,q}$ over U are

$$\Gamma(U, \mathcal{H}_{\mathbb{S}}^{p,q}) := \{\beta \in \Gamma(U, \mathcal{F}^p) \mid \int_\gamma \beta = 0 \ \forall\ \gamma \in \mathbb{S}_{q-1}\}.$$

We also define a space of *distinguished sections* of $\mathcal{H}_{\mathbb{S}}^{p,q}$ by

$$\Gamma(U, \mathcal{H}_{\mathbb{S}}^{p,q})_{\text{dist}} := \{\beta \in \Gamma(U, \mathcal{H}_{\mathbb{S}}^{p,q}) \mid d\left(\int_\gamma \beta\right) = 0 \ \forall\ \gamma \in \mathbb{S}_q\}.$$

(That is, the period integrals $\int_\gamma \beta$ are constant for all $\gamma \in \mathbb{S}_q$, and vanish for all $\gamma \in \mathbb{S}_{q-1}$.)

For each $\mathbb{S}_\bullet$, then, we can define specific B-model correlation functions, using the Ω coming from the distinguished section of $\mathcal{H}_{\mathbb{S}}^{n,0}$ (which is well defined up to a complex scalar multiple). This has the advantage that the correlation functions have been turned into actual functions on a parameter space (in accord with the physicists' interpretation) rather than sections of a bundle. The disadvantage is that further parameters—in the form of a choice of splitting—have been introduced. However, the necessity of considering further parameters such as these, on which the correlation functions will depend anti-holomorphically rather than holomorphically, was recently realized in the physics literature [**124**].

In addition to the distinguished n-form Ω, our choice of splitting determines a family of distinguished vector fields which when contracted with Ω yield the distinguished sections of $\mathcal{H}_{\mathbb{S}}^{n-1,1}$. These vector fields can be integrated into *canonical coordinates*, well-defined up to a $\mathrm{GL}(r, \mathbb{C})$ transformation. (The flexibility of that final $\mathrm{GL}(r, \mathbb{C})$ choice comes from the constants of integration, which must also be specified in order to completely determine a set of canonical coordinates.)

A bit more explicitly, if γ_0 spans $\mathbb{S}_0$ and $\gamma_0, \gamma_1, \ldots, \gamma_r$ span $\mathbb{S}_1$, then the distinguished Ω satisfies $\int_{\gamma_0} \Omega = \text{constant}$, and the coordinates are given by

$$\int_{\gamma_1} \Omega, \ldots, \int_{\gamma_r} \Omega.$$

If we start with an arbitrary n-form $\widetilde{\Omega}$, we can write the distinguished n-form as

$$\Omega := \frac{\widetilde{\Omega}}{\int_{\gamma_0} \widetilde{\Omega}}$$

and the canonical coordinates as

$$\frac{\int_{\gamma_1} \widetilde{\Omega}}{\int_{\gamma_0} \widetilde{\Omega}}, \ldots, \frac{\int_{\gamma_r} \widetilde{\Omega}}{\int_{\gamma_0} \widetilde{\Omega}}.$$

This is the most general possible form for canonical coordinates (and a distinguished n-form) needed for the physical theory, according to recent work in physics [**115**].

Let us fix a splitting filtration $\mathbb{S}_\bullet$. Consider a basis $\{\beta^i\}$ of $\mathcal{H}^n$ consisting of distinguished sections of the bundles $\mathcal{H}_{\mathbb{S}}^{p,q}$ (ordered so that the basis is also adapted to the Hodge filtration $\mathcal{F}^\bullet$), and a multi-valued basis $\{\gamma_j\}$ of the homology local system $\mathrm{Hom}(R^n\pi_*\mathbb{C}_{\mathcal{W}}, \mathbb{C}_S)$, adapted to the splitting filtration $\mathbb{S}_\bullet$. Then the period matrix $(\int_{\gamma_j}\beta^i)$ (which has multi-valued entries) will take a block upper triangular form with constant diagonal blocks. And if we calculate the connection matrix in the basis $\{\beta^i\}$, it takes the special form

$$\begin{pmatrix} 0 & A_0^1 & 0 & & \cdots & 0 \\ & 0 & A_1^1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & & 0 & A_{n-1}^1 \\ 0 & 0 & \cdots & & & 0 \end{pmatrix}$$

in which the only nonzero entries are in the first block superdiagonal of the matrix. The entries A_j^1 precisely contain the data for the B-model correlation functions, calculated in our distinguished basis.

LECTURE 7
The A-Variation of Hodge Structure

7.1. Variations of Hodge structure near the boundary of moduli

In this lecture, we begin by reviewing the asymptotic behavior of a variation of Hodge structure near the boundary of moduli space, and the behavior of the B-model correlation functions there. Comparing to the A-model correlation functions will reveal some similarities—this is one of the hints of mirror symmetry. We make the similarities even more apparent by using the A-model correlation functions to construct a new variation of Hodge structure, which we call the A-variation of Hodge structure.

Let $S = (\Delta^*)^r \subset \overline{S} = \Delta^r$, and suppose we are given a family $\pi : \mathcal{W} \to S$ of complex manifolds. We will assume that there is a way to complete this to a family $\bar{\pi} : \overline{\mathcal{W}} \to \overline{S}$ in which $\bar{\pi}$ is still proper (but no longer smooth). Thus, $0 \in \overline{S}$ is a boundary point in the parameter space. Pick a basepoint $s \in S$; then the fundamental group $\pi_1(S, s)$ is generated by loops $\gamma^{(1)}, \ldots, \gamma^{(r)}$ with $\gamma^{(j)}$ homotopic to the standard generator of $\pi_1(\Delta_j^*)$, where Δ_j^* is the j^{th} factor in $(\Delta^*)^r$.

The Monodromy Theorem (Landman [**65**]). *The action of each generator $\gamma^{(j)}$ gives a quasi-unipotent automorphism $T^{(j)}$ of $H^k(W_s, \mathbb{Q})$, i.e., $(((T^{(j)})^{b_j} - I)^{r_j} = 0$. (This is called* unipotent *if $b_j = 1$.)*

We will restrict attention to the unipotent case. This is partially for technical convenience, but in fact, in the examples which have been calculated for mirror symmetry purposes, only unipotent monodromy transformations have played a rôle.

When $T^{(j)}$ is unipotent, its logarithm can be defined by the following sum (which is finite).

$$N^{(j)} := \log T^{(j)} := (T^{(j)} - I) - \frac{1}{2}\,(T^{(j)} - I)^2 + \cdots .$$

(Note that the $T^{(j)}$'s and $N^{(j)}$'s all commute.) Let $z_1, \ldots, z_r$ be coordinates on $\overline{S}$, with z_j a coordinate on the j^{th} disk. Consider the operator

$$\mathcal{N} := \exp\left(-\frac{1}{2\pi i}\sum \log z_j\, N^{(j)}\right) = \\ I + \left(-\frac{1}{2\pi i}\sum \log z_j\, N^{(j)}\right) + \frac{1}{2!}\left(-\frac{1}{2\pi i}\sum \log z_j\, N^{(j)}\right)^2 + \cdots$$

(also a finite sum). For any section e of the local system $R^k\pi_*(\mathbb{C}_{\mathcal{W}})$, a simple calculation shows that

$$\nabla(\mathcal{N}(e)) = -\frac{1}{2\pi i}\sum \frac{dz_j}{z_j}\, N^{(j)}(e). \tag{7.1}$$

The key facts about the asymptotic behavior are as follows.

The Nilpotent Orbit Theorem (Schmid [**89**]). *Assume that each monodromy transformation $T^{(j)}$ is unipotent. Let $e_1(s), \ldots, e_r(s)$ be a multi-valued basis of $R^k\pi_*(\mathbb{C}_{\mathcal{W}})$, and let $\eta_\ell := \mathcal{N}(e_\ell)$. Then each η_ℓ is a single-valued section of $\mathcal{H}^k$ on S, and together they can be used to generate an extension $\overline{\mathcal{H}}^k$ of $\mathcal{H}^k$ to $\overline{S}$. By eq.* (7.1)*, the Gauss–Manin connection extends to a connection on $\overline{\mathcal{H}}^k$ (again denoted by ∇) with* regular singular points*, i.e., the extended connection is a map*

$$\nabla : \overline{\mathcal{H}}^k \to (T^{(1,0)}_{\overline{S}})^*(\log B) \otimes \overline{\mathcal{H}}^k$$

where $(T^{(1,0)}_{\overline{S}})^*(\log B)$ is the free $\mathcal{O}_{\overline{S}}$-module generated by $\frac{dz_j}{z_j}$, $j = 1, \ldots, r$. Moreover, the Hodge bundles $\mathcal{F}^p$ have locally free extensions to subbundles $\overline{\mathcal{F}}^p \subset \overline{\mathcal{H}}^k$ such that

$$\nabla(\overline{\mathcal{F}}^p) \subset (T^{(1,0)}_{\overline{S}})^*(\log B) \otimes \overline{\mathcal{F}}^{p-1}.$$

The asymptotic behavior as $z_j \to 0$ of the B-model correlation functions

$$\langle \theta\, \alpha\, \beta\rangle = \int_W \nabla_\theta(\alpha) \wedge \beta$$

can be deduced from this theorem. If we let $\theta_j = 2\pi i\, z_j\, \frac{d}{dz_j}$ (chosen to remove poles in the asymptotic expression for the correlation function) then the leading term in $\langle \theta_j\, \eta_\ell\, \beta\rangle$ is given by the monodromy:

$$\lim_{z_j \to 0} \langle \theta_j\, \eta_\ell\, \beta\rangle = -\int_W N^{(j)}(e_\ell) \wedge \beta. \tag{7.2}$$

The essential properties of the monodromy are captured by the *monodromy weight filtration* $\mathbb{W}_\bullet$ on the cohomology, which has the properties that $N^{(j)}\mathbb{W}_\ell \subset \mathbb{W}_{\ell-2}$, and that for any positive real numbers $a_1, \ldots, a_r$, the operator $N := \sum a_j N^{(j)}$ induces isomorphisms $N^\ell : \mathrm{Gr}^{\mathbb{W}}_{n+\ell} \to \mathrm{Gr}^{\mathbb{W}}_{n-\ell}$. Any splitting filtration which we use to make calculations of B-model correlation functions must be somehow compatible with this monodromy weight filtration, if those calculations are to make sense near the boundary.

If mirror symmetry is going to hold, there must be a correspondence between the limiting behaviors described in eqs. (5.2) and (7.2). In fact, the first thing to notice is that the natural flat coordinates on the A-model moduli space are multiple-valued, with the ambiguity precisely specified by $H^2_{\mathrm{DR}}(M, \mathbb{Z})$. So there must be some part of the monodromy weight filtration which matches that behavior. This motivated the following definition, first given in [**73, 74**] (cf. also [**38**]).

