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The Cycling of Partitions and Compositions under Repeated Shifts

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Abstract

In "Bulgarian Solitaire", a player divides a deck of n cards into piles. Each move consists of taking a card from each pile to form a single new pile. One is concerned only with how many piles there are of each size. Starting from any division into piles, one always reaches some cycle of partitions of n. Brandt proved that for $n = 1+2+\cdots+k$, the cycle is just the single partition into piles of distinct sizes $1, 2, \ldots, k$. Let $D_{\mathcal{B}}(n)$ denote the maximum number of moves required to reach a cycle. Igusa and Etienne proved that $D_{\mathcal{B}}(n) \leq k^2 - k$ whenever $n \leq 1+2+\cdots+k$, and equality holds when $n = 1+2+\cdots+k$. We present a simple new derivation of these facts. We improve the bound to $D_{\mathcal{B}}(n) \leq k^2 - 2k - 1$, whenever $n < 1+2+\cdots+k$ with $k \geq 4$. We present a lower bound for $D_{\mathcal{B}}(n)$ that is likely to be the actual value. We introduce a new version of the game, Carolina Solitaire, in which the piles are kept in order, so we work with compositions rather than partitions. Many analogous results can be obtained.

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Section 1. Introduction

In the early 1980's, an article by Martin Gardner brought widespread attention to a card game that had attracted the curiosity of some European mathematicians. Called *Bulgarian Solitaire*, the game works as follows: First divide a finite deck of n cards into piles. A move consists of removing one card from each pile and forming a new pile. The operation is repeated over and over. We pay attention only to how many piles there are of each positive size, ignoring the locations of the piles.

Thus, Bulgarian Solitaire can be viewed as a way of changing one partition into another. For instance, the partition into k parts of distinct sizes from 1 to k is preserved under the operation. Clearly, for any n, any start eventually leads to a cycle of partitions, since there are only finitely many partitions of n altogether. What is striking in playing the game is that starting from any partition of a deck of size $n = 1+2+\cdots+k$ cards, one always arrives eventually at this stable partition into sizes 1 through k. This effect is all the more dramatic in that it seems to take quite awhile in some cases with only moderately large k, say $k = 5$.

We don't know yet the reason for the appellation "Bulgarian." However, we were introduced to an ordered variation of the game by a Bulgarian visitor, Andrey Andreev. In his game, which we shall call *Carolina Solitaire*, one also maintains an ordering of the nonempty piles. Say we begin with n cards divided into a row of piles of sizes $c_1, \ldots, c_r > 0$, $\sum_i c_i = n$. One card is removed from each pile, and these r cards are then placed in a pile ahead of the others. Any exhausted pile (size 0) is ignored; only nonempty piles are considered. For a triangular number n, say $n = 1+2+\cdots+k$, this game also appears to arrive at a stable division, with piles of sizes $k, k-1, \ldots, 1$.

After proving this fact for Carolina Solitaire, as well as deriving bounds on how soon the cycling begins for arbitrary n, we asked experts for references to related work, and our search finally led us to the literature on Bulgarian Solitaire. Our methods are easily adapted to this simpler unordered game. In fact, we obtain improved bounds on the maximum number of shifts needed for any game on n cards to cycle. We shall also mention work on other variation(s) of Bulgarian Solitaire, most notably one called Montreal Solitaire.

Let us now define the games formally, present some examples and calculations, and describe the plan of the paper. For a positive integer n, we say $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a *partition of* n, and we write $\lambda \vdash n$, provided the nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ add up to n, If $\lambda_s > \lambda_{s+1} = 0$, we say λ has s parts, and we often drop the zeroes, writing $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$.

The *shift operation* $\mathcal{B}$ *on* λ is the partition of n, denoted $\mathcal{B}(\lambda)$, obtained by decreasing each part λ_i by one, inserting a part s, and discarding any zero parts. So $\mathcal{B}^{(i)}(\lambda)$ denotes the partition obtained by successively applying the shift operation $\mathcal{B}$ to λ a total of i times. Starting with a partition λ, we describe Bulgarian solitaire by repeatedly applying the shift operation to obtain the sequence of partitions

$$\lambda, \mathcal{B}(\lambda), \mathcal{B}^{(2)}(\lambda), \ldots .$$

For a couple of simple examples, we note that $\lambda = (2,1,1,1,1) \vdash 6$, gives the sequence $(2,1,1,1,1), (5,1), (4,2), (3,2,1), (3,2,1), \ldots$, while $\lambda = (6,1) \vdash 7$ yields $(6,1)$, $(5,2)$, $(4,2,1)$, $(3,3,1)$, $(3,2,2)$, $(3,2,1,1)$, $(4,2,1), \ldots$. The first example is fixed at the partition $(3,2,1)$ after three steps, while the second example reaches a cycle after two steps.

We say a partition $\mu \vdash n$ is *$\mathcal{B}$-cyclic* if $\mathcal{B}^{(i)}(\mu) = \mu$ for some $i > 0$. Brandt [3] characterized all $\mathcal{B}$-cyclic partitions for Bulgarian Solitaire. In particular, when n is a triangular number, $1 + 2 + \cdots + k$, he proved that $(k, k-1, ..., 2, 1)$ is the unique $\mathcal{B}$-cyclic partition of n, one that we end up with, no matter where we start. Note that if n is not triangular, there is no fixed partition under $\mathcal{B}$.

To measure how long it takes for Bulgarian Solitaire to cycle, for $\lambda \vdash n$ we let $d_{\mathcal{B}}(\lambda)$ denote the smallest integer $i \geq 0$ such that $\mathcal{B}^{(i)}(\lambda)$ is $\mathcal{B}$-cyclic . Let $D_{\mathcal{B}}(n) := \max\{d_{\mathcal{B}}(\lambda) : \lambda \vdash n\}$, so that for any $\lambda \vdash n$, $D_{\mathcal{B}}(n)$ steps reach a cycle. Trivially, $D_{\mathcal{B}}(1) = D_{\mathcal{B}}(2) = 0$ and $D_{\mathcal{B}}(3) = 2$. The values of $D_{\mathcal{B}}(n)$, $4 \leq n \leq 36$, are displayed in Figure 1, which we worked out by computer. The data are arranged using the representation $n = 1 + 2 + \cdots + (k-1) + r$, $1 \leq r \leq k$.

Brandt [3] conjectured that $D_{\mathcal{B}}(n) = k^2 - k$ for triangular n, $n = 1 + 2 + \cdots + k$. Hobby and Knuth [7] confirmed this for $k \leq 10$ by computer. It was first proven by Igusa [8] and Etienne [5], apparently independently at about the same time, although Etienne's proof was published much later. (Note that Etienne's paper is noted as having been received in 1984.) In 1987 Bentz [2] also gave a different proof. In Section 2 we present our proof of Brandt's result characterizing all $\mathcal{B}$-cyclic partitions. This leads to our simple, new proof of the value of $D_{\mathcal{B}}(n)$ for triangular numbers n, given in Section 3.

Igusa and Etienne went on to obtain the upper bound $D_{\mathcal{B}}(n) \leq k^2 - k$ for general n, represented as $1 + 2 + \cdots + (k-1) + r$, $1 \leq r \leq k$. It is by no means sharp for every n (see Figure 1). Igusa [8] mentioned that this bound follows easily from Brandt's conjecture using a comparison theorem by Akin and Davis [1, Theorem 3(c)]. We will describe this comparison theorem and show the implication in Section 4. We then prove a better general bound (Theorem 4.4), which is $D_{\mathcal{B}}(n) \leq k^2 - 2k - 1$, when $1 \leq r < k$ and $k \geq 4$. (Cases $n = 2, 5$ violate the bound.) We also present a lower bound for $D_{\mathcal{B}}(n)$ that we suspect is the correct value in general, although a proof of this claim has eluded us.

In Section 5 we consider *Carolina Solitaire*, the ordered version of Bulgarian Solitaire. Analogous with $d_{\mathcal{B}}(\lambda)$ and $D_{\mathcal{B}}(n)$ for Bulgarian Solitaire, we introduce $d_{\mathcal{C}}(\lambda)$ and $D_{\mathcal{C}}(n)$ for Carolina Solitaire. We show that for triangular $n = 1 + 2 + \cdots + k$, $D_{\mathcal{C}}(n) = k^2 - 1$. For other n of the form $n = 1 + 2 + \cdots + (k-1) + r$, $1 \leq r < k$, we prove $D_{\mathcal{C}}(n) \leq k^2 - k - 2$, provided that $k \geq 4$. We also present a lower bound for $D_{\mathcal{C}}(n)$ that we conjecture is the correct value in general.

Other variations of Bulgarian Solitaire have been introduced: Yeh and Servedio [9,10] studied a variation on circular compositions; Cannings and Haigh [4] investigated "Montreal Solitaire," in which game the rule from λ to $\mathcal{B}(\lambda)$ is changed when an exhausted pile (size 0) in λ occurs. Akin and Davis [1] also introduced "Austrian Solitaire." In this game, a special pile called the *bank* is reserved. Each move consists of taking a card from each pile into the bank, and then generating some new piles from the bank by a certain rule.

	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$r=1$	$D_{\mathcal{B}}(4)=2$	$D_{\mathcal{B}}(7)=4$	$D_{\mathcal{B}}(11)=8$	$D_{\mathcal{B}}(16)=15$	$D_{\mathcal{B}}(22)=24$	$D_{\mathcal{B}}(29)=35$
$r=2$	$D_{\mathcal{B}}(5)=3$	$D_{\mathcal{B}}(8)=5$	$D_{\mathcal{B}}(12)=8$	$D_{\mathcal{B}}(17)=12$	$D_{\mathcal{B}}(23)=18$	$D_{\mathcal{B}}(30)=28$
$r=3$	$D_{\mathcal{B}}(6)=6$	$D_{\mathcal{B}}(9)=7$	$D_{\mathcal{B}}(13)=9$	$D_{\mathcal{B}}(18)=13$	$D_{\mathcal{B}}(24)=18$	$D_{\mathcal{B}}(31)=24$
$r=4$		$D_{\mathcal{B}}(10)=12$	$D_{\mathcal{B}}(14)=14$	$D_{\mathcal{B}}(19)=16$	$D_{\mathcal{B}}(25)=19$	$D_{\mathcal{B}}(32)=25$
$r=5$			$D_{\mathcal{B}}(15)=20$	$D_{\mathcal{B}}(20)=23$	$D_{\mathcal{B}}(26)=26$	$D_{\mathcal{B}}(33)=29$
$r=6$				$D_{\mathcal{B}}(21)=30$	$D_{\mathcal{B}}(27)=34$	$D_{\mathcal{B}}(34)=38$
$r=7$					$D_{\mathcal{B}}(28)=42$	$D_{\mathcal{B}}(35)=47$
$r=8$						$D_{\mathcal{B}}(36)=56$

Figure 1. $D_{\mathcal{B}}(n)$ for $n=4,5,\ldots,36$.