We say that a boundary point is *maximally unipotent* if

$$\mathcal{H}_s = (\mathcal{F}^n)_s \oplus (\mathbb{W}_{2n-2})_s$$

and

$$\mathcal{H}_s = (\mathcal{F}^{n-1})_s \oplus (\mathbb{W}_{2n-4})_s$$

for all s near the point. With this definition, the distinguished holomorphic n-form and the canonical coordinates can be defined as in lecture six.

There is an alternate version of this "maximally unipotent monodromy" condition, which agrees with the original one for Calabi–Yau threefolds, but is more restrictive in higher dimension. We say that a boundary point is *strongly maximally unipotent* if the weight filtration $\mathbb{W}_\bullet$ has nontrivial graded pieces in even degree only, and if the induced filtration on homology defined by

$$\mathbb{S}_\ell := \mathrm{Ann}(\mathbb{W}_{2n-2\ell+2})$$

is a splitting filtration. (Note that the corresponding filtration on cohomology is then

$$\mathbb{S}^\ell := \mathrm{Ann}(\mathbb{S}_{\ell-1}) = \mathbb{W}_{2n-2\ell};$$

this is the filtration which should be opposite to the Hodge filtration.) In this case, we will be able to use distinguished sections to calculate B-model correlation functions, as explained earlier.

At the moment, only the original version of the definition has been justified to the satisfaction of physicists as an appropriate characterization of points which should be useful for mirror symmetry. To completely carry out a mirror symmetry type calculation, though, the second version would seem to be necessary. And as we shall see, that version has been extremely successful in examples.

Actually, even just at the level of the monodromy action, the parallels between the structure of the Lefschetz operators on the cohomology and the action of monodromy are rather striking, as was first observed by Cattani, Kaplan and Schmid [**35**]. The operators $\mathrm{ad}(e^j)$ describe Lefschetz decompositions of the cohomology of M, which have many structural parallels to the monodromy weight filtration at a maximally unipotent point.

7.2. Reinterpreting the A-model correlation functions

Let M be a Calabi–Yau manifold on which a complex structure and Kähler metric have been fixed. Inspired by some of the similarities between the two different types of correlation functions, we wish to improve the analogy by translating the A-model correlation functions into data describing a variation of Hodge structure. Consider the moduli space $\mathcal{D}/\Gamma$ for A-model correlation functions, and a coordinate chart specified by a cone σ:

$$\begin{array}{ccccc} \mathcal{D}/\Gamma & \leftarrow & \mathcal{D}_\sigma/H^2(M,\mathbb{Z}) & \cong & (\Delta^*)^r \\ & & \cap\!| & & \cap\!| \\ & & (\mathcal{D}_\sigma/H^2(M,\mathbb{Z}))^- & \cong & \Delta^r \end{array}$$

We assume that the cone σ (which we call a *framing*) is generated by a basis e^1, $\dots$, e^r of $H^2(M,\mathbb{Z})$. Let t_1, $\dots$, t_r be coordinates on $H^2(M,\mathbb{C})$ dual to this basis (so that elements of $H^2(M,\mathbb{C})$ take the form $\sum t_j e^j$). The natural vector fields for making calculations of correlation functions which involve a term from the tangent space $H^2(M,\mathbb{C})$ are the vector fields $\partial/\partial t_j$. These are the analogues of the distinguished vector fields which we had on the B-model side.

On the other hand, natural coordinates on $\mathcal{D}_\sigma/H^2(M,\mathbb{Z})$ are furnished by $q_j = \exp(2\pi i\, t_j)$. Then

$$\frac{\partial}{\partial t_j} = 2\pi i\, q_j \frac{\partial}{\partial q_j},$$

from which we conclude that those correlation functions should naturally be evaluated on the basis $2\pi i\, q_j\, \partial/\partial q_j$ of the sheaf of logarithmic vector fields on the space $(\mathcal{D}_\sigma/H^2(M,\mathbb{Z}))^-$.

We identify $\partial/\partial t_j$ with the operation of taking the quantum product with the basis element $e^j \in H^2(M,\mathbb{Q})$. The resulting map is determined by the correlation functions of the form $\langle e^j\, \alpha\, \beta\rangle$. We had a particularly simple form for these correlation functions, given in example 4.4, in terms of the Gromov–Witten maps Γ_η. We now wish to reinterpret that formula in the following way. We will describe a holomorphic bundle[1] $\mathcal{E} := \left(\bigoplus H^{\ell,\ell}(M)\right) \otimes \mathcal{O}_{(\mathcal{D}_\sigma/H^2(M,\mathbb{Z}))^-}$ with a connection[2] (with regular singular points)

$$\nabla := \frac{1}{2\pi i}\left(\sum d\log q_j \otimes \mathrm{ad}(e^j) + \sum_{0\neq\eta\in H_2(M,\mathbb{Z})} d\log\left(\frac{1}{1-q^\eta}\right)\otimes\Gamma_\eta\right)$$

which was derived from the formulas for $e^j\star$, where $\mathrm{ad}(e^j) : H^k(M) \to H^{k+2}(M)$ is defined by $\mathrm{ad}(e^j)(A) = e^j \cup A$. We also define a "Hodge filtration"

$$\mathcal{E}^p := \left(\bigoplus_{0\le\ell\le m-p} H^{\ell,\ell}(M)\right)\otimes\mathcal{O}_{(\mathcal{D}_\sigma/H^2(M,\mathbb{Z}))^-}.$$

This describes a structure we call the *framed A-variation of Hodge structure with framing σ*. To be a bit more precise, we should study "formally degenerating variations of Hodge structure," since the series used to define ∇ is only formal. (We won't formulate that theory in detail here.)

The connection ∇ which we defined from the Gromov–Witten invariants is in fact a *flat* holomorphic connection [**151**]. The flatness follows from the associativity (and commutativity) of the binary operation. In fact, since the directional derivatives with respect to ∇ corresponded to binary products $e^j \star \zeta$ (where e^j describes the direction of the derivative), iterated directional derivatives have the form $e^k \star (e^j \star \zeta)$. We would simply need to know that reversing the order of j and k produces the same result, and this is guaranteed by the commutativity and associativity.

In particular, the flatness is automatic when $\dim_{\mathbb{C}} M = 3$, a case in which there is no issue of associativity. The recent proofs of associativity of quantum cohomology [**87, 67, 12**] guarantee that this connection is flat in arbitrary dimension.

---

[1] There are a few variants to this construction, in which one uses slightly different bundles. Essentially, one can restrict to any subbundle of $\bigoplus H^{\ell,\ell}(M)$ which is preserved by cup products with the part of $H^{1,1}(M)$ which it contains.

[2] I am indebted to P. Deligne for advice which led to this form of the formula (cf. [**38**]).

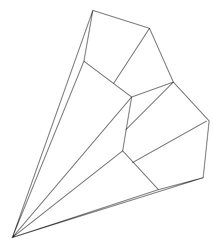


Figure 1. Adjacent Kähler cones

As in the geometric case, there is an additional structure associated to this variation of Hodge structure: a local system. The local system on homology takes the simple form

$$\mathbb{S}_\ell := H_{0,0} \oplus H_{1,1} \oplus \cdots \oplus H_{\ell,\ell},$$

and the corresponding local system on cohomology then becomes

$$\mathbb{S}^\ell = H^{\ell,\ell} \oplus H^{\ell+1,\ell+1} \oplus \cdots \oplus H^{n,n}.$$

The logarithms of the monodromy actions which define these local systems are specified by the topological pairings, and coincide with the cup-product maps

$$H^2(M,\mathbb{Z}) \otimes \mathbb{S}^\ell \to \mathbb{S}^{\ell+1}.$$

In the next lecture, we will formulate a precise conjecture which equates this A-variation of Hodge structure with the geometric variation of Hodge structure on a mirror partner.

7.3. Beyond the Kähler cone

We indicated in lecture five that the conformal field theory moduli space is actually *larger* than the nonlinear σ-model moduli space. We can now explain how this comes about—it is due to an analysis of the effect of flops on the conformal field theory.

Flops are birational transformations among Calabi–Yau threefolds which have been studied extensively as part of the minimal model program (see for example [**1**]). The effect of flops on the Kähler cone of a Calabi–Yau threefold is as follows. Given a Calabi–Yau threefold X with a complex structure J, and a linear system $|L|$ inducing a flopping contraction from X_J to $\widehat{X}_{\widehat{J}}$, the Kähler cones $\mathcal{K}_J$ and $\widehat{\mathcal{K}}_{\widehat{J}}$ share a common wall, which contains the class of $|L|$, as depicted in figure 1. The Kähler cone has already occurred in our discussion of the moduli spaces of σ-models. The natural question arises: suppose we attempt to "attach" the moduli spaces $\mathcal{D}/\Gamma$ and $\widehat{\mathcal{D}}/\widehat{\Gamma}$ along (the images of) their common wall? In fact, it now appears likely that the conformal field theory moduli spaces of X and $\widehat{X}$ are analytic continuations of each other, and that this "attached space" is a part of the full conformal field theory moduli space [**106, 153**]. (This at least seems to happen in examples—the arguments for this rely on mirror symmetry, and involve finding regions in the mirror's moduli space which correspond to the X_J and $\widehat{X}_{\widehat{J}}$ theories, respectively.) One of the consequences of this would be an analytic continuation of correlation functions from $\mathcal{D}/\Gamma$ to $\widehat{\mathcal{D}}/\widehat{\Gamma}$.

Here is a formal calculation from [**153, 108**] which supports this analytic continuation idea (see also [**76**] for a more mathematical treatment). The union of all of the Kähler cones of birational models of X_J is known as the *movable cone* $\operatorname{Mov} X_J$ [**61**]. We compute in the formal semigroup ring $\mathbb{Q}[q; \operatorname{Mov}(X_J)^\vee]$ (which we identify canonically with the same ring for $\widehat{X}_{\widehat{J}}$), and so the computation is purely formal.

Consider the simplest flop: the flop based on a collection of disjoint holomorphic rational curves $\Gamma_i \subset X_J$ (in a common homology class $[\Gamma]$) such that the normal bundle is $N_{\Gamma_i/X_J} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. (These curves must be flopped simultaneously in order to ensure that the flopped variety is Kähler.) A reasonable genericity assumption about the *other* rational curves on X_J is this: all (pseudo-)holomorphic curves in classes $\eta \notin \mathbb{R}_{>0}[\Gamma]$ are disjoint from the Γ_i's. Since there is a proper transform map on divisors, the Gromov–Witten invariants (which in this case are determined entirely by intersection properties of η and the number of elements in $\mathcal{M}^*_{(\eta,J)}$) do not change when passing from X_J to $\widehat{X}_{\widehat{J}}$, except for the invariants $\Phi_{[\Gamma]}$ themselves. The cup product can also change.

The A-model correlation functions on X_J can be written in the form

$$\begin{aligned}\langle A\, B\, C\rangle = A\cdot B\cdot C &+ \frac{q^{[\Gamma]}}{1-q^{[\Gamma]}}\,(A\cdot\Gamma)(B\cdot\Gamma)(C\cdot\Gamma)\,n_\Gamma \\ &+ \sum_{\substack{\eta\in H_2(X,\mathbb{Z})\\ \eta\neq\lambda\Gamma}} \frac{q^\eta}{1-q^\eta}\,\Phi_\eta(A,B,C).\end{aligned}$$

Only the first terms change when passing to $\widehat{X}_{\widehat{J}}$ and in fact we claim that

$$\begin{aligned}A\cdot B\cdot C &+ \frac{q^{[\Gamma]}}{1-q^{[\Gamma]}}\,(A\cdot\Gamma)(B\cdot\Gamma)(C\cdot\Gamma)\,n_\Gamma \\ &= \widehat{A}\cdot\widehat{B}\cdot\widehat{C} + \frac{q^{[\widehat{\Gamma}]}}{1-q^{[\widehat{\Gamma}]}}\,(\widehat{A}\cdot\widehat{\Gamma})(\widehat{B}\cdot\widehat{\Gamma})(\widehat{C}\cdot\widehat{\Gamma})\,n_{\widehat{\Gamma}},\end{aligned}$$

where $\widehat{A}$, $\widehat{B}$, and $\widehat{C}$ are the proper transforms of A, B, and C. (In other words, the change in the topological term is precisely compensated for by the change in the $q^{[\Gamma]}$ term.)

We will check this formula in the case in which A and B meet one of the curves Γ transversally at a and b points, respectively, and $\widehat{C}$ meets $\widehat{\Gamma}$ transversally at c points. (The general case can be deduced from this one.) Then C must contain Γ with multiplicity c, and the configuration of divisors is as in figure 2 (which illustrates the case $a = b = c = 1$ for simplicity). A and B have no intersection points along Γ, but both $\widehat{A}$ and $\widehat{B}$ contain $\widehat{\Gamma}$, and they meet $\widehat{C}$. The total number of intersection points of $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ (counted with multiplicity) which lie in $\widehat{\Gamma}$ is thus abc.

Since a similar thing happens for each curve Γ_i in the numerical equivalence class, we see that

$$\widehat{A}\cdot\widehat{B}\cdot\widehat{C} - A\cdot B\cdot C = abc\, n_\Gamma = -(A\cdot\Gamma)(B\cdot\Gamma)(C\cdot\Gamma)\,n_\Gamma \tag{7.3}$$

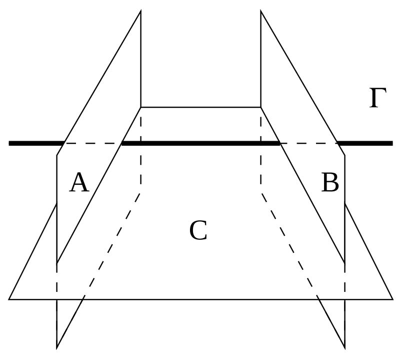


Figure 2a. Before the flop.

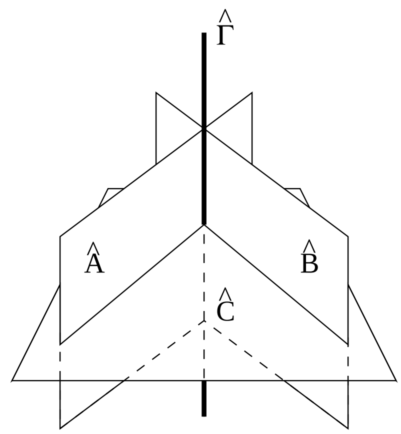


Figure 2b. After the flop.

(using $A \cdot \Gamma = a$, $B \cdot \Gamma = b$, $C \cdot \Gamma = -c$). On the other hand, since $[\widehat{\Gamma}] = -[\Gamma]$ and $n_{\widehat{\Gamma}} = n_\Gamma$, we can compute:

$$
\begin{aligned}
&\frac{q^{[\Gamma]}}{1-q^{[\Gamma]}}\,(A\cdot\Gamma)(B\cdot\Gamma)(C\cdot\Gamma)\,n_\Gamma - \frac{q^{[\widehat{\Gamma}]}}{1-q^{[\widehat{\Gamma}]}}\,(\widehat{A}\cdot\widehat{\Gamma})(\widehat{B}\cdot\widehat{\Gamma})(\widehat{C}\cdot\widehat{\Gamma})\,n_{\widehat{\Gamma}}\\
&\quad=\frac{q^{[\Gamma]}}{1-q^{[\Gamma]}}\,(A\cdot\Gamma)(B\cdot\Gamma)(C\cdot\Gamma)\,n_\Gamma + \frac{q^{-[\Gamma]}}{1-q^{-[\Gamma]}}(A\cdot\Gamma)(B\cdot\Gamma)(C\cdot\Gamma)\,n_\Gamma\\
&\quad=\left(\frac{q^{[\Gamma]}}{1-q^{[\Gamma]}}+\frac{1}{q^{[\Gamma]}-1}\right)(A\cdot\Gamma)(B\cdot\Gamma)(C\cdot\Gamma)\,n_\Gamma\\
&\quad=-\,(A\cdot\Gamma)(B\cdot\Gamma)(C\cdot\Gamma)\,n_\Gamma.
\end{aligned}
\tag{7.4}
$$

Adding eqs. (7.3) and (7.4) proves the desired formula.

The conclusion from all of this should be that the mirror symmetry phenomenon is really about birational equivalence classes. For if there is any analytic continuation of the correlation function from the region associated to $\mathcal{K}_J$ out into the next cone $\widehat{\mathcal{K}}_{\widehat{J}}$, the calculation above shows that this analytic continuation must in fact reproduce the correlation function of the flopped model $\widehat{X}_{\widehat{J}}$.

It is tempting to think that if we combined the σ-model moduli spaces for all birational models of X we would fill out the entire conformal field theory moduli space. However, some examples that have been worked out by Witten [**153**] and by Aspinwall, Greene and the author [**107**] show that this is not the case. In those examples, there are other regions in the moduli space which correspond to rather different kinds of physical model, including some called *Landau–Ginzburg theories* which will play a rôle again in the next lecture.

LECTURE 8
Mirror Symmetry

8.1. Mirror manifold constructions

The original speculations about mirror symmetry were based on the appearance of arbitrariness of a choice that was made in identifying certain constituents of the conformal field theory associated to a Calabi–Yau manifold with geometric objects on the manifold. The distinction between vertex operators which appear in the A-model and B-model correlation functions is simply a difference in sign of a certain quantum number; if that sign is changed, the geometric interpretation is altered dramatically. This led Dixon [**128**] and Lerche–Vafa–Warner [**140**] to propose that there might be a second Calabi–Yau manifold producing essentially the same physical theory as the first, but implementing this change of sign.

Some time later,[1] an explicit construction was made by Greene and Plesser [**135**] which showed that this phenomenon does indeed occur in physics. The construction rests on a chain of equivalences which are believed to hold among different physical models, as follows.

1. Certain σ-models on Calabi–Yau manifolds are believed to correspond to so-called Landau–Ginzburg theories [**137**]. (It has recently been recognized [**153**] that this correspondence is not direct, but involves analytic continuation on the moduli space.) Roughly speaking, the class of Calabi–Yau manifolds for which this correspondence can be made is the class of ample anti-canonical hypersurfaces in toric varieties. Such a hypersurface will have an equation of the form $\Phi(x_1, \dots, x_{n+1}) = 0$ (in some appropriate coordinates on the torus), and this same polynomial is used as a "superpotential" in constructing the Landau–Ginzburg theory.
2. Certain Landau–Ginzburg theories—quotients of the ones for which the superpotential is of "Fermat type"

$$\Phi(x_1, \dots, x_{n+1}) = x_1^{d_1} + \cdots + x_{n+1}^{d_{n+1}}$$

[1] At about the same time, another important piece of evidence for mirror symmetry was given by Candelas, Lynker, and Schimmrigk [**123**], who found an almost perfect symmetry under the exchange $h^{1,1} \leftrightarrow h^{2,1}$ on the set of Hodge numbers coming from Calabi–Yau threefolds which can be realized as weighted projective hypersurfaces.

by certain finite groups Γ—are believed to correspond to yet another type of conformal field theory. This other theory is described in terms of discrete series representations $V^{(k)}$ of the "N=2 superconformal algebra," and it takes the form

$$\left(\bigotimes_j V^{(d_j+2)}\right)/G$$

where G is a slight enlargement of the group Γ. (Note that the case of Γ being trivial is allowed, but then G is not trivial.)

The representation theory of the N=2 superconformal algebra is related to these things by analyzing the conformal field theory on an infinite cylinder. (The superconformal algebra can be described in terms of automorphisms of the cylinder.)

3. By studying the representation theory, Greene and Plesser find a kind of duality among the finite groups G: there is a dual group $\widehat{G}$ and an isomorphism

$$\left(\bigotimes_j V^{(d_j+2)}\right)/G \cong \left(\bigotimes_j V^{(d_j+2)}\right)/\widehat{G}$$

which has the "sign-reversing property" of mirror symmetry.
4. The duality can be extended to the groups Γ, and the mirror Landau–Ginzburg theory of Φ/Γ is $\Phi/\widehat{\Gamma}$. This looks a bit asymmetric, since for example the case Γ trivial leads to a rather large group $\widehat{\Gamma}$. But the group Γ continues to act as a group of "quantum symmetries" on the quotient theory, in a way that restores symmetry to this construction.
5. Finally, the Calabi–Yau which is the quotient of the Fermat hypersurface by Γ should have as its mirror the one which is the quotient by $\widehat{\Gamma}$.

This is called the *Greene–Plesser orbifolding construction.*

There is a conjectural generalization of this construction, which as of yet has no basis in conformal field theory—it is simply a mathematician's guess. This generalization would work for an arbitrary family of Calabi–Yau hypersurfaces in toric varieties. The construction is due to V. Batyrev [**21**].[2]

Take an ample anticanonical hypersurface M in a toric variety V, and let $\{M_t\}$ be the family of such. This family is determined by the Newton polygon of the corresponding equations—that is a polygon $P \subset L_{\mathbb{R}} := L \otimes \mathbb{R}$, where L is the *monomial lattice* of the torus T (of which V is a compactification).

Batyrev shows that the Calabi–Yau condition admits a particularly simple characterization in terms of P: the polyhedron P is *reflexive*, which means that each hyperplane H which supports a face of codimension one of P can be written in the form

$$H = \{y \in L_{\mathbb{R}} \mid (\ell, y) = -1\}$$

for some appropriate vector $v \in \operatorname{Hom}(L, \mathbb{Z})$. (The key property here is the *integrality* of the vector v—there would always be some $v \in \operatorname{Hom}(L, \mathbb{R})$ to define H.)

[2]We restrict ourselves to the hypersurface case here; further generalizations—to complete intersections—were subsequently given by Borisov and Batyrev–Borisov [**32, 22**].