Section 2. Characterizing Cyclic Partitions

We begin by describing $\mathcal{B}$-cyclic partitions of n, a result discovered by Brandt [3]. Etienne [5] as well as Akin and Davis [1] also gave proofs. Although Etienne proved the result for triangular n only, the idea in his proof is simple and can be applied to general n. The approach presented here is similar to Etienne's and we include it for the reader's benefit.

Theorem 2.1. [3] *Let $n=1+2+\cdots+(k-1)+r$, $1\le r\le k$. Then $\lambda\vdash n$ is $\mathcal{B}$-cyclic if and only if λ has the form*

$$(k-1+\delta_{k-1},k-2+\delta_{k-2},\ldots,2+\delta_2,1+\delta_1,\delta_0),$$

where each δ_i is 0 or 1 and $\sum_{i=0}^{k-1}\delta_i=r$.

Corollary 2.2. *If $n=1+2+\cdots+k$, then $(k,k-1,\ldots,2,1)$ is the unique $\mathcal{B}$-cyclic partition of n.* ∎

In order to prove the theorem, we introduce a natural array representation of a partition λ. We are particularly interested in the how repeatedly shifting λ affects the position of the 1's in the array.

For a partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_s)\vdash n$, we associate a $(0,1)$-array

$$M_\lambda=[m_{ij}]_{i,j=1}^{\infty},\quad \text{where}\quad m_{ij}=\begin{cases}1, & \text{if } j\le s \quad\text{and}\quad i\le\lambda_j;\\ 0, & \text{otherwise}.\end{cases}$$

The columns of M_λ correspond to the parts of λ.

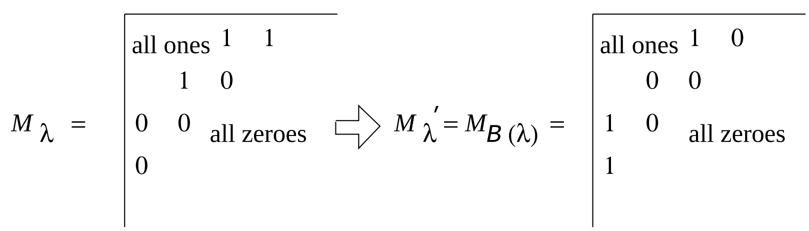
We notice that $M_{\mathcal{B}(\lambda)}$ can be obtained directly from M_λ by a shifting process on a $(0,1)$-array. In later sections, such a shifting process can help us evaluate $d_{\mathcal{B}}(\lambda)$ easily for some particular partitions λ. We describe this shifting process as follows: For a $(0,1)$-array M, we say that the w-th diagonal of M consists of entries m_{ij} where $i+j-1=w$. Assume $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_s)\vdash n$ and its associated array M_λ is given.

Step 1. Diagonally Circular Shifting: For each diagonal of M_λ, say $v_1, v_2, \ldots,$ and v_w are the entries (listed from left to right) in this diagonal, we replace them by $v_w, v_1, v_2, \ldots,$ and v_{w-1} respectively. Then we obtain a new array, denote it $M_\lambda{}'$. Note that the numbers of 1's in columns of $M_\lambda{}'$ are $s, \lambda_1-1, \lambda_2-1, \ldots, \lambda_s-1, 0, 0, 0, \ldots$. If $s \geq \lambda_1-1$ then $M_\lambda{}'$ is the array $M_{\mathcal{B}(\lambda)}$. If $s<\lambda_1-1$ then we need an extra step to obtain $M_{\mathcal{B}(\lambda)}$.

Step 2. Left Shifting: We remove all zero entries in the first column of $M_\lambda{}'$ and shift each entry at the (i,j)-position, $i \geq s+1$, $j \geq 2$, to the $(i,j-1)$-position. The new array is $M_{\mathcal{B}(\lambda)}$.

Figure 2 shows two examples of the shifting process described above.

$\lambda=(2,2,1,1)$:



$\lambda=(4,2)$:

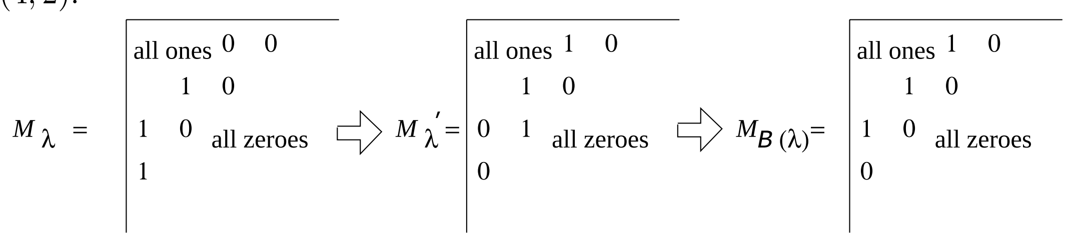


Figure 2. A shifting process on the arrays.

Proof of Theorem 2.1. $(\Leftarrow)$ Assume that λ has the stated form. The array M_λ has all ones on the first $k-1$ diagonals and all zeroes beyond diagonal k. Each shift just shifts entries on diagonal k. Thus, $\mathcal{B}^{(k)}(\lambda)=\lambda$, and λ is $\mathcal{B}$-cyclic .

$(\Rightarrow)$ In the shifting process, we notice that Step 1 keeps every entry on the same diagonal, while Step 2 always brings the entry 1 at the $(s+1,2)$-position to a diagonal with smaller index. Thus, for any $\mathcal{B}$-cyclic partition μ, the shifting process from M_μ to $M_{\mathcal{B}(\mu)}$ should involve Step 1 only.

Now assume that λ is $\mathcal{B}$-cyclic and λ cannot be expressed in the stated form. Then for some w, there is a 0 in the w-th diagonal and a 1 in the $(w+1)$-th diagonal. Since the series of shifting processes from M_λ to $M_{\mathcal{B}(\lambda)}$ to $M_{\mathcal{B}^{(2)}(\lambda)}$, and so on involves Step 1 (Diagonally Circular Shifting) only, and since the integers w and $w+1$ are relatively prime, we will reach (in at most $w^2 - w - 1$ steps) an array M_μ which has 0 as $(1, w)$-entry and 1 as $(w+1, 1)$-entry. (Figure 3 shows an example for such a series of shifting processes.) Then the shifting process from M_μ to $M_{\mathcal{B}(\mu)}$ involves Step 2, a contradiction. ∎

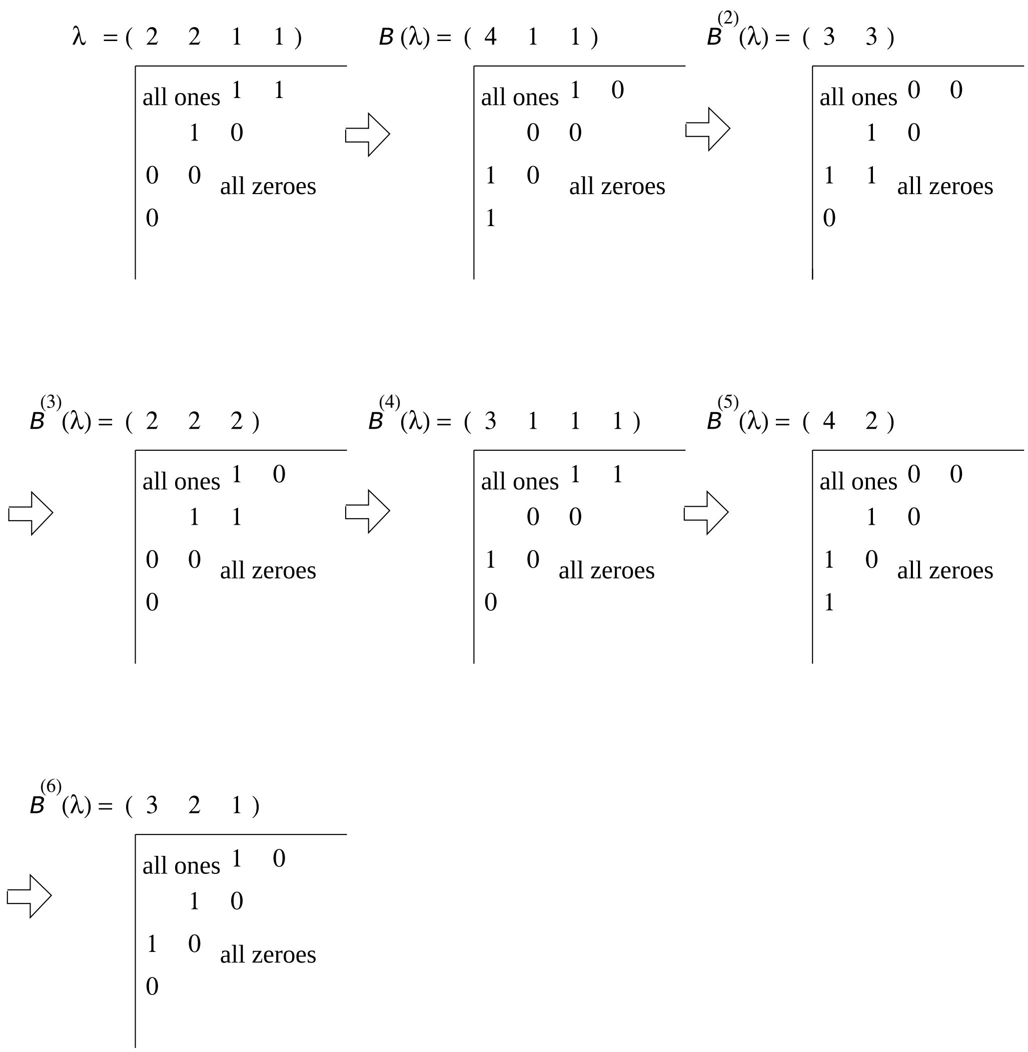


Figure 3. A series of shifting processes on the associated array of $\lambda = (2, 2, 1, 1)$.

It is easy to see from Theorem 2.1 that for a given n with $r = k$, $k-1$, or 1, starting from any partition of n and repeatedly applying the shift operation $\mathcal{B}$ always reaches a unique cycle of partitions of n.

For general n, Brandt [3] also showed that the number of cycles for Bulgarian Solitaire is $\frac{1}{k}\sum_{d|\gcd(k,r)}\phi(d)\binom{k/d}{r/d}$, where $\phi(d)$ is the Euler ϕ-function. The derivation of this formula was explained in detail later in the paper [1] of Akin and Davis.

Section 3. Evaluating $D_{\mathcal{B}}(n)$ for Triangular Numbers n

The partition $\lambda = (2,2,1,1)$ in Figure 3 gives $D_{\mathcal{B}}(3) \geq 3^2 - 3$. In general, we have

Theorem 3.1. [1, 2, 5] *If* $n = 1+2+\cdots+k$, *then* $D_{\mathcal{B}}(n) \geq k^2 - k$.