Lemma 8.1 (Batyrev)**.** *If P is reflexive, then the* polar polyhedron

$$P^o := \{x \in \operatorname{Hom}(L, \mathbb{R}) \mid (x, y) \geq -1 \textit{ for all } y \in P\}$$

is also reflexive.

The conjectured generalization is that the mirror of the family $\{M_t\}$ of hypersurfaces determined by P should be the family $\{W_s\}$ of hypersurfaces (in a compactification of the dual torus of T) determined by the polar polyhedron P^o.

One of the pieces of evidence for this conjecture is

Theorem 8.2 (Batyrev)**.**

$$\dim H^{\pm 1,1}(\widehat{M}) = \dim H^{\mp 1,1}(\widehat{W}),$$

where $\widehat{M}$ and $\widehat{W}$ are $\mathbb{Q}$-factorial terminalizations of M and W respectively.

Batyrev and collaborators have also explored the Hodge structures of these hypersurfaces in considerable detail [**19, 24, 23**].

A refinement of Batyrev's theorem called the *monomial-divisor mirror map* was introduced in [**18**]. This map gives an explicit combinatorial correspondence between (appropriate subspaces of) $H^{\pm 1,1}(\widehat{M})$ and $H^{\mp 1,1}(\widehat{W})$, and is expected to correctly determine the derivative of the mirror map near the large radius limit point. That derivative data is precisely what one needs in order to evaluate the "constants of integration" in finding the canonical coordinates q_j.

There is another mirror manifold construction for a class of threefolds which has been proposed by Voisin [**97**] and Borcea [**31**]. Let S be a K3 surface with an involution ι such that $\iota^*(\Omega) = -\Omega$ for any holomorphic two-form Ω on S, and let E be an elliptic curve. The quotient $\overline{M} = (S \times E)/(\iota \times (-1))$ has singularities along the fixed curves of the involution $\iota \times (-1)$, but they can be resolved by a simple blowing up to produce a Calabi–Yau threefold M.

Involutions of this type on K3 surfaces have been classified by Nikulin [**81**], who found that they fall into a pattern with a remarkable symmetry; when the Hodge numbers of the associated Calabi–Yau threefold are calculated, this symmetry becomes the expected mirror relation among Hodge numbers. The detailed knowledge which is available concerning the variations of Hodge structure on K3 surfaces can be used to study the correlation functions in detail for these models [**97**], which provides further evidence that mirror partners have been correctly identified. In fact, there is also a physics argument explaining why these pairs of conformal field theories are actually mirror to each other [**113**], based on the physics of mirror symmetry for K3 surfaces.

8.2. Hodge-theoretic mirror conjectures

We can now formulate the main conjecture in the mathematical study of mirror symmetry.

The Hodge-Theoretic Mirror Symmetry Conjecture. *Given a boundary point $P \in \overline{\mathcal{M}}_W$ with maximally unipotent monodromy (or perhaps with strongly maximally unipotent monodromy), there should exist a mirror partner M of W, a framing σ of M, a neighborhood U of P in $\mathcal{M}_W$, and a "mirror map"*

$$\mu : U \to (\mathcal{D}_\sigma/L)^-$$

which is determined up to constants of integration by the property that

$$\mu^*(d\log q_j) = d\left(\frac{\int_{\gamma_j}\Omega}{\int_{\gamma_0}\Omega}\right),$$

such that μ induces an isomorphism between appropriate sub-variations of Hodge structure of

1. *the formal completion of the geometric variation of Hodge structure at P, and*
2. *the framed A-variation of Hodge structure with framing σ.*

(The sub-variations of Hodge structure should contain the entire first two terms of the Hodge filtration on both sides.)

There are additional conjectures one wants to make about the relationship between M and W: there should also be isomorphisms

$$H^{p,q}(W) \cong H^{-p,q}(M) \quad p \geq 0,$$

and these should preserve all correlation functions. (In particular, the "reverse" mirror isomorphism should hold, and there should also be isomorphisms between correlation functions which do not come from variations of Hodge structure.) Of course, such isomorphisms only make sense if we have specified the constants of integration. In fact, one wants to conjecture that the entire conformal field theory moduli spaces are isomorphic, but this is a difficult conjecture to make precisely at present since we do not have a complete mathematical understanding of conformal field theory moduli spaces.

If we start with the A-variation of Hodge structure, there is another conjecture we can make.

Converse Conjecture. *Conversely, given (M, σ), the corresponding A-variation of Hodge structure comes from geometry, in the sense that there is a family $\mathcal{Z} \to \overline{S}$ of varieties degenerating at $0 \in \overline{S}$ such that the framed A-variation of Hodge structure is isomorphic to the formal completion at 0 of a (Tate-twisted) sub-variation of Hodge structures of the variation of Hodge structures on some cohomology of Z_s.*

Due to the phenomenon of rigid Calabi–Yau manifolds, we can't assume any stronger properties about Z_s: Calabi–Yau threefolds with $h^{2,1} = 0$ cannot have mirror partners in the usual sense, since such a mirror partner would satisfy $h^{1,1} = 0$, which is absurd. However, there is an example in the physics literature of a rigid Calabi–Yau manifold, known as the "Z-orbifold," which has a mirror physical theory that was worked out recently by Candelas, Derrick and Parkes [**119**] (see also [**105**]). In this example, the variation of Hodge structure associated to the mirror theory can be described by the family of cubic sevenfolds in $\mathbb{P}^8$ (with a suitable Tate twist).

8.3. Some computations

We explain some of the evidence in favor of the mirror symmetry conjectures which has been accumulated through specific computations.[3] We will compute

[3] The computations presented here are taken from the original paper of Candelas, de la Ossa, Green and Parkes [**118**] on the quintic threefold, and a paper of Greene, Plesser and the present author [**133**] on higher dimensional mirror manifolds. A survey of other calculations of this type (and the methods for making them) can be found in [**77**].

with Calabi–Yau hypersurfaces of dimension $n \geq 3$ in ordinary projective space $\mathbb{CP}^{n+1}$; the degree of the hypersurface must be $n+2$. The family of such hypersurface includes a Fermat hypersurface, which is part of the "Dwork pencil" with defining equation:

$$x_0^{n+2} + \cdots + x_{n+1}^{n+2} - (n+2)\psi\, x_0 \cdots x_{n+1} = 0,$$

where ψ is a parameter. The group

$$\Gamma := \{(\alpha_0, \ldots, \alpha_{n+1}) \mid \alpha_j \in \mu_{n+2}, \prod \alpha_j = 1\}/\{(\alpha, \ldots, \alpha)\}$$

acts on the fibers of this family by componentwise multiplication on the coordinates.

Using either the Greene–Plesser orbifolding construction, or Batyrev's polar polyhedron construction, one sees that the family $\{M\}$ of hypersurfaces of degree $n+2$ in $\mathbb{CP}^{n+1}$ has as its predicted mirror the family $\{W\}$ described as the Dwork pencil modulo Γ (living in the quotient space $\mathbb{CP}^{n+1}/\Gamma$). In fact, we can describe the moduli space of this mirrored family in terms of the parameter ψ^{n+2}—the reason for passing to a power is the existence of an additional automorphism, acting on the family as a whole, generated by componentwise multiplication in the x's by $(\alpha, 1, \ldots, 1)$ while simultaneously multiplying ψ by α^{-1}.

It is not difficult to compute where this family becomes singular. The partial derivatives of the defining equation are all of the form

$$(n+2)\left(x_j^{n+1} - x_j^{-1}\psi\, x_0 \cdots x_{n+1}\right)$$

and for these to vanish simultaneously we must have $\psi^{n+2} = 1$. Moreover, the additional automorphism of the family fixed the fiber $\psi = 0$, and so causes additional singularities there. Thus, we can describe the moduli space as $\mathbb{CP}^1 - \{0, 1, \infty\}$, with its natural compactification being $\mathbb{CP}^1$.

What is the monodromy behavior at the boundary points? (We label the monodromy transformations according to the point.) At $\psi^{n+2} = 0$, we find that the monodromy has finite order, at $\psi^{n+2} = 1$ it is unipotent but $(T_1 - I)^2 = 0$ so the order is not maximal (since $n \neq 1$), and at $\psi^{n+2} = \infty$ we find maximal order of unipotency $(T_\infty - I)^n \neq 0$. In fact, this point is maximally unipotent, and even strongly maximally unipotent, in the terminology established earlier.

To compute canonical coordinates and correlation functions near $\psi^{n+2} = \infty$ we need to know the period functions there. These can be found by studying the differential equations which they satisfy. In this case of toric hypersurfaces, we have a special method available—the representation of cohomology by means of residues of differential forms on the ambient space with poles along the hypersurface. A basis for the primitive cohomology can be written (in the affine chart $x_0 = 1$, say) as

$$\beta_j := \operatorname{Res}\left(\frac{\psi^{j+1}\,(x_1 \cdots x_{n+1})^j\, dx_1 \wedge \cdots \wedge dx_{n+1}}{\left(1 + x_1^{n+2} + \cdots + x_{n+1}^{n+2} - (n+2)\psi\, x_1 \cdots x_{n+1}\right)^{j+1}}\right)$$

The connection matrix in this basis can then be found using Griffiths' "reduction of pole order" lemma [**50**] to calculate coefficients θ_{ij} such that

$$\nabla(\beta_i) = \sum \theta_{ij}\beta_j.$$

To find the period matrix from the connection matrix, one must solve some differential equations. For if $\{e_k\}$ is a basis for the local system and we write $e_k = \sum \eta_{ki}\beta_i$

n	n-point function
3	$5 + 2875\,q + 4876875\,q^2 + 8564575000\,q^3 + 15517926796875\,q^4$ $+ 28663236110956000\,q^5 + 53621944306062201000\,q^6$ $+ 101216230345800061125625\,q^7 + 192323666400003538944396875\,q^8$ $+ 367299732093982242625847031250\,q^9$ $+ 704288164978454714776724365580000\,q^{10}$ $+ 1354842473951260627644461070753075500\,q^{11}$ $+ 2613295702542192770504516764304958585000\,q^{12}$ $+ 5051976384195377826370376750184667397150000\,q^{13}$ $+ 9784992122065556293839548184561593434114765625\,q^{14}$ $+ 18983216783256131050355758292004110332155634496875\,q^{15}$ $+ 36880398908911843175757970052077286676680907186572875\,q^{16}$ $+ 71739993072775923425756947313710004388338109828244718125\,q^{17}$ $+ 139702324572802672116486725324237666156179096139345867681250\,q^{18}$ $+ \ldots$
4	$6 + 120960\,q + 4136832000\,q^2 + 148146924602880\,q^3 + 5420219848911544320\,q^4$ $+ 200623934537137119778560\,q^5 + 7478994517395643259712737280\,q^6$ $+ 280135301818357004749298146851840\,q^7$ $+ 10528167289356385699173014219946393600\,q^8$ $+ 396658819202496234945300681212382224722560\,q^9$ $+ 14972930462574202465673643937107499992165427200\,q^{10}$ $+ 566037069767251121484562070892662863943365345190400\,q^{11}$ $+ 21424151141341932048068067497996096856724987411324108800\,q^{12} + \ldots$
5	$7 + 3727381\,q + 2637885990187\,q^2 + 1927092954108108787\,q^3$ $+ 1425153551321014327663291\,q^4 + 1060347883438857662557634869906\,q^5$ $+ 791661306374088776109692880989252173\,q^6$ $+ 592348256908461616176898022359492565546566\,q^7$ $+ 443865568545713063761643598030194801299861575595\,q^8$ $+ 332947403131697202086626568381790256001850741509664373\,q^9 + \ldots$
6	$8 + 106975232\,q + 1672023727001600\,q^2 + 26611692333081695092736\,q^3$ $+ 426129121674687823674948571136\,q^4$ $+ 6842148599241293047857339542861643776\,q^5$ $+ 110018992594692024449889564415904439556898816\,q^6$ $+ 1770551943055574073245974844490813198478975912902656\,q^7$ $+ 28508925683951911989843155602330000507452539542539447947264\,q^8$ $+ \cdots$

Table 1. n-point functions in dimension n

then

$$0 = \nabla(e_k) = \sum d\eta_{ki}\,\beta_i + \sum \eta_{ki}\theta_{ij}\beta_j$$

gives differential equations for the unknown coefficient functions η_{ki}:

$$d\eta_{ki} = -\sum \eta_{k\ell}\theta_{\ell i}.$$

The flatness of ∇ is equivalent to the integrability of these equations, which can therefore be solved.

$$
\boxed{\begin{aligned}
Y_1^1 = 5 &+ 2875\,\tfrac{1^3 q}{1-q} + 609250\,\tfrac{2^3 q^2}{1-q^2} + 317206375\,\tfrac{3^3 q^3}{1-q^3} + 242467530000\,\tfrac{4^3 q^4}{1-q^4} \\
&+ 229305888887625\,\tfrac{5^3 q^5}{1-q^5} + 248249742118022000\,\tfrac{6^3 q^6}{1-q^6} \\
&+ 295091050570845659250\,\tfrac{7^3 q^7}{1-q^7} + 375632160937476603550000\,\tfrac{8^3 q^8}{1-q^8} \\
&+ 503840510416985243645106250\,\tfrac{9^3 q^9}{1-q^9} \\
&+ 704288164978454686113488249750\,\tfrac{10^3 q^{10}}{1-q^{10}} \\
&+ 1017913203569692432490203659468875\,\tfrac{11^3 q^{11}}{1-q^{11}} \\
&+ 1512323901934139334751675234074638000\,\tfrac{12^3 q^{12}}{1-q^{12}} \\
&+ 2299488568136266648325160104772265542625\,\tfrac{13^3 q^{13}}{1-q^{13}} \\
&+ 3565959228158001564810294084668822024070250\,\tfrac{14^3 q^{14}}{1-q^{14}} \\
&+ 5624656824668483274179483938371579753751395250\,\tfrac{15^3 q^{15}}{1-q^{15}} \\
&+ 9004003639871055462831535610291411200360685606000\,\tfrac{16^3 q^{16}}{1-q^{16}} + \dots
\end{aligned}}
$$

Table 2. Three-point function in dimension three

$$
\boxed{\begin{aligned}
Y_1^1 = 6 &+ 60480\,\tfrac{1^2 q}{1-q} + 440884080\,\tfrac{2^2 q^2}{1-q^2} + 6255156277440\,\tfrac{3^2 q^3}{1-q^3} \\
&+ 117715791990353760\,\tfrac{4^2 q^4}{1-q^4} + 2591176156368821985600 \cdot 5^2\ \tfrac{5^2 q^5}{1-q^5} + \dots
\end{aligned}}
$$

Table 3. Three-point function in dimension four

If we work in a local coordinate $z = \psi^{-n-2}$ near $\psi^{n+2} = \infty$, we find that a basis $e_0(z), \dots, e_{n+1}(z)$ of local solutions can be found such that $e_0(z)$ is single-valued near $z = 0$, and

$$e_{j+1}(z) = (\log z)\, e_j(z) + \text{single-valued function}.$$

(This is a consequence of the maximally unipotent monodromy.) The vectors $e_j(z)$ form the columns of the period matrix.

One can then use row operations to put the period matrix in upper triangular form, with constant diagonal elements. (Let us choose the diagonal elements to all be $n + 2$.) This implements the change of basis to a basis consisting of distinguished sections of $\mathcal{H}^{p,q}_{\mathbb{S}}$. The nonzero entries A_j^1 in the connection matrix are then calculated by differentiating rows of the period matrix, and writing the result as a multiple of a subsequent row. Each such entry takes the form

$$A_j^1 = Y_j^1\, \frac{dq}{q},$$

and the functions Y_j^1 represent correlation functions $\langle (\partial/\partial t)\, \beta_j\, \beta_{n-j-1} \rangle$.

This can all be done very explicitly, using power series expansions of the unknown single-valued functions, in these examples. (I advise using MAPLE or MATHEMATICA if you would like to try it for yourself.) We show two kinds of calculations in the tables. For the first, only the "maximally unipotent" assumption is required, since the calculation requires only the distinguished n-form and the canonical coordinates. What is computed in table 1 is the "n-point function," which iterates the differential of the period map n times. (This was introduced some years ago in the variation of Hodge structures context by Carlson, Green, Griffiths and Harris [**34**].)

$$\begin{aligned} Y_1^1 &= 7 + 1009792\,\tfrac{1^2 q}{1-q} + 122239786088\,\tfrac{2^2 q^2}{1-q^2} + 30528671745480104\,\tfrac{3^2 q^3}{1-q^3} \\ &\quad + 10378199509395886153216\,\tfrac{4^2 q^4}{1-q^4} + \dots \\ Y_2^1 &= 7 + 1707797\,\tfrac{1\cdot q^1}{1-q^1} + 510787745643\,\tfrac{2\cdot q^2}{1-q^2} + 222548537108926490\,\tfrac{3\cdot q^3}{1-q^3} \\ &\quad + 113635631482486991647224\,\tfrac{4\cdot q^4}{1-q^4} + \dots \end{aligned}$$

Table 4. Three-point functions in dimension five

$$\begin{aligned} Y_1^1 &= 8 + 15984640\,\tfrac{1^2 q}{1-q} + 33397159706624\,\tfrac{2^2 q^2}{1-q^2} + 154090254047541417984\,\tfrac{3^2 q^3}{1-q^3} \\ &\quad + 1000674891265872131899670528\,\tfrac{4^2 q^4}{1-q^4} + \dots \\ Y_2^1 &= 8 + 37502976\,\tfrac{1\cdot q^1}{1-q^1} + 224340704157696\,\tfrac{2\cdot q^2}{1-q^2} + 2000750410187341381632\,\tfrac{3\cdot q^3}{1-q^3} \\ &\quad + 21122119007324663457380794368\,\tfrac{4\cdot q^4}{1-q^4} + \dots \\ Y_2^2 &= 8 + 59021312\,\tfrac{q}{1-q} + 821654025830400\,\tfrac{q^2}{1-q^2} + 12197109744970010814464\,\tfrac{q^3}{1-q^3} \\ &\quad + 186083410628492378226388631552\,\tfrac{q^4}{1-q^4} + \dots \end{aligned}$$

Table 5. Three-point functions in dimension six

The other computations, displayed in tables 2–5, are of three-point functions Y_b^a, read off of the connection matrix in a distinguished basis. (There is a symmetry $Y_b^a = Y_{n-a-b}^a = Y_{n-a-b}^b$ so we only show some of these.) The coefficients in the series expansions are the predicted values of the Gromov–Witten invariants. The three-point function Y_0^1 has the value $n+2$ (a constant, due to the definition of canonical coordinates) and is not shown in the tables. The other functions Y_j^1 come directly from the connection matrix. In dimension six, there is also a "secondary" function, which (by the B-model version of the associativity, which is simply the associativity of the "sheaf cup product" pairing) can be calculated as $Y_2^2 = (Y_2^1)^2/Y_1^1$.

There is a relation between the computations in table 1, and those in tables 2–5, which can be explicitly verified from these tables: it is

$$n\text{-point function } = \frac{Y_0^1 \cdot Y_1^1 \cdot \, \cdots \, \cdot Y_{n-1}^1}{(n+2)^n}.$$

The functions Y_b^a are predicted to agree with quantum products on the mirror manifolds

$$\zeta^a \star \zeta^b \star \zeta^{n-a-b},$$

where ζ^j is the class of a linear space (in $\mathbb{CP}^{n+1}$) of complex codimension j. In fact, we have displayed things in tables 2–5 with this in mind, writing series in terms of $q^k/(1-q^k)$.

Also in tables 2–5, we have pulled out some factors of the degree of the rational curve. If there are ℓ occurrences of "1" among $\{a, b, n-a-b\}$, then there will be ℓ of the linear spaces of codimension one, and each meets a given rational curve Γ in $\deg(\Gamma)$ points, giving rise to a factor of $(\deg(\Gamma))^\ell$ in the Gromov–Witten invariants. Pulling out those factors makes the comparison with "counting" problems more transparent.

All of the predicted Gromov–Witten invariants in degrees one and two in these tables have been verified by Katz [**60**]; most of the invariants in degree three have been verified by Ellingsrud and Strømme [**40, 41**].

POSTSCRIPT:
Recent Developments

As mentioned in the introduction, the subject of mirror symmetry is a rapidly developing one, and much has happened since the lectures on which these notes are based were delivered. We will briefly sketch some of these developments in this postscript.

The Gromov–Witten invariants and their generalizations have been studied particularly intensively. The definition of Ruan [**86**] which we presented in the lectures has been supplanted by other definitions drawn from symplectic geometry (cf. [**12, 87**]) which work directly in cohomology (avoiding the bordism technicalities) and are also more general. In full generality these extended Gromov–Witten invariants are not only associated to curves of genus zero with three vertex operators, but also to curves of arbitrary genus g with k vertex operators (provided that $2g-2+k>0$) and even to some non-topological correlation functions.[4] There are at least three proofs of the associativity relations for these symplectic Gromov–Witten invariants [**87, 67, 12**], including proofs of a stronger form of associativity known as the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations [**151, 125, 101, 129**] which are relevant in the case of higher genus. As in the genus zero case, these higher genus invariants can be used to encode a kind of quantum cohomology ring (somewhat larger than the one we studied here); it is also possible to interpret the WDVV associativity relation as the flatness of a certain connection [**129**]. A very accessible exposition of this circle of ideas has been written by McDuff and Salamon [**12**].