Proof. It suffices to present $\lambda \vdash n$ such that $d_{\mathcal{B}}(\lambda) = k^2 - k$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{k+1})$ where $\lambda_1 = k-1$, $\lambda_i = k-i+1$ for $i = 2, 3, \ldots, k$, and $\lambda_{k+1} = 1$. By repeatedly applying the shifting process (described in previous section) to the arrays $M_\lambda, M_{\mathcal{B}(\lambda)}, \ldots$, it is easy to see that

$$\begin{aligned}
\mathcal{B}^{(k)}(\lambda) &= (\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k + 1),\\
\mathcal{B}^{(2k)}(\lambda) &= (\lambda_1, \lambda_2, \ldots, \lambda_{k-1} + 1, \lambda_k),\\
&\ \ \vdots\\
\mathcal{B}^{((k-2)k)}(\lambda) &= (\lambda_1, \lambda_2, \lambda_3 + 1, \lambda_4, \lambda_5, \ldots, \lambda_k),\\
\mathcal{B}^{((k-1)k-1)}(\lambda) &= (\lambda_1 + 2, \lambda_2, \lambda_3, \ldots, \lambda_{k-1}), \text{and}\\
\mathcal{B}^{((k-1)k)}(\lambda) &= (k, k-1, k-2, \ldots, 2, 1).
\end{aligned}$$

Therefore $d_{\mathcal{B}}(\lambda) = k^2 - k$. ∎

Before we prove in Theorem 3.7 that for triangular n, the lower bound $k^2 - k$ is the actual value of $D_{\mathcal{B}}(n)$, we need some notation and lemmas. Let us start with the example for $n = 15$: The 176 partitions of 15 can be arranged in a tree illustrated in Figure 4 (from [3]) so that the vertices correspond to the partitions and going down corresponds to applying $\mathcal{B}$. (Thus, the root of the tree is the partition $(5,4,3,2,1)$.) In this figure, each vertex is labeled with the number of parts for its corresponding partition.

For a partition $\lambda \vdash n$, we associate a sequence $\mathrm{seq}_{\mathcal{B}}(\lambda) = \langle c_1, c_2, \ldots\rangle$, where c_i is the number of parts in $\mathcal{B}^{(i-1)}(\lambda)$. For instance, the top left vertex in Figure 4 (corresponding to the partition $\lambda = (4,4,3,2,1,1) \vdash 15$) has the associated sequence $\mathrm{seq}_{\mathcal{B}}(\lambda) = \langle 6,5,5,5,\underline{4,5,6},5,5,\underline{4,5,5,6},5,\underline{4,5,5,5,6},4,5,5,5,5,5,\ldots\rangle$.

In Figure 4, we observe that the pattern "$\underline{x-1, x, x, \ldots, x, x+1}$" appears quite often in most associated sequences. It will play a pivotal role in our process for evaluating $D_{\mathcal{B}}(n)$. To study this pattern, we associate each partition with a diagram: Given a

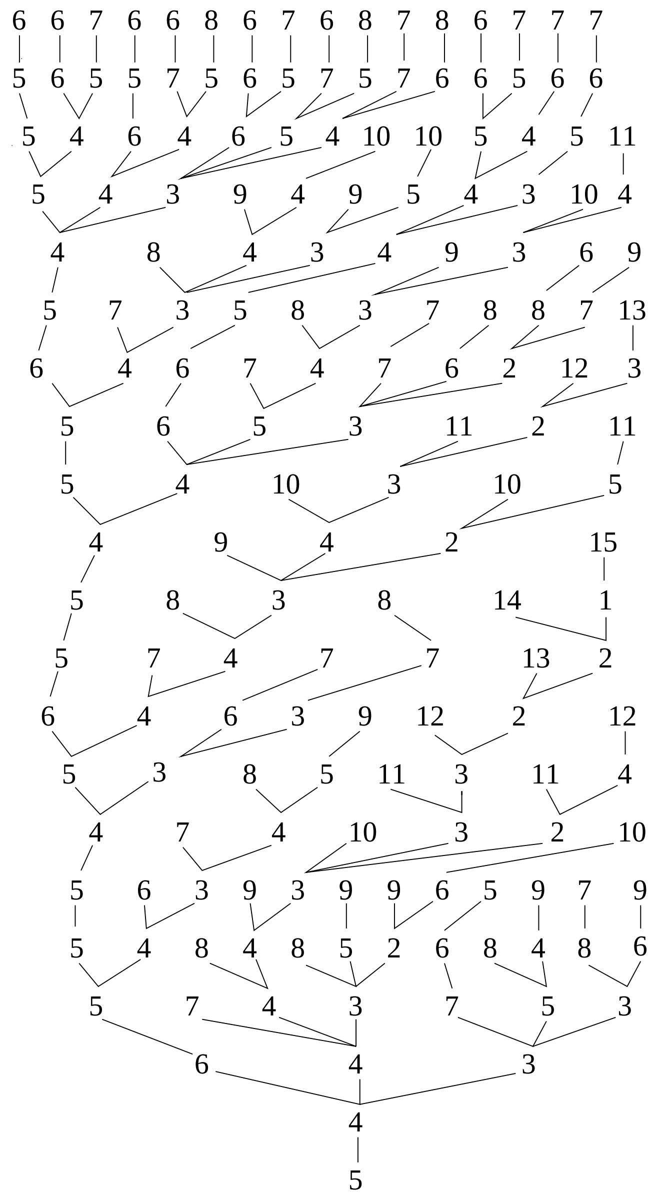


Figure 4. A tree for the partitions of 15.

partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) \vdash n$, let $\mathrm{seq}_{\mathcal{B}}(\lambda) = \langle c_1, c_2, \ldots \rangle$ be its associated sequence. Then $\mathrm{diagram}_{\mathcal{B}}(\lambda)$ is defined to be the diagram shown in Figure 5, where each λ_i-column (resp., c_i-column) has λ_i (resp., c_i) 1's on the top and has infinitely many 0's on all other entries. For example, if $\lambda = (4, 2, 2, 2)$ then $\mathrm{diagram}_{\mathcal{B}}(\lambda)$ is shown in Figure 6(a).

We notice that there are c_i 1's in the row corresponding to c_i. Furthermore, $\mathrm{diagram}_{\mathcal{B}}(\lambda)$ can be regarded as bookkeeping for the shift operation $\mathcal{B}$ on λ, since we can easily obtain $\mathcal{B}^{(i)}(\lambda)$ from $\mathrm{diagram}_{\mathcal{B}}(\lambda)$ for any integer i. For example, the

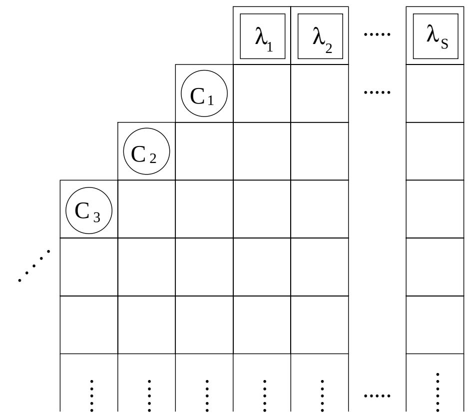


Figure 5. $\text{diagram}_{\mathcal{B}}(\lambda)$.

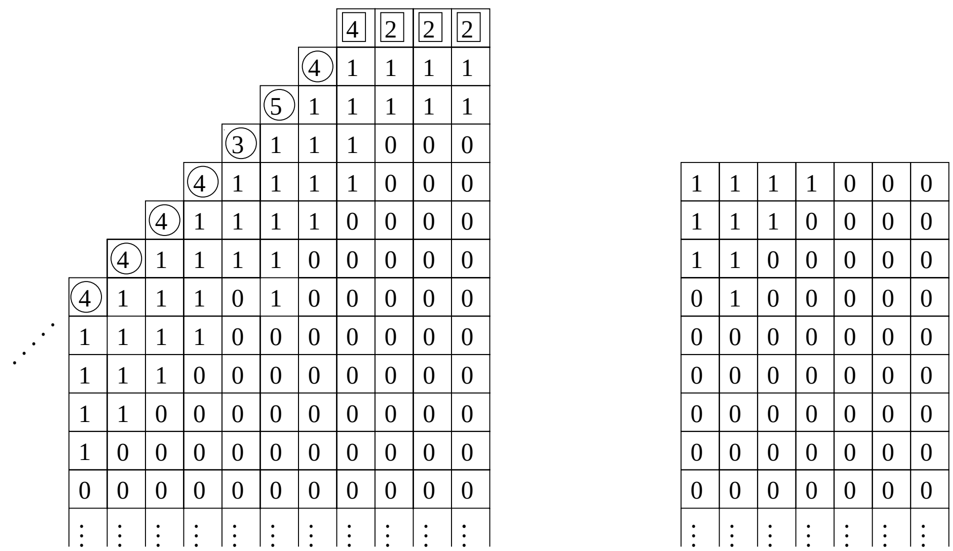


Figure 6. (a) $\text{diagram}_{\mathcal{B}}(\lambda)$ for $\lambda = (4, 2, 2, 2)$ (b) The rectangle below c_3.

rectangle below c_3 (see Figure 6(b)) gives $\mathcal{B}^{(3)}(\lambda) = (4, 3, 2, 1)$ and the rectangle below c_6 gives $\mathcal{B}^{(6)}(\lambda) = (4, 3, 2, 1)$.

Proposition 3.2. *Assume* $n \geq 1, \lambda \vdash n$, *and* $\text{seq}_{\mathcal{B}}(\lambda) = \langle c_1, c_2, \ldots \rangle$. *Then*

(1) $c_{i+1} \leq c_i + 1$ *for* $i \geq 1$.

(2) For $n = 1 + 2 + \cdots + (k-1) + r$, *where* $1 \leq r \leq k$, *the sum of any* k *consecutive terms in the sequence is at most* $k(k-1) + r$.

(3) (The Sandwich Property) If $i < j$ *and* $c_i < x < c_j$, *then there exist integers* p *and* q *such that* $i \leq p < q \leq j$, $q \geq p+2$, *and* $(c_p, c_{p+1}, \ldots, c_q) = (x-1, x, x, \ldots, x, x+1)$.

Proof. (1) follows from the definition of the shift operation $\mathcal{B}$. To prove (2), we see for $i > 0$ in $\text{diagram}_{\mathcal{B}}(\lambda)$ that $c_{i+1} + c_{i+2} + \cdots + c_{i+k}$ is the number of 1's in the k rows labelled by $c_{i+1}, c_{i+2}, \ldots$. These come from $\mathcal{B}^{(i)}(\lambda)$ (in the rectangle below c_i), or from the staircase consisting of $j-1$ squares to the right of c_{i+j}, $1 \le j \le k$. Thus, $c_{i+1} + \cdots + c_{i+k} \le n + 1 + 2 + \cdots + (k-1) = k(k-1) + r$.

For (3), choose p as large as possible such that $i \le p < j$ and $c_p \le x - 1$. ∎

We next require a series of lemmas that rely on Proposition 3.2 and $\text{diagram}_{\mathcal{B}}(\lambda)$.

Lemma 3.3. *Let $\lambda \vdash n = 1 + 2 + \cdots + k$ and $\text{seq}_{\mathcal{B}}(\lambda) = \langle c_1, c_2, \ldots \rangle$. Then*

(1) $d_{\mathcal{B}}(\lambda) = t$, where $\langle c_t, c_{t+1}, \ldots \rangle = \langle k-1, k, k, \ldots \rangle$.