Parallel to this development, Gromov–Witten invariants have also been defined purely within algebraic geometry. The methods of Katz described in the lectures were developed further (see [**139**] and the appendix to [**114**]), and similar methods based on the construction of a "virtual moduli cycle" were developed independently by Li and Tian [**66**]. The foundations for an algebraic theory of Gromov–Witten invariants were carefully laid by Kontsevich and Manin [**64**] (again, the higher genus

[4]There have also been investigations into the physical interpretation of these higher genus invariants, and how they should transform under mirror symmetry (in the case of Calabi–Yau threefolds) [**114, 115**]. At one time, it had been expected that for Calabi–Yau threefolds the genus zero topological correlation functions would completely determine the conformal field theory, but now it is known that higher genus invariants are needed as well [**111**].

invariants and the WDVV equations play an important rôle), and the program they initiated was ultimately carried out [**28, 27, 26**], producing a definition of Gromov–Witten invariants based on stable maps. (The work of Li–Tian mentioned above [**66**] is also closely related to this program.) Even before this program was complete, Kontsevich had applied it to obtain some spectacular results in enumerative geometry, including a verification of the predicted number 242467530000 of rational quartics on the general quintic threefold [**62**]. The stable map theory is nicely explained, with further references, in [**45**].

Kontsevich has also formulated a "homological" version of the mirror conjecture [**63**] involving what are known as A^∞-categories (cf. [**91**]), which is related to the "extended moduli space" introduced by Witten [**152**]. By a construction of Fukaya [**44**], to every compact symplectic manifold (Y, ω) with vanishing first Chern class, one can associate an A^∞-category whose objects are essentially the Lagrangian submanifolds of Y, and whose morphisms are determined by the intersections of pairs of submanifolds. Kontsevich's conjecture relates the bounded derived category of the Fukaya category of Y (playing the rôle of the B-model) to the bounded derived category of the category of coherent sheaves on a mirror partner X (playing the rôle of the A-model). I must refer the reader to [**63**] for further details concerning this fascinating conjecture.

The art of making predictions about enumerative geometry from calculations with the variation of Hodge structure on a candidate mirror partner has been considerably refined: see [**77**] for a survey and references to the literature. The era of numerical experiments in mirror symmetry seems to be largely over, and has been supplanted by a more analytical period. Witten's analysis of the physics related to Calabi–Yau manifolds which are hypersurfaces in toric varieties [**153**] was further developed in [**143**], where techniques were found—somewhat related to methods introduced by Batyrev [**20**] for the study of quantum cohomology of toric varieties—for precisely calculating a variant of the quantum cohomology ring of the Calabi–Yau manifold. (The variant is derived from enumerative problems on the ambient space rather than directly on the Calabi–Yau manifold.) There is a physics argument, but not a complete mathematics argument, which explains why this variant should coincide with the usual quantum cohomology ring after a change of coordinates in the coefficient ring. This variant *can* be rigorously shown to agree with the correlation functions of the mirror Calabi–Yau manifold, again calculated in the "wrong" coordinates. In this way, the results of [**143**] provided the first analytical proof that some kind of enumerative problem on one side of the mirror could be related to a variation of Hodge structure calculation on the other side. Further development of these ideas in [**144**] led to a preliminary argument to the effect that the physical theories associated to a Batyrev–Borisov pair should actually be mirror to each other.

In a striking recent development, Givental has proved [**46, 47, 48**] that for Calabi–Yau complete intersections in projective spaces, the "predicted" enumerative formulas which one calculates by using a Batyrev–Borisov candidate mirror partner are in fact correct evaluations of the Gromov–Witten invariants. This establishes, for example, the accuracy of *all* of the predictions about the general quintic threefold made by Candelas et al. [**118**] (and which we listed in table 2). Givental's remarkable proof actually has very little to do with mirror symmetry *per se*: in studying an equivariant version of quantum cohomology, he finds enough structure to enable a calculation which is formally similar to (and certainly inspired by) the variation of Hodge structure calculations on the candidate mirror partner.

The last several years have also been a period of dramatic developments in string theory. There are new techniques which go by the names of "duality" and "nonperturbative methods," and a number of the recent results have been closely related to Calabi–Yau manifolds and mirror symmetry. One of the earliest nonperturbative results [**146, 134**] was the discovery[5] that the string theory moduli spaces associated to Calabi–Yau manifolds should be attached along loci corresponding to "conifold transitions"—a process in which a collection of rational curves is contracted to ordinary double points and the resulting space is then smoothed to produce another Calabi–Yau manifold. This new attaching procedure supplements, but is rather different from, the gluing of Kähler cones which we discussed in section 7.3. In the new procedure, a moduli space of a different dimension (corresponding to a Calabi–Yau manifold with different Hodge numbers than the original) is cemented on at the same point where the two like-dimensional pieces (Kähler cones differing by a flop) have been glued together. The "cement" which holds these two spaces together (i.e., the physical process responsible) is a phase transition between charged black holes on one component of the moduli space and elementary particles on the other.

The string theory moduli spaces mentioned above are actually somewhat larger than the conformal field theory moduli spaces which were one of the primary subjects of these lectures. There are two variants of string theory which are relevant, called type IIA and type IIB string theories, and the additional parameters which must be added to the conformal field theory moduli space differs between the two. In the case of type IIA, the extra parameters are a choice of holomorphic 3-form and the choice of an element in the intermediate Jacobian of the Calabi–Yau threefold. (Some of the mathematical structure of these spaces related to the intermediate Jacobians was anticipated in work of Donagi and Markman [**39**].) In the case of type IIB, the new parameters are similar, but related to the even cohomology of the manifold. These two types of parameters should be mapped to each other under mirror symmetry [**112, 142**]. In fact, a large number of other related structures called "D-brane moduli spaces" should also correspond under mirror symmetry—the precise implications of this correspondence (which appears to be connected to Kontsevich's homological mirror symmetry conjecture) are still being worked out.

Finally, in a very exciting recent development, a completely new geometric aspect of mirror symmetry has been discovered by Strominger, Yau and Zaslow [**148**]. A Calabi–Yau manifold X of real dimension $2n$ on which a complex structure J and Kähler form ω have been fixed has a natural class of n-dimensional submanifolds M defined by the property that $\omega|_M \equiv 0$ and $\mathrm{Im}(\Omega)|_M \equiv 0$ for some choice of holomorphic n-form Ω. These *special Lagrangian submanifolds* were introduced by Harvey and Lawson [**53**] as a natural class of volume-minimizing submanifolds; they have many other interesting properties, including an exceptionally well-behaved deformation theory [**72**]. Strominger, Yau and Zaslow argue on physical grounds (using the correspondence of D-brane moduli spaces mentioned above) that whenever X has a mirror partner, then X must admit a map $\rho : X^{2n} \to B^n$ whose generic fiber

[5] This had been anticipated some time earlier in the physics literature [**116, 132, 121, 117**] based on the discovery of and speculations about conifold transitions in the mathematics literature [**36, 42, 54, 94, 85, 43**], but an understanding of the physical mechanism behind the attachment of the moduli spaces was lacking.

is a special Lagrangian n-torus, and which has a section $\sigma : B \to X$ whose image is itself a special Lagrangian submanifold. Given this structure, the mirror partner of X is then predicted to be a compactification of the family of dual tori of the fibers of ρ. (The section specifies a point $p_b := \sigma(b)$ on each torus $T_b := \rho^{-1}(b)$; the dual torus is then $\mathrm{Hom}(\pi_1(T_b, p_b), \mathrm{U}(1))$.) There is also an argument—quite similar in nature to [**144**]—that such a structure should suffice for producing a mirror isomorphism between the corresponding physical theories. A mathematical account of this construction can be found in [**78**], which attempts to make the mathematical implications of this story precise: given a "special Lagrangian m-torus fibration," all of the structure we have seen relating the quantum cohomology and the variation of Hodge structure should (conjecturally) follow as a consequence. For the Voisin–Borcea threefolds, the structure of these special Lagrangian torus fibrations (using a mildly degenerate metric) has been worked out in complete detail by Gross and Wilson [**52**], who find compatibility with the previously observed mirror phenomena in a beautiful geometric form.

BIBLIOGRAPHY

Books

1. H. Clemens, J. Kollár, and S. Mori, Higher-Dimensional Complex Geometry, Astérisque, vol. 166, Société Mathématique de France, 1988.
2. P. E. Conner and E. E. Floyd, Differentiable Periodic Maps, Ergeb. Math. Grenzgeb. N. F., vol. 33, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1964.
3. D. Cox and S. Katz, Mirror Symmetry and Algebraic Geometry, in preparation.
4. C. W. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Interscience Publishers, New York, 1962.
5. P. Deligne, Equations Différentielles à Points Singuliers Réguliers, Lecture Notes in Math., vol. 163, Springer-Verlag, Berlin, Heidelberg, New York, 1970.
6. W. Fulton, Intersection Theory, Ergeb. Math. Grenzgeb. (3), vol. 2, Springer-Verlag, Berlin, New York, 1984.
7. B. Greene and S.-T. Yau (eds.), Mirror Symmetry II, International Press, Cambridge, 1996.
8. P. Griffiths et al., Topics in Transcendental Algebraic Geometry, Annals of Math. Studies, vol. 106, Princeton University Press, Princeton, 1984.
9. R. Hartshorne, Algebraic Geometry, Springer-Verlag, New York, Heidelberg, Berlin, 1977.
10. T. Hübsch, Calabi–Yau Manifolds: A Bestiary for Physicists, World Scientific, Singapore, New Jersey, London, Hong Kong, 1992.
11. G. Karpilovsky, Symmetric and G-Algebras, With Applications to Group Representations, Kluwer Academic Publishers, 1990.
12. D. McDuff and D. Salamon, J-Holomorphic Curves and Quantum Cohomology, University Lecture Series, vol. 6, American Mathematical Society, Providence, 1994.
13. D. Mumford and J. Fogarty, Geometric Invariant Theory, 2nd ed., Ergeb. Math. Grenzgeb. N. F., vol. 34, Springer-Verlag, Berlin, Heidelberg, New York, 1982.
14. C. Nash, Differential Topology and Quantum Field Theory, Academic Press, London, San Diego, 1991.

15. C. Voisin, Symétrie Miroir, Panoramas et Synthèses, vol. 2, Société Mathématique de France, Paris, 1996.
16. S.-T. Yau (ed.), Mathematical Aspects of String Theory, World Scientific, Singapore, 1987.
17. ———, Essays on Mirror Manifolds, International Press, Hong Kong, 1992.

Mathematics Articles

18. P. S. Aspinwall, B. R. Greene, and D. R. Morrison, *The monomial-divisor mirror map*, Internat. Math. Res. Notices (1993), 319–337.
19. V. V. Batyrev, *Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori*, Duke Math. J. **69** (1993), 349–409.
20. ———, *Quantum cohomology rings of toric manifolds*, Journées de Géométrie Algébrique d'Orsay (Juillet 1992) (A. Beauville et al., eds.), Astérisque, vol. 218, Société Mathématique de France, 1993, pp. 9–34.
21. ———, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Algebraic Geom. **3** (1994), 493–535.
22. V. V. Batyrev and L. A. Borisov, *Dual cones and mirror symmetry for generalized Calabi–Yau manifolds*, Mirror Symmetry II (B. Greene and S.-T. Yau, eds.), International Press, Cambridge, 1996, pp. 71–86.
23. V. V. Batyrev and D. A. Cox, *On the Hodge structure of projective hypersurfaces in toric varieties*, Duke Math. J. **75** (1994), 293–338.
24. V. V. Batyrev and D. van Straten, *Generalized hypergeometric functions and rational curves on Calabi–Yau complete intersections in toric varieties*, Comm. Math. Phys. **168** (1995), 493–533.
25. A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom. **18** (1983), 755–782.
26. K. Behrend, *Gromov–Witten invariants in algebraic geometry*, preprint, 1996 (alg-geom/9601011).
27. K. Behrend and B. Fantechi, *The intrinsic normal cone*, preprint, 1996 (alg-geom/9601010).
28. K. Behrend and Yu. Manin, *Stacks of stable maps and Gromov–Witten invariants*, Duke Math. J., to appear.
29. F. A. Bogomolov, *Hamiltonian Kähler manifolds*, Dokl. Akad. Nauk SSSR **243** (1978), no. 5, 1101–1104.
30. C. Borcea, *On desingularized Horrocks–Mumford quintics*, J. Reine Angew. Math. **421** (1991), 23–41.
31. ———, *K3 surfaces with involution and mirror pairs of Calabi–Yau manifolds*, Mirror Symmetry II (B. Greene and S.-T. Yau, eds.), International Press, Cambridge, 1996, pp. 717–744.
32. L. Borisov, *Towards the mirror symmetry for Calabi–Yau complete intersections in toric Fano varieties*, preprint, 1993 (alg-geom/9310001).
33. E. Calabi, *On Kähler manifolds with vanishing canonical class*, Algebraic Geometry and Topology, A Symposium in Honor of S. Lefschetz (R. H. Fox et al., eds.), Princeton University Press, Princeton, 1957, pp. 78–89.
34. J. Carlson, M. Green, P. Griffiths, and J. Harris, *Infinitesimal variations of Hodge structure, I*, Compositio Math. **50** (1983), 109–205.
35. E. Cattani, A. Kaplan, and W. Schmid, *L^2 and intersection cohomologies for a polarizable variation of Hodge structure*, Invent. Math. **87** (1987), 217–252.

36. C. H. Clemens, *Double solids*, Adv. Math. **47** (1983), 107–230.
37. ———, *Some results on Abel–Jacobi mappings*, Topics in Transcendental Algebraic Geometry (P. Griffiths, ed.), Princeton University Press, Princeton, 1984, pp. 289–304.
38. P. Deligne, *Local behavior of Hodge structures at infinity*, Mirror Symmetry II (B. Greene and S.-T. Yau, eds.), International Press, Cambridge, 1996, pp. 683–700.
39. R. Donagi and E. Markman, *Cubics, integrable systems, and Calabi–Yau threefolds*, Proc. of the Hirzebruch 65 Conference on Algebraic Geometry (M. Teicher, ed.), Israel Math. Conf. Proc., vol. 9, Bar-Ilan University, 1996, pp. 199–221.
40. G. Ellingsrud and S. A. Strømme, *The number of twisted cubic curves on the general quintic threefold*, Math. Scand. **76** (1995), 5–34.
41. ———, *Bott's formula and enumerative geometry*, J. Amer. Math. Soc. **9** (1996), 175–193.
42. R. Friedman, *Simultaneous resolution of threefold double points*, Math. Ann. **274** (1986), 671–689.
43. ———, *On threefolds with trivial canonical bundle*, Complex Geometry and Lie Theory (J. A. Carlson et al., eds.), Proc. Symp. Pure. Math., vol. 53, American Mathematical Society, Providence, 1991, pp. 103–134.
44. K. Fukaya, *Morse homotopy, A^{∞}-category, and Floer homologies*, Proceedings of GARC Workshop on Geometry and Topology '93, Lecture Notes Ser., 18, Seoul Nat. Univ., Seoul, 1993, pp. 1–102.
45. W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, Algebraic Geometry—Santa Cruz 1995, Proc. Symp. Pure Math., American Mathematical Society, Providence, to appear.
46. A. B. Givental, *Homological geometry I: Projective hypersurfaces*, Selecta Math. (N.S.) **1** (1995), 325–345.
47. ———, *Homological geometry and mirror symmetry*, Proc. Internat. Congr. Math. Zürich 1994 (S. D. Chatterji, ed.), vol. 1, Birkäuser Verlag, Basel, Boston, Berlin, 1995, pp. 472–480.
48. ———, *Equivariant Gromov–Witten invariants*, Internat. Math. Res. Notices (1996), 613–663.
49. A. Grassi and D. R. Morrison, *Automorphisms and the Kähler cone of certain Calabi–Yau manifolds*, Duke Math. J. **71** (1993), 831–838.
50. P. A. Griffiths, *On the periods of certain rational integrals, I*, Annals of Math. (2) **90** (1969), 460–495.
51. M. Gromov, *Pseudo-holomorphic curves on almost complex manifolds*, Invent. Math. **82** (1985), 307–347.
52. M. Gross and P. M. H. Wilson, *Mirror symmetry via 3-tori for a class of Calabi–Yau threefolds*, preprint, 1996 (alg-geom/9608004).
53. R. Harvey and H. B. Lawson, Jr., *Calibrated geometries*, Acta Math. **148** (1982), 47–157.
54. F. Hirzebruch (notes by J. Werner), *Some examples of threefolds with trivial canonical bundle*, Gesammelte Abhandlungen, Band II, Springer-Verlag, Berlin, New York, 1987, pp. 757–770.
55. H. Hofer and D. A. Salamon, *Floer homology and Novikov rings*, The Floer Memorial Volume (H. Hofer et al., eds.), Birkhäuser Verlag, Basel, Boston, Berlin, 1995, pp. 483–524.

56. T. Johnsen and S. Kleiman, *Rational curves of degree at most* 9 *on a general quintic threefold*, Comm. Algebra **24** (1996), 2721–2753.
57. ______, *Toward Clemens' conjecture in degrees between* 10 *and* 24, preprint, 1996, (alg-geom/9601024).
58. S. Katz, *On the finiteness of rational curves on quintic threefolds*, Compositio Math. **60** (1986), 151–162.
59. ______, *Rational curves on Calabi–Yau threefolds*, Essays on Mirror Manifolds (S.-T. Yau, ed.), International Press, Hong Kong, 1992, pp. 168–180.
60. ______, *Rational curves on Calabi–Yau manifolds: Verifying predictions of mirror symmetry*, Projective Geometry with Applications, Lecture Notes in Pure and Appl. Math., vol. 166, Marcel Dekker, New York, 1994, pp. 231–239.
61. Y. Kawamata, *Crepent blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces*, Annals of Math. (2) **127** (1988), 93–163.
62. M. Kontsevich, *Enumeration of rational curves via torus actions*, The Moduli Space of Curves (R. Dijkgraaf et al., eds.), Progress in Math., vol. 129, Birkhäuser, Boston, 1995, pp. 335–368.
63. ______, *Homological algebra of mirror symmetry*, Proc. Internat. Congr. Math. Zürich 1994 (S. D. Chatterji, ed.), vol. 1, Birkäuser Verlag, Basel, Boston, Berlin, 1995, pp. 120–139.
64. M. Kontsevich and Yu. Manin, *Gromov–Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys. **164** (1994), 525–562.
65. A. Landman, *On the Picard–Lefschetz transformations*, Trans. Amer. Math. Soc. **181** (1973), 89–126.
66. J. Li and G. Tian, *Virtual moduli cycles and Gromov–Witten invariants of algebraic varieties*, preprint, 1996 (alg-geom/9602007).
67. G. Liu, *Associativity of quantum multiplication*, preprint, 1994.
68. E. Looijenga, *New compactifications of locally symmetric varieties*, Proceedings of the 1984 Vancouver Conference in Algebraic Geometry (J. Carrell et al., eds.), CMS Conference Proceedings, vol. 6, American Mathematical Society, Providence, 1986, pp. 341–364.
69. Yu. I. Manin, *Generating functions in algebraic geometry and sums over trees*, The Moduli Space of Curves (R. Dijkgraaf et al., eds.), Progress in Math., vol. 129, Birkhäuser, Boston, 1995, pp. 401–417.
70. D. McDuff, *Examples of symplectic structures*, Invent. Math. **89** (1987), 13–36.
71. ______, *Symplectic manifolds with contact type boundaries*, Invent. Math. **103** (1991), 651–671.
72. R. C. McLean, *Deformations of calibrated submanifolds*, J. Differential Geom., to appear.
73. D. R. Morrison, *Mirror symmetry and rational curves on quintic threefolds: A guide for mathematicians*, J. Amer. Math. Soc. **6** (1993), 223–247.
74. ______, *Compactifications of moduli spaces inspired by mirror symmetry*, Journées de Géométrie Algébrique d'Orsay (Juillet 1992), Astérisque, vol. 218, Société Mathématique de France, 1993, pp. 243–271.
75. ______, *Mirror symmetry and moduli spaces of superconformal field theories*, Proc. Internat. Congr. Math. Zürich 1994 (S. D. Chatterji, ed.), vol. 2, Birkäuser Verlag, Basel, Boston, Berlin, 1995, pp. 1304–1314.