(2) If $d_{\mathcal{B}}(\lambda) = t \ge k+1$ then at least one of the following holds:

(i) $(c_p, c_{p+1}, \ldots, c_q) = (k-1, k, k, \ldots, k, k+1)$ for some p, q with $t-k \le p < q \le t-1$ and $q \ge p+2$;

(ii) $(c_p, c_{p+1}, \ldots, c_q) = (k-2, k-1, k-1, \ldots, k-1, k)$ for some p, q with $t-k+1 \le p < q \le t+1$ and $q \ge p+2$.

Proof. (1) By Theorem 2.1, we have $d_{\mathcal{B}}(\lambda) = t$ where $\text{seq}_{\mathcal{B}}(\lambda) = \langle c_1, \ldots, c_t, k, k, k, \ldots \rangle$ and $c_t \ne k$. Then Proposition 3.2 forces $c_t = k-1$.

(2) For $i > j$, let $e_{i,j}$ denote the entry in the c_i-row and the c_j-column of $\text{diagram}_{\mathcal{B}}(\lambda)$. Then $e_{t,t-1} = 1$. Since $c_t = k-1$, we have $e_{t,i} = 0$ for some i, $t-k \le i \le t-2$. We choose such i as large as possible. Then $e_{t,i+1} = 1$, and, since $c_{t+1} = k = c_t + 1$, comparing rows for c_t and c_{t+1} in $\text{diagram}_{\mathcal{B}}(\lambda)$, we have $e_{t+1,i+1} = 1$ as well. The column below c_{i+1} then has 1's down at least as far as the row for c_{t+1}. Proposition 3.2(1) gives $c_i \ge c_{i+1} - 1$, which forces all entries in the c_i-column above the 0 at $e_{t,i}$ to be 1's. Therefore, $c_i = t-i-1$ and $c_{i+1} = t-i$.

If $i \ge t-k+1$, then $c_i \le k-2$, and (ii) holds by the Sandwich Property. Else, we have $i = t-k$, $c_{t-k} = k-1$, and $c_{t-k+1} = k$. If $c_j \le k-2$ for some j, $t-k+2 \le j \le t-1$, (ii) holds by the Sandwich Property, while if $c_j \ge k+1$ for some such j, (i) holds. Else, suppose for contradiction that $k-1 \le c_j \le k$, for $j = t-k+2, t-k+3, \ldots, t-1$: Since $i = t-k$, the row for c_t in $\text{diagram}_{\mathcal{B}}(\lambda)$ consists of $k-1$ 1's followed by all 0's. The row for c_{t+1} is then k 1's followed by all 0's. There are k 1's at the tops of columns $c_{t+1}, c_{t+2}, \ldots$. Hence, in the rectangle below c_{t-1}, we have just $k-1$ 1's in the first row, followed by $k-j+1$ 1's in row j, $2 \le j \le k$. Rows below that are all 0. In total, we have $n-1$ 1's in the rectangle, which is a contradiction since this rectangle represents, after reordering the columns if necessary, a partition of n. ∎

Lemma 3.4. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) \vdash n \ge 1$ and $\text{seq}_{\mathcal{B}}(\lambda) = \langle c_1, c_2, \ldots \rangle$. If for some p, k,*

$$(c_p, \ldots, c_{p+k}) = (k-2, k-1, k-1, \ldots, k-1, k),$$

then $p + k \le n + 1$.

Proof. Let $e_{i,j}$ (resp., $f_{i,j}$) denote the entry in the c_i-row and the c_j-column (resp., λ_j-column) of $\text{diagram}_{\mathcal{B}}(\lambda)$. Then we have $e_{p+k-2,p} = 1$ and $e_{p+k-1,p} = 0$, since $c_p = k-2$. By the given condition, in the c_{p+k}-row we also have $e_{p+k,j} = 1$ for $j = p+1, p+2, \ldots, p+k-1$. Note that in addition to these $k-1$ entries of 1, there is exactly one more 1 in the c_{p+k}-row, since $c_{p+k} = k$.

If $f_{p+k,j} = 1$ for some j, $1 \le j \le s$, then $p+k \le \lambda_j \le n$. Else, $e_{p+k,j} = 1$ for some j, $1 \le j \le p-1$. In fact, $j = 1$, for, otherwise, we have $e_{p+k,j-1} = 0$, and, since $c_{p+k} = k = c_{p+k-1} + 1$, comparing rows for c_{p+k} and c_{p+k-1} in $\mathrm{diagram}_{\mathcal{B}}(\lambda)$, we also have $e_{p+k-1,j-1} = 0$. Similarly, since $c_{p+k-1} = c_{p+k-2}$, $e_{p+k-2,p} = 1$, and $e_{p+k-1,p} = 0$, and comparing rows for c_{p+k-1} and c_{p+k-2} in $\mathrm{diagram}_{\mathcal{B}}(\lambda)$, we have $e_{p+k-2,j-1} = 0$. Then $c_j > c_{j-1} + 1$, a contradiction. So $j = 1$, and then $p+k-1 \le c_1 \le n$. ∎

Lemma 3.5. *Let $\lambda \vdash n \ge 1$ and $\mathrm{seq}_{\mathcal{B}}(\lambda) = \langle c_1, c_2, \ldots\rangle$. If for some integers x, p, q, $q \ge p+3$ and*

$$(c_p, c_{p+1}, \ldots, c_q) = (x-1, x, x, \ldots, x, x+1),$$

then either $p \le x$ or

$$(c_{p'}, c_{p'+1}, \ldots, c_{q'}) = (x'-1, x', x', \ldots, x', x'+1)$$

for some integers x', p', q' with $x' \le x$, $2 \le q'-p' < q-p$, and $p-x \le p' < q' \le p+1$.

Proof. We assume $p \ge x+1$ and prove the existence of x', p', and q'. For $i > j$, let $e_{i,j}$ denote the entry in the c_i-row and the c_j-column of $\mathrm{diagram}_{\mathcal{B}}(\lambda)$. Then $e_{p,p-1} = 1$. Since $c_p = x-1$, we have $e_{p,p'} = 0$ for some p', $p-x \le p' \le p-2$. We choose such p' as large as possible. Then $e_{p,j} = 1$ for $j = p'+1, p'+2, \ldots, p-1$, and, since $c_{p+1} = x = c_p + 1$, comparing rows for c_{p+1} and c_p in $\mathrm{diagram}_{\mathcal{B}}(\lambda)$, we also have $e_{p+1,j} = 1$ for $j = p'+1, p'+2, \ldots, p$. The column below $c_{p'+1}$ then has $1'$s down at least as far as the row at c_{p+1}. Proposition 3.2(1) gives $c_{p'} \ge c_{p'+1} - 1$, which forces all entries above the zero at $e_{p,p'}$ to be 1's. Therefore $c_{p'} = p-p'-1$, $c_{p'+1} = p-p'$, and $e_{p+2,p'+1} = 0$.

Since $c_{p+2} = x = c_{p+1}$, comparing rows for c_{p+2} and c_{p+1} in $\mathrm{diagram}_{\mathcal{B}}(\lambda)$, we have $e_{p+2,j} = 1$ for $j = p'+2, p'+3, \ldots, p+1$. If $e_{p+3,p'+2} = 1$ then $c_{p'+2} = p-p'+1$, and we are finished. So we may assume $e_{p+3,p'+2} = 0$ and $c_{p'+2} = p-p'$.

Since $c_{p+3} = x = c_{p+2}$, comparing rows for c_{p+3} and c_{p+2} in $\mathrm{diagram}_{\mathcal{B}}(\lambda)$, we have $e_{p+3,j} = 1$ for $j = p'+3, p'+4, \ldots, p+2$. If $e_{p+4,p'+3} = 1$ then $c_{p'+3} = p-p'+1$, and we are finished. So we may assume $e_{p+4,p'+3} = 0$ and $c_{p'+3} = p-p'$. Continue this process ....

So we may assume $e_{q-1,p'+q-p-2} = 0$, $c_{p'+q-p-2} = p-p'$, and $e_{q-1,j} = 1$ for $j = p'+q-p-1, p'+q-p, \ldots, q-2$. Since $c_q = x+1 = c_{q-1}+1$, comparing rows for c_q and c_{q-1} in $\mathrm{diagram}_{\mathcal{B}}(\lambda)$, we have $e_{q,p'+q-p-1} = 1$, and hence $c_{p'+q-p-1} = p-p'+1$. This completes the proof. ∎

Lemma 3.6. *Let $\lambda \vdash n \ge 1$ and $\mathrm{seq}_{\mathcal{B}}(\lambda) = \langle c_1, c_2, \ldots\rangle$. If*

$$(c_p, c_{p+1}, c_{p+2}) = (x-1, x, x+1),$$

then $p \le x$.

Proof. For $i > j$, let $e_{i,j}$ denote the entry in the c_i-row and the c_j-column of $\mathrm{diagram}_{\mathcal{B}}(\lambda)$. Then $e_{p,p-1} = 1$. Assume the contrary, *i.e.*, $p > x$. Then $c_p = x-1 < p-1$

implies $e_{p,i} = 0$ for some i, $1 \leq i \leq p-2$. Choose such i as large as possible. Then $e_{p,i+1} = 1$. Since $c_{p+1} = x = c_p + 1$ and $c_{p+2} = x + 1 = c_{p+1} + 1$, comparing rows for c_p, c_{p+1}, and c_{p+2} in $\text{diagram}_{\mathcal{B}}(\lambda)$, we also have $e_{p+2,i+1} = e_{p+1,i+1} = 1$. Thus $c_{i+1} > c_i + 1$, a contradiction. ∎

We are now in a position to prove

Theorem 3.7. [5, 8] *If $n = 1 + 2 + \cdots + k$, then*

$$D_{\mathcal{B}}(n) = k^2 - k.$$

Proof. Note that from Theorem 3.1 we have $D_{\mathcal{B}}(n) \geq k^2 - k$. Now we prove $D_{\mathcal{B}}(n) \leq k^2 - k$. Let $\lambda \vdash n$. It suffices to show $d_{\mathcal{B}}(\lambda) \leq k^2 - k$. We may assume $k \geq 3$, since it is easy to verify that $d_{\mathcal{B}}(\lambda) \leq k^2 - k$ for $k = 1, 2$. Further, we may also assume $d_{\mathcal{B}}(\lambda) \geq k+1$. (Otherwise, $d_{\mathcal{B}}(\lambda) \leq k < k^2 - k$, since $k \geq 3$.) Let $\text{seq}_{\mathcal{B}}(\lambda) = \langle c_1, c_2, \ldots \rangle$ where $\langle c_t, c_{t+1}, \ldots \rangle = \langle k-1, k, k, k, \ldots \rangle$. Then $d_{\mathcal{B}}(\lambda) = t$ and at least one of (i), (ii) of Lemma 3.3 holds.

If (ii) holds with $q = p + k$, we have $p = t - k + 1$ and $q = t + 1$. Note that Lemma 3.4 gives $p + k \leq n + 1$. Then $d_{\mathcal{B}}(\lambda) = t = p + k - 1 \leq n \leq k^2 - k$, since $k \geq 3$.

Else, (i) or (ii) holds and $q \leq p + k - 1$. By Lemmas 3.5 and 3.6, $\text{seq}_{\mathcal{B}}(\lambda)$ has at most $k^2 - 2k - 1$ terms before the c_p-term. Then $d_{\mathcal{B}}(\lambda) = t \leq (p-1) + k + 1 \leq (k^2 - 2k - 1) + k + 1 \leq k^2 - k$. ∎

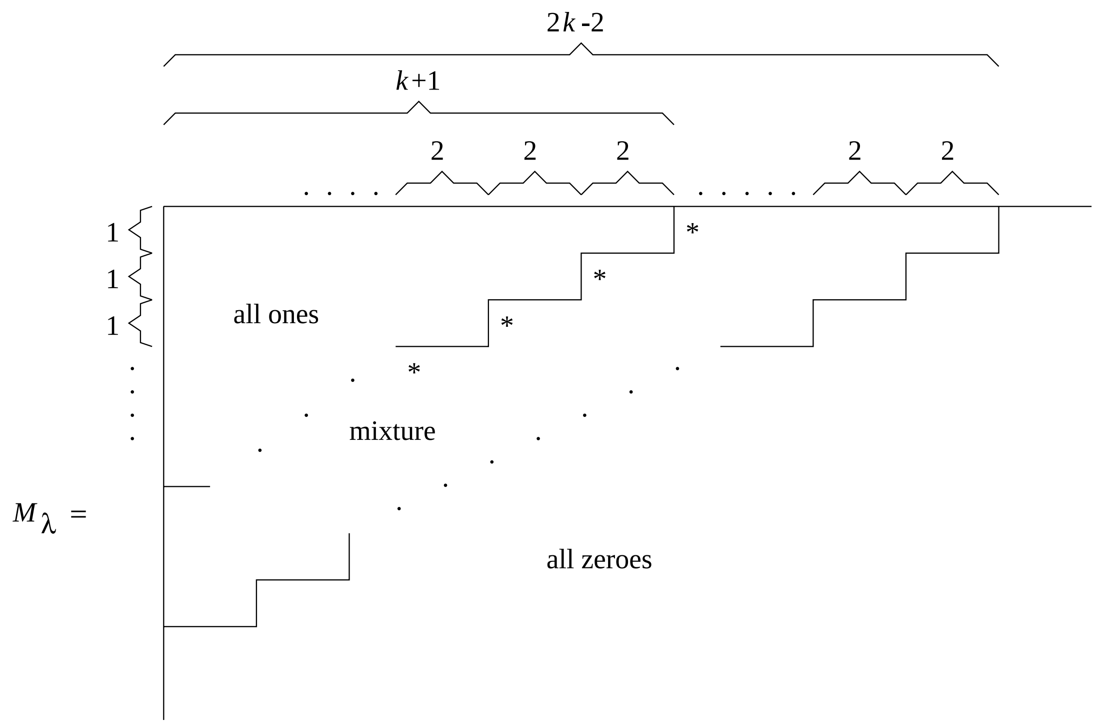
We observe that for $k = 2$ (resp., $k = 3$), $\lambda = (1, 1, 1)$ (resp., $\lambda = (2, 2, 1, 1)$) is the only partition of $n = 1 + 2 + \cdots + k$ achieving $d_{\mathcal{B}}(\lambda) = k^2 - k$. For $k \geq 4$, we have

Theorem 3.8. *Let $k \geq 4$ and $n = 1 + 2 + \cdots + k$. If $\lambda \vdash n$ with $d_{\mathcal{B}}(\lambda) = k^2 - k$, then the associated array $M_\lambda = [m_{ij}]_{i,j=1}^{\infty}$ satisfies the following conditions (illustrated in Figure 7):*
(1) $m_{ij} = 1$ *for* $j \leq k + 3 - 2i$,
(2) $m_{ij} = 0$ *for* $j \geq 2k + 1 - 2i$, *and*
(3) $|\{(i,j) : j = k + 4 - 2i, m_{ij} = 0\}| \leq 2$.

Proof. If $k \geq 4$ and $d_{\mathcal{B}}(\lambda) = k^2 - k$, the proof of Theorem 3.7 implies (i) of Lemma 3.3 holds and $q = p + k - 1$. Further, we have $\text{seq}_{\mathcal{B}}(\lambda) = \langle c_1, c_2, \ldots \rangle$ where $(c_k, c_{k+1}, c_{k+2}) = (k-1, k, k+1)$ and $(c_{2k}, c_{2k+1}, c_{2k+2}, c_{2k+3}) = (k-1, k, k, k+1)$.

For $i > j$, let $e_{i,j}$ denote the entry in the c_i-row and the c_j-column of $\text{diagram}_{\mathcal{B}}(\lambda)$. Then $e_{k+2,k+1} = 1$ and $e_{k+2,k} = 1$. We also have $e_{k+2,i} = 1$ for $i = 1, 2, \ldots, k-1$. (Otherwise, we choose i as large as possible such that $1 \leq i \leq k-1$ and $e_{k+2,i} = 0$. By comparing rows for c_{k+2}, c_{k+1}, and c_k in $\text{diagram}_{\mathcal{B}}(\lambda)$, we have $e_{k,i} = e_{k+1,i} = 0$, and hence $c_{i+1} > c_i + 1$, a contradiction.) Therefore, $c_i \geq k + 2 - i$ for $i = 1, 2, \ldots, k+2$, and hence Condition (1) holds.

We also have $e_{2k,i} = 1$ for $i = k+1, k+2, \ldots, 2k-1$. (Otherwise, we choose i as large as possible such that $k + 3 \leq i \leq 2k - 2$ and $e_{2k,i} = 0$. Then $e_{2k,i+1} = 1$. We



Among the $\lfloor (k+3)/2 \rfloor$ entries marked by $*$, at most two are zeroes.

Figure 7. Conditions for Theorem 3.8.

note that $e_{2k+1,k+1} = 1$ and $e_{2k+2,k+1} = 0$, since $c_{k+1} = k$. By comparing rows for c_{2k}, c_{2k+1}, and c_{2k+2} in diagram$_{\mathcal{B}}(\lambda)$, we have $e_{2k+2,i+1} = e_{2k+1,i+1} = 1$, and hence $c_{i+1} > c_i + 1$, a contradiction.) Therefore, $e_{2k,i} = 0$ for $i = 1, 2, \ldots, k$, and Condition (2) holds.

Since $e_{2k,k+3} = 1$, by comparing rows for c_{2k}, c_{2k+1}, c_{2k+2}, and c_{2k+3} in diagram$_{\mathcal{B}}(\lambda)$, we have $e_{2k+3,k+3} = e_{2k+2,k+3} = e_{2k+1,k+3} = 1$, and hence $c_{k+3} \geq k$. So we have $\sum_{j=1}^{k+2} e_{k+3,j} \geq k$, and hence Condition (3) holds. ∎

We have checked by computer that the converse of this theorem is true for $k = 4, 5, 6, 7$. (In particular, the converse with Condition (3) removed is also true for $k = 4, 5, 6$.) When $k = 8$, there are 1276 partitions satisfying all three conditions in Theorem 3.8, but 9 partitions among them have $d_{\mathcal{B}}(\lambda) = 20 \neq k^2 - k$. So the three conditions given in Theorem 3.8 are necessary, but not sufficient, for the associated array of extremal partitions.

Section 4. Evaluating Bounds on $D_{\mathcal{B}}(n)$ for Non-Triangular Numbers n

For any non-triangular n, we can write $n = 1 + 2 + \cdots + (k-1) + r$, where $1 \leq r < k$. In this section we first prove $D_{\mathcal{B}}(n) \leq k^2 - k$ based on Theorem 3.7 and a comparison theorem from [1]. Then we improve this bound to $k^2 - 2k - 1$ for $k \geq 4$. We also present a lower bound that is likely the actual value.

Let $\Lambda = \cup_{i=1}^{\infty} \Lambda_i$ where Λ_i denotes the set containing all partitions of i. For $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ in Λ, we define a *partial ordering* (*i.e.,* a reflexive, antisymmetric and transitive relation) by:

$$\lambda \leq \mu \quad \Leftrightarrow \quad \lambda_i \leq \mu_i \quad \text{for} \quad i = 1, 2, \ldots.$$

Then we have the following theorem due to Akin and Davis [1, Theorem 3(c)]:

Theorem 4.1. *If λ, $\mu \in \Lambda$ and $\lambda \leq \mu$, then $\mathcal{B}(\lambda) \leq \mathcal{B}(\mu)$.*

Proof. Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ (not necessarily in non-increasing order) be the parts of two partitions x and y respectively. We observe that if $x_i \leq y_i$ for $i \geq 1$, then $x \leq y$. (To verify this, we may assume further, without loss of generality, that the x_i's are already in non-increasing order. If $y_1 < y_2$, we interchange y_1 and y_2, then $x_i \leq y_i$ for $i \geq 1$ still holds. If $y_2 < y_3$, we interchange y_2 and y_3, then $x_i \leq y_i$ for $i \geq 1$ still holds. Continuing this process, eventually we can arrange all y_i's in non-increasing order and $x_i \leq y_i$ for $i \geq 1$ still holds. Therefore $x \leq y$.)

Now we assume λ, $\mu \in \Lambda$ and $\lambda \leq \mu$. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$, where $\lambda_1 \geq \cdots \geq \lambda_s > \lambda_{s+1} = 0$, $\mu_1 \geq \cdots \geq \mu_t > \mu_{t+1} = 0$, and $\lambda_i \leq \mu_i$ for $i \geq 1$. Then the parts of $\mathcal{B}(\lambda)$ are $x_1 = s$, $x_2 = \lambda_1 - 1, x_3 = \lambda_2 - 1, \ldots, x_{s+1} = \lambda_s - 1, x_{s+2} = 0, x_{s+3} = 0, \ldots$. Similarly, the parts of $\mathcal{B}(\mu)$ are $y_1 = t$, $y_2 = \mu_1 - 1, y_3 = \mu_2 - 1, \ldots, y_{t+1} = \mu_t - 1, y_{t+2} = 0, y_{t+3} = 0, \ldots$. Since $x_i \leq y_i$ for $i \geq 1$, we have $\mathcal{B}(\lambda) \leq \mathcal{B}(\mu)$ by the observation above. ∎

Combining this theorem with Theorem 3.7, we can show

Theorem 4.2. [5, 8] *Let $n = 1 + 2 + \cdots + (k-1) + r, 1 \leq r < k$. Then*

$$D_{\mathcal{B}}(n) \leq k^2 - k.$$

Proof. Let $\lambda \vdash n$. It suffices to show $d_{\mathcal{B}}(\lambda) \leq k^2 - k$. We choose μ, ν such that $\mu \vdash (1 + 2 + \cdots + (k-1))$, $\nu \vdash (1 + 2 + \cdots + k)$, and $\mu \leq \lambda \leq \nu$. By Theorem 4.1, we have that $\mathcal{B}^{(k^2-k)}(\mu) \leq \mathcal{B}^{(k^2-k)}(\lambda) \leq \mathcal{B}^{(k^2-k)}(\nu)$. Note that Theorem 3.7 gives $\mathcal{B}^{(k^2-k)}(\mu) = (k-1, k-2, \ldots, 2, 1)$ and $\mathcal{B}^{(k^2-k)}(\nu) = (k, k-1, \ldots, 2, 1)$. Thus $\mathcal{B}^{(k^2-k)}(\lambda)$ has the form stated in Theorem 2.1 and $d_{\mathcal{B}}(\lambda) \leq k^2 - k$. ∎

We prove an analogue of Lemma 3.3 for non-triangular n. Applying it along with Lemmas 3.4, 3.5, and 3.6 leads to an improved upper bound on $D_{\mathcal{B}}(n)$ for non-triangular n.