76. ———, *Beyond the Kähler cone*, Proc. of the Hirzebruch 65 Conference on Algebraic Geometry (M. Teicher, ed.), Israel Math. Conf. Proc., vol. 9, Bar-Ilan University, 1996, pp. 361–376.
77. ———, *Making enumerative predictions by means of mirror symmetry*, Mirror Symmetry II (B. Greene and S.-T. Yau, eds.), International Press, Cambridge, 1996, pp. 457–482.
78. ———, *The geometry underlying mirror symmetry*, preprint, 1996 (alg-geom/9608006).
79. T. Nakayama, *On Frobeniusean algebras I*, Annals of Math. (2) **40** (1939), 611–633.
80. P. Nijsse, *Clemens' conjecture for octic and nonic curves*, Indag. Math. (N. S.) **6** (1995), 213–221.
81. V. V. Nikulin, *Discrete reflection groups in Lobachevsky spaces and algebraic surfaces*, Proc. Internat. Congr. Math. Berkeley 1986 (A. M. Gleason, ed.), vol. 1, American Mathematical Society, Providence, 1987, pp. 654–671.
82. S. P. Novikov, *Multivalued functions and functionals — an analogue of the Morse theory*, Soviet Math. Dokl. **24** (1981), 222–225.
83. K. Oguiso, *On algebraic fiber space structures on a Calabi–Yau 3-fold*, Internat. J. Math. **4** (1993), 439–465.
84. T. H. Parker and J. G. Wolfson, *Pseudo-holomorphic maps and bubble trees*, J. Geom. Anal. **3** (1993), 63–98.
85. M. Reid, *The moduli space of* 3*-folds with* $K{=}0$ *may nevertheless be irreducible*, Math. Ann. **278** (1987), 329–334.
86. Y. Ruan, *Topological sigma model and Donaldson type invariants in Gromov theory*, Duke Math. J. **83** (1996), 461–500.
87. Y. Ruan and G. Tian, *A mathematical theory of quantum cohomology*, J. Differential Geom. **42** (1995), 259–367.
88. J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of* 2*-spheres*, Annals of Math. (2) **113** (1981), 1–24.
89. W. Schmid, *Variation of Hodge structure: The singularities of the period mapping*, Invent. Math. **22** (1973), 211–319.
90. C. Schoen, *On fiber products of rational elliptic surfaces with section*, Math. Z. **197** (1988), 177–199.
91. J. D. Stasheff, *Homotopy associativity of* H*-spaces, I, II*, Trans. Amer. Math. Soc. **108** (1963), 275–312.
92. R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86.
93. G. Tian, *Smoothness of the universal deformation space of compact Calabi–Yau manifolds and its Peterson–Weil metric*, Mathematical Aspects of String Theory (S.-T. Yau, ed.), World Scientific, Singapore, 1987, pp. 629–646.
94. G. Tian and S.-T. Yau, *Three dimensional algebraic manifolds with* $c_1 = 0$ *and* $\chi = -6$, Mathematical Aspects of String Theory (S.-T. Yau, ed.), World Scientific, Singapore, 1987, pp. 629–646.
95. A. N. Todorov, *The Weil–Petersson geometry of the moduli space of* $SU(n{\geq}3)$ *(Calabi–Yau) manifolds, I*, Comm. Math. Phys. **126** (1989), 325–246.
96. E. Viehweg, *Weak positivity and the stability of certain Hilbert points, III*, Invent. Math. **101** (1990), 521–543.
97. C. Voisin, *Miroirs et les involutions sur les surfaces K3*, Journées de Géométrie Algébrique d'Orsay (Juillet 1992), Astérisque, vol. 218, Société Mathématique

de France, 1993, pp. 273–323.

98. ———, *A mathematical proof of the Aspinwall–Morrison formula*, Compositio Math., to appear.
99. P. M. H. Wilson, *The Kähler cone on Calabi–Yau threefolds*, Invent. Math. **107** (1992), 561–583, *Erratum*, ibid. **114** (1993), 231–233.
100. E. Witten, *Physics and geometry*, Proc. Internat. Congr. Math. Berkeley 1986 (A. M. Gleason, ed.), vol. 1, American Mathematical Society, Providence, 1987, pp. 267–303.
101. ———, *Two dimensional gravity and intersection theory on moduli space*, Surveys in Differential Geom. **1** (1991), 243–310.
102. J. G. Wolfson, *Gromov's compactness of pseudo-holomorphic curves and symplectic geometry*, J. Differential Geom. **28** (1988), 383–405.
103. S.-T. Yau, *On Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), 1798–1799.
104. R. Ye, *Gromov's compactness theorem for pseudo-holomorphic curves*, Trans. Amer. Math. Soc. **342** (1994), 671–694.

Physics Articles

105. P. S. Aspinwall and B. R. Greene, *On the geometric interpretation of $N = 2$ superconformal theories*, Nuclear Phys. B **437** (1995), 205–230.
106. P. S. Aspinwall, B. R. Greene, and D. R. Morrison, *Multiple mirror manifolds and topology change in string theory*, Phys. Lett. B **303** (1993), 249–259.
107. ———, *Calabi–Yau moduli space, mirror manifolds and spacetime topology change in string theory*, Nuclear Phys. B **416** (1994), 414–480.
108. ———, *Measuring small distances in N=2 sigma models*, Nuclear Phys. B **420** (1994), 184–242.
109. P. S. Aspinwall and C. A. Lütken, *Quantum algebraic geometry of superstring compactifications*, Nuclear Phys. B **355** (1991), 482–510.
110. P. S. Aspinwall and D. R. Morrison, *Topological field theory and rational curves*, Comm. Math. Phys. **151** (1993), 245–262.
111. ———, *Chiral rings do not suffice: N=$(2,2)$ theories with nonzero fundamental group*, Phys. Lett. B **334** (1994), 79–86.
112. ———, *U-duality and integral structures*, Phys. Lett. B **355** (1995), 141–149.
113. ———, *String theory on K3 surfaces*, Mirror Symmetry II (B. Greene and S.-T. Yau, eds.), International Press, Cambridge, 1996, pp. 703–716.
114. M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, *Holomorphic anomalies in topological field theories* (with an appendix by S. Katz), Nuclear Phys. B **405** (1993), 279–304.
115. ———, *Kodaira–Spencer theory of gravity*, Comm. Math. Phys. **165** (1994), 311–427.
116. P. Candelas, A.M. Dale, C. A. Lütken, and R. Schimmrigk, *Complete intersection Calabi–Yau manifolds*, Nuclear Phys. B **298** (1988), 493–525.
117. P. Candelas and X. C. de la Ossa, *Comments on conifolds*, Nuclear Phys. B **342** (1990), 246–268.
118. P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes, *A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory*, Nuclear Phys. B **359** (1991), 21–74.

119. P. Candelas, E. Derrick, and L. Parkes, *Generalized Calabi–Yau manifolds and the mirror of a rigid manifold*, Nuclear Phys. B **407** (1993), 115–154.
120. P. Candelas, A. Font, S. Katz, and D. R. Morrison, *Mirror symmetry for two parameter models – II*, Nuclear Phys. B **429** (1994), 626–674.
121. P. Candelas, P. S. Green, and T. Hübsch, *Rolling among Calabi–Yau vacua*, Nuclear Phys. B **330** (1990), 49–102.
122. P. Candelas, G. Horowitz, A. Strominger, and E. Witten, *Vacuum configuration for superstrings*, Nuclear Phys. B **258** (1985), 46–74.
123. P. Candelas, M. Lynker, and R. Schimmrigk, *Calabi–Yau manifolds in weighted* $\mathbb{P}_4$, Nuclear Phys. B **341** (1990), 383–402.
124. S. Cecotti and C. Vafa, *Topological anti-topological fusion*, Nuclear Phys. B **367** (1991), 359–461.
125. R. Dijkgraaf, E. Verlinde, and H. Verlinde, *Topological strings in* $D < 1$, Nuclear Phys. B **352** (1991), 59–86.
126. M. Dine, N. Seiberg, X.-G. Wen, and E. Witten, *Nonperturbative effects on the string world sheet, I, II*, Nuclear Phys. B **278** (1986), 769–789, ibid. **289** (1987), 319–363.
127. J. Distler and B. Greene, *Some exact results on the superpotential from Calabi–Yau compactifications*, Nuclear Phys. B **309** (1988), 295–316.
128. L. J. Dixon, *Some world-sheet properties of superstring compactifications, on orbifolds and otherwise*, Superstrings, Unified Theories, and Cosmology 1987 (G. Furlan et al., eds.), World Scientific, Singapore, 1988, pp. 67–126.
129. B. Dubrovin, *Integrable systems in topological field theory*, Nuclear Phys. B **379** (1992), 627–689.
130. P. Ginsparg, *Applied conformal field theory*, Fields, Strings, and Critical Phenomena, Les Houches Summer School in Theoretical Physics, 1988 (E. Brézin and J. Zinn-Justin, eds.), North-Holland, Amsterdam, 1990, pp. 1–168.
131. M. Green and J. Schwarz, *Anomaly cancellations in supersymmetric* D=10 *gauge theory and superstring theory*, Phys. Lett. B **149** (1984), 117–122.
132. P. S. Green and T. Hübsch, *Connecting moduli spaces of Calabi–Yau threefolds*, Comm. Math. Phys. **119** (1988), 431–441.
133. B. R. Greene, D. R. Morrison, and M. R. Plesser, *Mirror manifolds in higher dimension*, Comm. Math. Phys. **173** (1995), 559–598.
134. B. R. Greene, D. R. Morrison, and A. Strominger, *Black hole condensation and the unification of string vacua*, Nuclear Phys. B **451** (1995), 109–120.
135. B. R. Greene and M. R. Plesser, *Duality in Calabi–Yau moduli space*, Nuclear Phys. B **338** (1990), 15–37.
136. ______, *An introduction to mirror manifolds*, Essays on Mirror Manifolds (S.-T. Yau, ed.), International Press, Hong Kong, 1992, pp. 1–27.
137. B. R. Greene, C. Vafa, and N. Warner, *Calabi–Yau manifolds and renormalization group flows*, Nuclear Phys. B **324** (1989), 371–390.
138. K. Intriligator, *Fusion residues*, Mod. Phys. Lett. A **6** (1991), 3543–3556.
139. S. Katz, *Gromov–Witten invariants via algebraic geometry*, Trieste Conference on S-Duality and Mirror Symmetry, Nuclear Phys. B Proc. Suppl., vol. 46, 1996, pp. 96–107.
140. W. Lerche, C. Vafa, and N. P. Warner, *Chiral rings in* $N = 2$ *superconformal theories*, Nuclear Phys. B **324** (1989), 427–474.
141. G. Moore and N. Seiberg, *Classical and quantum conformal field theory*, Comm. Math. Phys. **123** (1989), 177–254.

142. D. R. Morrison, *Mirror symmetry and the type II string*, Trieste Conference on S-Duality and Mirror Symmetry, Nuclear Phys. B Proc. Suppl., vol. 46, 1996, pp. 146–155.
143. D. R. Morrison and M. R. Plesser, *Summing the instantons: Quantum cohomology and mirror symmetry in toric varieties*, Nuclear Phys. B **440** (1995), 279–354.
144. ______, *Towards mirror symmetry as duality for two-dimensional abelian gauge theories*, Trieste Conference on S-Duality and Mirror Symmetry, Nuclear Phys. B Proc. Suppl., vol. 46, 1996, pp. 177–186.
145. A. Strominger, *Yukawa couplings in superstring compactification*, Phys. Rev. Lett. **55** (1985), 2547–2550.
146. ______, *Massless black holes and conifolds in string theory*, Nuclear Phys. B **451** (1995), 96–108.
147. A. Strominger and E. Witten, *New manifolds for superstring compactification*, Comm. Math. Phys. **101** (1985), 341–361.
148. A. Strominger, S.-T. Yau, and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. B **479** (1996), 243–259.
149. C. Vafa, *Topological mirrors and quantum rings*, Essays on Mirror Manifolds (S.-T. Yau, ed.), International Press, Hong Kong, 1992, pp. 96–119.
150. E. Witten, *Topological sigma models*, Comm. Math. Phys. **118** (1988), 411–449.
151. ______, *On the structure of the topological phase of two-dimensional gravity*, Nuclear Phys. B **340** (1990), 281–332.
152. ______, *Mirror manifolds and topological field theory*, Essays on Mirror Manifolds (S.-T. Yau, ed.), International Press, Hong Kong, 1992, pp. 120–159.
153. ______, *Phases of $N=2$ theories in two dimensions*, Nuclear Phys. B **403** (1993), 159–222.

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