Lemma 4.3. *Let $\lambda \vdash n = 1 + 2 + \cdots + k + r, 1 \leq r < k$, and $\mathrm{seq}_{\mathcal{B}}(\lambda) = \langle c_1, c_2, \ldots \rangle$.*

(1) If $(c_t, \ldots, c_{t+k-1}) = (c_{t+k}, \ldots, c_{t+2k-1}) = \cdots$, i.e., $c_{t+i} = c_{t+i+k}$ for $i \geq 0$, then $d_{\mathcal{B}}(\lambda) \leq t - 1$. (Thus we can find $d_{\mathcal{B}}(\lambda)$ by choosing such t as small as possible.)

(2) If $c_t = k-1, c_{t+1} = k, c_{t+i} = c_{t+i+k}$ for $i \geq 0$, and $(c_{t-k}, \ldots, c_{t-1}) \neq (c_t, \ldots, c_{t+k-1})$, then at least one of (i), (ii) of Lemma 3.3 holds.

Proof. (1) If $c_{t+i} = c_{t+i+k}$ for $i \geq 0$, then each c_{t+i} is the number of parts for some $\mathcal{B}$-cyclic partition, and hence Theorem 2.1 gives $k-1 \leq c_{t+i} \leq k$ for $i \geq 0$. In particular, we note that the rows for $c_t, c_{t+1}, \ldots, c_{t+k-1}$ in diagram$_{\mathcal{B}}(\lambda)$ each consists of $k-1$ or k 1's. Then $\mathcal{B}^{(t-1)}(\lambda)$ has the stated form of Theorem 2.1, since it is represented by the rectangle below c_{t-1}, after reordering the columns if necessary. Therefore, $\mathcal{B}^{(t-1)}(\lambda)$ is $\mathcal{B}$-cyclic , and hence $d_{\mathcal{B}}(\lambda) \leq t-1$.

(2) The proof is similar to that of Lemma 3.3(2). In the last case, we suppose for contradiction that $c_{t-k} = k-1$, $c_{t-k+1} = k$, and $k-1 \leq c_j \leq k$, for $j = t-k+2, t-k+3, \ldots, t-1$. Similar to the proof of (1), then the rectangle below c_{t-k-1} in diagram$_{\mathcal{B}}(\lambda)$ gives that $\mathcal{B}^{(t-k-1)}(\lambda)$ is $\mathcal{B}$-cyclic . Thus, by Theorem 2.1, $\mathcal{B}^{(t-k-1+i)}(\lambda) = \mathcal{B}^{(t-1+i)}(\lambda)$ for $i \geq 0$. Counting the number of parts in each partition, we have $(c_{t-k}, \ldots, c_{t-1}) = (c_t, \ldots, c_{t+k-1})$, a contradiction. ∎

Theorem 4.4. *Let n be non-triangular and $n = 1+2+\cdots+(k-1)+r$, $1 \leq r < k$. Then*
(1) $D_{\mathcal{B}}(n) \leq k^2-2k-1$ for $k \geq 4$.
(2) Furthermore, equality holds when $k \geq 4$ and $r = k-1$.

Proof. (1) We may assume $k \geq 5$, since we have $D_{\mathcal{B}}(n) \leq 7$ for $k = 4$ from Figure 1. Let $\lambda \vdash n$. It suffices to show $d_{\mathcal{B}}(\lambda) \leq k^2-2k-1$. Let seq$_{\mathcal{B}}(\lambda) = \langle c_1, c_2, \ldots \rangle$. Let $\mathcal{B}^{(t-1)}(\lambda)$ be $\mathcal{B}$-cyclic for some $t \geq 1$. Then Theorem 2.1 gives $c_{t+i} = c_{t+i+k}$ and $k-1 \leq c_{t+i} \leq k$ for $i \geq 0$. Further, we may assume $c_t = k-1$ and $c_{t+1} = k$, since $r \neq k$. We choose such t as small as possible. If $t \leq k$, then $d_{\mathcal{B}}(\lambda) \leq t-1 \leq k-1 < k^2-2k-1$, since $k \geq 5$. So we may assume $t \geq k+1$ and $(c_{t-k}, \ldots, c_{t-1}) \neq (c_t, \ldots, c_{t+k-1})$. Then, by Lemma 4.3, at least one of (i), (ii) of Lemma 3.3 holds.

If (ii) holds with $q = p+k$, we have $(p,q) = (t-k+1, t+1)$. Note that Lemma 3.4 gives $p+k \leq n+1$. Then $d_{\mathcal{B}}(\lambda) \leq t-1 = p+k-2 \leq n-1 < k^2-2k-1$, since $k \geq 5$.

Else, if (i) or (ii) holds and $q \leq p+k-2$, by Lemmas 3.5 and 3.6, seq$_{\mathcal{B}}(\lambda)$ has at most k^2-3k-1 terms before the c_p-term. Then $d_{\mathcal{B}}(\lambda) \leq t-1 \leq (p-1)+k \leq (k^2-3k-1)+k = k^2-2k-1$.

Else, suppose for contradiction that (i) or (ii) holds with $q = p+k-1$: if (i) holds with $q = p+k-1$, we have $(p,q) = (t-k, t-1)$ and $c_{t-1} = k+1$, which is a contradiction, since, by Proposition 3.2(2), we have $c_{t-1} \leq k$; else, (ii) holds with $q = p+k-1$, we have $(p,q) = (t-k+2, t+1)$ and $c_{t-k+2} = k-2$, which is also a contradiction, since, by comparing rows for c_{t+1} and c_t in diagram$_{\mathcal{B}}(\lambda)$, we have $c_{t-k+2} \neq k-2$.

(2) Assume $k \geq 4$ and $r = k-1$. By (1), it suffices to present $\lambda \vdash n$ such that $d_{\mathcal{B}}(\lambda) = k^2-2k-1$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{k+1})$ where $\lambda_1 = k-1$, $\lambda_2 = k-2$, $\lambda_i = k-i+1$ for $i = 3, 4, \ldots, k$, and $\lambda_{k+1} = 1$. Imitating the proof of Theorem 3.1, we can show that $d_{\mathcal{B}}(\lambda) = k^2-2k-1$. ∎

So far we have obtained an upper bound for $D_{\mathcal{B}}(n)$ in Theorem 4.4. Next we will find a lower bound for $D_{\mathcal{B}}(n)$ and conjecture this lower bound is the actual value of $D_{\mathcal{B}}(n)$.

Lower Bound Theorem 4.5. *For $n \geq 3$ and $n = 1+2+\cdots+(k-1)+r$, $1 \leq r \leq k$,*

$$D_{\mathcal{B}}(n) \geq \begin{cases} (k-3-r)k+r+2 & \text{for} \quad 1 \leq r < \lfloor \frac{k-1}{2} \rfloor \\ n-k+1 & \text{for} \quad r = \lfloor \frac{k-1}{2} \rfloor \text{ or } \lfloor \frac{k+1}{2} \rfloor \\ (r-2)k+r & \text{for} \quad \lfloor \frac{k+1}{2} \rfloor < r \leq k. \end{cases}$$

Proof. It suffices to present $\lambda \vdash n$ such that $d_{\mathcal{B}}(\lambda)$ is the stated lower bound on $D_{\mathcal{B}}(n)$. We consider three cases: (Associated arrays for two extremal partitions in Case (1) and (3) are illustrated in Figure 8.)

Case (1) $1 \leq r < \lfloor \frac{k-1}{2} \rfloor$: Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ where $\lambda_1 = k-2$, $\lambda_i = k-i$ for $i = 2, 3, \ldots, k-r-1$, and $\lambda_i = k-i+1$ for $i = k-r, k-r+1, \ldots, k$. Imitating the proof of Theorem 3.1, we can show that $d_{\mathcal{B}}(\lambda) = (k-3-r)k+r+2$.

Case (2) $r = \lfloor \frac{k-1}{2} \rfloor$ or $\lfloor \frac{k+1}{2} \rfloor$: Let $\lambda = (1, 1, \ldots, 1)$ with n parts of 1. By applying induction on i, we can show that $\mathcal{B}^{(1+1+2+3+\cdots+i)}(\lambda) = (n-(1+2+\cdots+i), i, i-1, i-2, \ldots, 2, 1)$ for $i = 1, 2, \ldots, k-2$. Thus $d_{\mathcal{B}}(\lambda) = 1+1+2+3+\cdots+(k-2)+(r-1) = n-k+1$.

Case (3) $\lfloor \frac{k+1}{2} \rfloor < r \leq k$: Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{k+1})$ where $\lambda_i = k-i$ for $i = 1, 2, \ldots, k-r+1$, $\lambda_i = k-i+1$ for $i = k-r+2, k-r+3, \ldots, k$, and $\lambda_{k+1} = 1$. Imitating the proof of Theorem 3.1, we can show that $d_{\mathcal{B}}(\lambda) = (r-2)k+r$. ∎

Conjecture 4.6. [5] *Assume $n \geq 3$.*

(1) If $n \in [1+2+\cdots+(K-1), 1+2+\cdots+(K-1)+\lfloor \frac{K-2}{2} \rfloor]$, then $D_{\mathcal{B}}(n)$ strictly decreases from $(K-1)^2-(K-1)$ to $\lceil \frac{K^2-2K}{2} \rceil$.

(2) If $n \in [1+2+\cdots+(K-1)+\lceil \frac{K-2}{2} \rceil, 1+2+\cdots+(K-1)+K]$, then $D_{\mathcal{B}}(n)$ strictly increases from $\lceil \frac{K^2-2K}{2} \rceil$ to K^2-K .

Actually Etienne gave K^2-K in stead of $(K-1)^2-(K-1)$ in (1) and gave $(K+1)^2-(K+1)$ in stead of K^2-K in (2). Both of these seem to be mistakes.

According to [5], this conjecture was confirmed for $n = 3, 4, 5, \ldots, 55$. Here we suggest a stronger conjecture that is confirmed for $n = 3, 4, 5, \ldots, 36$ (see Figure 1), and also for $r = k-1, k$.

Conjecture 4.7. *In Theorem 4.5 "$D_{\mathcal{B}}(n) \geq$" can be replaced with "$D_{\mathcal{B}}(n) =$."*

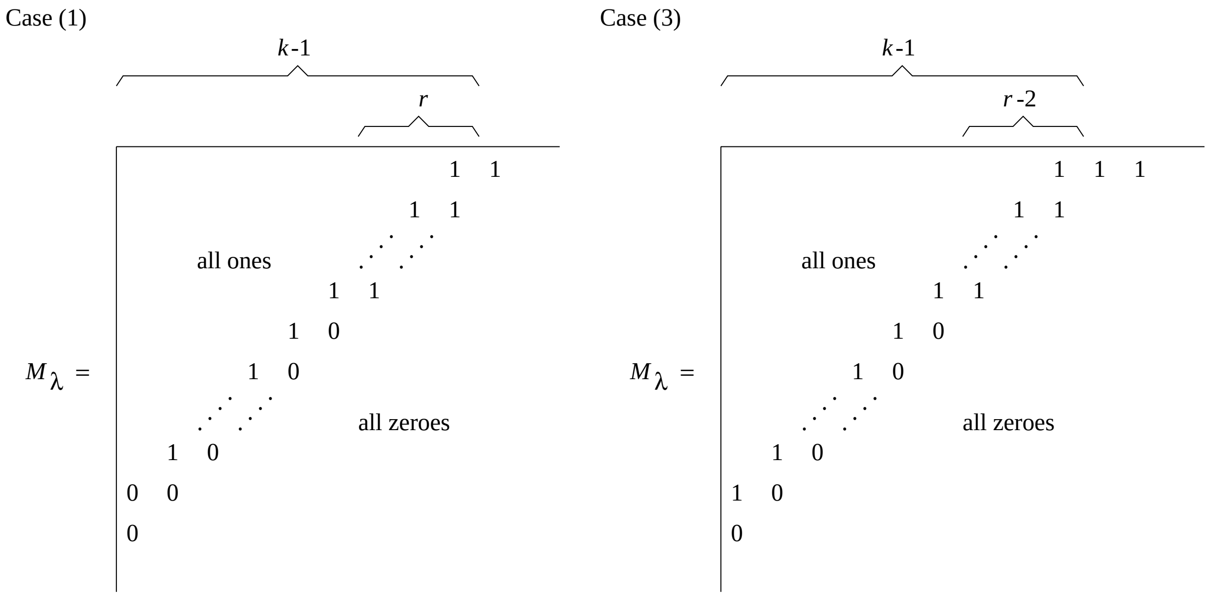


Figure 8. Associated arrays for two extremal partitions in Theorem 4.5.

Section 5. Carolina Solitaire: A Variation of Bulgarian Solitaire

Let us now define Carolina Solitaire formally, and then derive analogues of some of the results for Bulgarian Solitaire. For a positive integer n, we say $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a *composition of* n, and we write $\lambda \models n$, provided that the positive integers $\lambda_1, \lambda_2, \cdots, \lambda_s$ (not necessarily in non-increasing order) add up to n and $\lambda_i = 0$ for $i \geq s+1$. We say such λ has s (positive) parts, and we often drop the zeroes, writing $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$.

The *shift operation* $\mathcal{C}$ *on* λ is the composition of n, denoted $\mathcal{C}(\lambda)$, obtained by decreasing each part λ_i by one, inserting a part $= s$ as the first part, and discarding any zero parts. So $\mathcal{C}^{(i)}(\lambda)$ denotes the composition obtained by successively applying the shift operation $\mathcal{C}$ to λ a total of i times. Starting with a composition λ, we describe Carolina Solitaire by repeatedly applying the shift operation $\mathcal{C}$ to obtain the sequence of compositions

$$\lambda, \mathcal{C}(\lambda), \mathcal{C}^{(2)}(\lambda), \ldots.$$

For a couple of simple examples, we note that $\lambda = (2,1,1,1,1) \models 6$ gives the sequence $(2,1,1,1,1), (5,1), (2,4), (2,1,3), (3,1,2), (3,2,1), (3,2,1), \ldots,$ while $\lambda = (6,1) \models 7$ yields $(6,1)$, $(2,5)$, $(2,1,4)$, $(3,1,3)$, $(3,2,2)$, $(3,2,1,1)$, $(4,2,1)$, $(3,3,1)$, $(3,2,2)$, $(3,2,1,1)$, $(4,2,1)$, $\ldots$. The first example is fixed at the composition $(3,2,1)$ after five steps, while the second example reaches a cycle after four steps.

We say a composition $\mu = (\mu_1, \mu_2, \ldots, \mu_s) \models n$ is a *permutation* of a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) \models n$ if the parts μ_i of μ are a permutation of the parts λ_i of λ. We notice the connection between Bulgarian Solitaire and Carolina Solitaire: If $\mu \models n$ is a permutation of $\lambda \vdash n$ then, for $i > 0$, $\mathcal{C}^{(i)}(\mu) \models n$ is also a permutation of $\mathcal{B}^{(i)}(\lambda) \vdash n$.

Similar to $\mathcal{B}$-cyclic partition, $d_{\mathcal{B}}(\lambda)$, and $D_{\mathcal{B}}(n)$ in Bulgarian Solitaire, we can define $\mathcal{C}$-cyclic composition, $d_{\mathcal{C}}(\lambda)$, and $D_{\mathcal{C}}(n)$ in Carolina Solitaire: We say a composition $\mu \models n$ is *$\mathcal{C}$-cyclic* if $\mathcal{C}^{(i)}(\mu) = \mu$ for some $i > 0$. We will prove in Theorem 5.1 that $\lambda \models n$ is $\mathcal{C}$-cyclic if and only if $\lambda \vdash n$ is $\mathcal{B}$-cyclic . To measure how long it takes for Carolina Solitaire to cycle, for $\lambda \models n$ we let $d_{\mathcal{C}}(\lambda)$ denote the smallest integer $i \geq 0$ such that $\mathcal{C}^{(i)}(\lambda)$ is $\mathcal{C}$-cyclic . Let $D_{\mathcal{C}}(n) := \max\{d_{\mathcal{C}}(\lambda) : \lambda \models n\}$, so that for any $\lambda \models n$, $D_{\mathcal{C}}(n)$ steps reach a cycle. Trivially, $D_{\mathcal{C}}(1) = D_{\mathcal{C}}(2) = 0$ and $D_{\mathcal{C}}(3) = 3$. The values of $D_{\mathcal{C}}(n)$, $4 \leq n \leq 36$, are displayed in Figure 9, which we worked out by computer. The data are arranged using the representation $n = 1 + 2 + \cdots + (k-1) + r$, $1 \leq r \leq k$.

We will show in Theorem 5.5 that $D_{\mathcal{C}}(n) = k^2 - 1$ whenever $n = 1 + 2 + \cdots + k$. When n is a non-triangular number, we will prove in Theorem 5.8 that $D_{\mathcal{C}}(n) \leq k^2 - k - 2$ for $k \geq 4$. We also present in Theorem 5.9 a lower bound for $D_{\mathcal{C}}(n)$ that we conjecture is the correct value in general.

	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$r=1$	$D_{\mathcal{C}}(4)=3$	$D_{\mathcal{C}}(7)=6$	$D_{\mathcal{C}}(11)=11$	$D_{\mathcal{C}}(16)=19$	$D_{\mathcal{C}}(22)=29$	$D_{\mathcal{C}}(29)=41$
$r=2$	$D_{\mathcal{C}}(5)=5$	$D_{\mathcal{C}}(8)=8$	$D_{\mathcal{C}}(12)=12$	$D_{\mathcal{C}}(17)=17$	$D_{\mathcal{C}}(23)=23$	$D_{\mathcal{C}}(30)=34$
$r=3$	$D_{\mathcal{C}}(6)=8$	$D_{\mathcal{C}}(9)=10$	$D_{\mathcal{C}}(13)=13$	$D_{\mathcal{C}}(18)=18$	$D_{\mathcal{C}}(24)=24$	$D_{\mathcal{C}}(31)=31$
$r=4$		$D_{\mathcal{C}}(10)=15$	$D_{\mathcal{C}}(14)=18$	$D_{\mathcal{C}}(19)=21$	$D_{\mathcal{C}}(25)=25$	$D_{\mathcal{C}}(32)=32$
$r=5$			$D_{\mathcal{C}}(15)=24$	$D_{\mathcal{C}}(20)=28$	$D_{\mathcal{C}}(26)=32$	$D_{\mathcal{C}}(33)=36$
$r=6$				$D_{\mathcal{C}}(21)=35$	$D_{\mathcal{C}}(27)=40$	$D_{\mathcal{C}}(34)=45$
$r=7$					$D_{\mathcal{C}}(28)=48$	$D_{\mathcal{C}}(35)=54$
$r=8$						$D_{\mathcal{C}}(36)=63$

Figure 9. $D_{\mathcal{C}}(n)$ for $n = 4, 5, \ldots, 36$.

Now we start with the characterization of $\mathcal{C}$-cyclic compositions. Similar to the associated $(0,1)$-array for a partition, we can define the associated $(0,1)$-array for a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) \models n$:

$$M_\lambda = [m_{ij}]_{i,j=1}^{\infty}, \quad \text{where} \quad m_{ij} = \begin{cases} 1, & \text{if } j \leq s \quad \text{and} \quad i \leq \lambda_j; \\ 0, & \text{otherwise .} \end{cases}$$

The columns of M_λ correspond to the parts of λ.

We notice that $M_{\mathcal{C}(\lambda)}$ can be obtained directly from M_λ by a shifting process on a $(0,1)$-array: Assume $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) \models n$ and its associated array M_λ is given.
Step 1. Diagonally Circular Shifting: This is the same as Step 1 for Bulgarian Solitaire described in Section 2. After Step 1, we obtain a new array, denote it $M_\lambda{}'$. Note that the numbers of 1's in columns of $M_\lambda{}'$ are $s, \lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_s - 1, 0, 0, 0, \ldots$. If $\lambda_i - 1 = 0$ and $\lambda_{i+1} - 1 > 0$ for some i, then we need an extra step to obtain $M_{\mathcal{C}(\lambda)}$. Otherwise, $M_\lambda{}'$ is the array $M_{\mathcal{C}(\lambda)}$.

Step 2. Left Shifting: Whenever there is some integer j such that the j-th column of $M_\lambda{}'$ is an all zero column and the $(j+1)$-th column of $M_\lambda{}'$ is not an all zero column, we remove the j-th column and shift each column with larger index one column to the left. We repeat this shifting until there does not exist such an integer j.

Using the shifting process described above and imitating the proof of Theorem 2.1, we can show

Theorem 5.1. *Let $n = 1 + 2 + \cdots + (k-1) + r$, $1 \le r \le k$. Then $\lambda \models n$ is $\mathcal{C}$-cyclic if and only if λ has the form*

$$(k-1+\delta_{k-1}, k-2+\delta_{k-2}, \ldots, 2+\delta_2, 1+\delta_1, \delta_0),$$

where each δ_i is 0 or 1 and $\sum_{i=0}^{k-1} \delta_i = r$. ∎

Corollary 5.2. *If $n = 1 + 2 + \cdots + k$, then $(k, k-1, \ldots, 2, 1)$ is the unique $\mathcal{C}$-cyclic composition of n.* ∎

By Theorems 2.1 and 5.1, $\lambda \models n$ is $\mathcal{C}$-cyclic if and only if $\lambda \vdash n$ is $\mathcal{B}$-cyclic . So the number of cycles for Carolina Solitaire is the same as that for Bulgarian Solitaire.

Next we shall prove $D_{\mathcal{C}}(n) = k^2 - 1$ for triangular n.

Proposition 5.3. *Let $n = 1 + 2 + \cdots + k$ and $\lambda \models n$. Then λ is a permutation of $(k, k-1, \ldots, 2, 1)$ if and only if $d_{\mathcal{C}}(\lambda) \le k-1$.*

Proof. If λ is a permutation of $(k, k-1, \ldots, 1)$, it is easy to see that $\mathcal{C}^{(k-1)}(\lambda) = (k, k-1, \ldots, 1)$ and $d_{\mathcal{C}}(\lambda) \le k-1$. If λ is not a permutation of $(k, k-1, \ldots, 1)$, then there exists $\mu = \mathcal{C}^{(i)}(\lambda)$, $i \ge 0$, such that μ is not a permutation of $(k, k-1, \ldots, 1)$ and $\mathcal{C}(\mu)$ is a permutation of $(k, k-1, \ldots, 1)$. We note that μ must be a permutation of $(k+1, k-1, k-2, k-3, \ldots, 2)$, and hence $\mathcal{C}^{(k-1)}(\mu) = (k, k-1, \ldots, 3, 1, 2)$ is not $\mathcal{C}$-cyclic . Therefore, $d_{\mathcal{C}}(\mu) \ge k$ and $d_{\mathcal{C}}(\lambda) \ge k$. ∎

Combining Proposition 5.3 with Theorem 2.1, we have the following corollary that describes the relation between Carolina Solitaire and Bulgarian Solitaire when n is a triangular number.

Corollary 5.4. *Let $n = 1 + 2 + \cdots + k$ and let $\lambda \models n$ be a permutation of $\lambda' \vdash n$.*
(1) $d_{\mathcal{C}}(\lambda) \le k-1$ if and only if $d_{\mathcal{B}}(\lambda') = 0$.
(2) If $d_{\mathcal{C}}(\lambda) \ge k$ then $d_{\mathcal{C}}(\lambda) - d_{\mathcal{B}}(\lambda') = k-1$. ∎

Combining Corollary 5.4 with Theorem 3.7, we can prove:

Theorem 5.5. *If $n = 1 + 2 + \cdots + k$, then*

$$D_{\mathcal{C}}(n) = k^2 - 1. \quad ∎$$

Now we turn our attention to the value of $D_{\mathcal{C}}(n)$ for non-triangular n.

Proposition 5.6. *Let $\lambda \models n$ and $n = 1 + 2 + \cdots + (k-1) + r$, $1 \le r < k$.*
(1) If λ is a permutation of some $\mathcal{C}$-cyclic composition of n, then $d_{\mathcal{C}}(\lambda) \le k-1$.
(2) If λ is not a permutation of any $\mathcal{C}$-cyclic composition of n, then $d_{\mathcal{C}}(\lambda) \ge k-1$.

Proof. (1) Assume λ is a permutation of $\mu \models n$, where μ has the stated form of Theorem 2.1. Then we note that $\mathcal{C}^{(k-1)}(\lambda)$ also has the stated form of Theorem 2.1. Thus $\mathcal{C}^{(k-1)}(\lambda)$ is $\mathcal{C}$-cyclic and $d_{\mathcal{C}}(\lambda) \le k-1$.

(2) It is easy to verify the statement for $k \le 3$. So we may assume $k \ge 4$. If λ is not a permutation of any $\mathcal{C}$-cyclic composition of n, then there exists $\mu = \mathcal{C}^{(i)}(\lambda)$, $i \ge 0$, such that μ is not a permutation of any $\mathcal{C}$-cyclic composition and $\mathcal{C}(\mu)$ is a permutation of some $\mathcal{C}$-cyclic composition. By Theorem 2.1, we note that μ must be a permutation of $(k + \delta_{k-1}, k-2+\delta_{k-3}, k-3+\delta_{k-4}, \ldots, 3+\delta_2, 2+\delta_1, \delta_0+\delta_{k-2})$, where each δ_i is 0 or 1, $\delta_0 \le \delta_{k-2}$, and $(\delta_{k-1}, \delta_{k-2}) \ne (0,1)$. Then $\mathcal{C}^{(k-2)}(\mu)$ does not have the stated form of Theorem 2.1. So $d_{\mathcal{C}}(\mu) \ge k-1$, and hence $d_{\mathcal{C}}(\lambda) \ge k-1$. ∎

We note that from the condition "$d_{\mathcal{C}}(\lambda) = k-1$", we cannot determine whether λ is a permutation of some $\mathcal{C}$-cyclic composition of n. For example, when $k = 4$, $\lambda = (3,2,4) \models 9$ with $d_{\mathcal{C}}(\lambda) = 3$ is a permutation of the $\mathcal{C}$-cyclic composition $(4,3,2)$; however, $\lambda = (5,4) \models 9$ with $d_{\mathcal{C}}(\lambda) = 3$ is not a permutation of any $\mathcal{C}$-cyclic composition.

Similar to Corollary 5.4 for triangular n, we have the following corollary that describes the relation between Carolina Solitaire and Bulgarian Solitaire when n is not a triangular number:

Corollary 5.7. *Let $n = 1 + 2 + \cdots + (k-1) + r$, $1 \le r < k$, and let $\lambda \models n$ be a permutation of $\lambda' \vdash n$.*
(1) If $d_{\mathcal{C}}(\lambda) \le k-2$ then $d_{\mathcal{B}}(\lambda') = 0$.
(2) If $d_{\mathcal{C}}(\lambda) \ge k-1$ then $d_{\mathcal{C}}(\lambda) - d_{\mathcal{B}}(\lambda') = k-2$ or $k-1$.

Proof. (1) follows immediately from Proposition 5.6(2) and we can prove (2) by applying induction on $d_{\mathcal{C}}(\lambda)$. (For the induction basis, we note that if $d_{\mathcal{C}}(\lambda) = k-1$, then $d_{\mathcal{B}}(\lambda') = 0$ or 1 by Proposition 5.6.) ∎

We are now in a position to prove

Theorem 5.8. *Let n be non-triangular and $n = 1 + 2 + \cdots + (k-1) + r$, $1 \le r < k$. Then*
(1) $D_{\mathcal{C}}(n) \le k^2 - k - 2$ for $k \ge 4$.
(2) Furthermore, equality holds when $k \ge 4$ and $r = k-1$.

Proof. (1) follows from Theorem 4.4 and Proposition 5.7. To prove (2), it suffices to present $\lambda \models n$ such that $d_{\mathcal{C}}(\lambda) = k^2 - k - 2$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{k+1})$ where $\lambda_1 = k-1$, $\lambda_2 = k-2$, $\lambda_i = k - i + 1$ for $i = 3, 4, \ldots, k$, and $\lambda_{k+1} = 1$. Imitating the proof of Theorem 3.1, we can show that $d_{\mathcal{C}}(\lambda) = k^2 - k - 2$. ∎

Using the results for $D_{\mathcal{B}}(n)$ and considering the relations between $d_{\mathcal{C}}(\lambda)$ and $d_{\mathcal{B}}(\lambda)$, we have proven Theorems 5.5 and 5.8 for $D_{\mathcal{C}}(n)$. We can also prove these two theorems in another way: Note that we have Lemmas 3.3, 3.4, 3.5, 3.6, and 4.3 for Bulgarian Solitaire. If we consider analogous lemmas for Carolina Solitaire, then we can prove Theorems 5.5 and 5.8 directly.

Similar to Theorem 4.5 for $D_{\mathcal{B}}(n)$, we have the following "Lower Bound Theorem" for $D_{\mathcal{C}}(n)$:

Lower Bound Theorem 5.9. *For $n \geq 3$ and $n = 1+2+\cdots+(k-1)+r$, $1 \leq r \leq k$,*

$$D_{\mathcal{C}}(n) \geq \begin{cases} (k-2-r)k+r & \text{for} \quad 1 \leq r < \lfloor \frac{k-1}{2} \rfloor \\ n-1 & \text{for} \quad \lfloor \frac{k-1}{2} \rfloor \leq r \leq \lfloor \frac{k+1}{2} \rfloor \text{ and } r = 1 \\ n & \text{for} \quad \lfloor \frac{k-1}{2} \rfloor \leq r \leq \lfloor \frac{k+1}{2} \rfloor \text{ and } r \geq 2 \\ (r-1)k+r-1 & \text{for} \quad \lfloor \frac{k+1}{2} \rfloor < r \leq k. \end{cases}$$

Proof. Similar to the proof of Theorem 4.5, we consider three cases: $1 \leq r < \lfloor \frac{k-1}{2} \rfloor$, $\lfloor \frac{k-1}{2} \rfloor \leq r \leq \lfloor \frac{k+1}{2} \rfloor$, and $\lfloor \frac{k+1}{2} \rfloor < r \leq k$. For each case, we take the same extremal partition (composition) as that in the proof of Theorem 4.5. ∎

We conclude this paper by giving the following conjecture that is confirmed for $n = 3, 4, \ldots, 36$ (see Figure 9), and also for $r = k-1, k$.

Conjecture 5.10. *In Theorem 5.9 "$D_{\mathcal{C}}(n) \geq$" can be replaced with "$D_{\mathcal{C}}(n) =$."*

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References

1. E. Akin and M. Davis, Bulgarian solitaire, *Amer. Math. Monthly* **4** (1985), 237–250.

2. H-J. Bentz, Proof of the Bulgarian Solitaire conjectures, *Ars Combin.* **23** (1987), 151–170.

3. J. Brandt, Cycles of partitions, *Proc. Amer. Math. Soc.* **85** (1982), 483–486.

4. C. Cannings and J. Haigh, Montreal solitaire, *J. Combin. Theory Ser. A* **60** (1992), 50–66.

5. G. Etienne, Tableaux de Young et solitaire bulgare, *J. Combin. Theory Ser. A* **58** (1991), 181–197.

6. M. Gardner, Mathematical games, *Scientific American* **249** (1983), 12–21.

7. J. D. Hobby and D. Knuth, Problem 1: Bulgarian Solitaire, *A Programming and Problem-Solving Seminar, Dept. of Comp. Sci., Stanford University, Stanford* (December 1983), 6–13.

8. K. Igusa, Solution of the Bulgarian solitaire conjecture, *Math. Mag.* **58** (1985), 259–271.

9. R. Servedio and Y. N. Yeh, A bijective proof on circular compositions, *Bull. Inst. Math. Acad. Sinica* **23** (1995), 283–293.

10. Y. N. Yeh, A remarkable endofunction involving compositions, *Studies in Appl. Math.* (1995), 419–432