Greetings from The On-Line Encyclopedia of Integer Sequences!

a000594

Search

Hints

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A000594 Ramanujan's tau function (or tau numbers). +30
(Formerly M5153 N2237) 91

1, -24, 252, -1472, 4830, -6048, -16744, 84480, -113643, -115920, 534612, -370944, -577738, 401856, 1217160, 987136, -6905934, 2727432, 10661420, -7109760, -4219488, -12830688, 18643272, 21288960, -25499225, 13865712, -73279080, 24647168 (list; graph; listen)

OFFSET 1,2

COMMENT Coefficients of the cusp form of weight 12 for the full modular group.
It is conjectured that tau(n) is never zero.
Number of partitions of n into an even number of distinct parts - partitions of n into an odd number of distinct parts, with 24 types of each part. - Jon Perry (perry(AT)globalnet.co.uk), Apr 04 2004
M. J. Hopkins mentions that the only known primes p for which tau(p) == 1 mod p are 11, 23 and 691, that it is an open problem to decide if there are infinitely many such p, and that no others are known below 35000. Simon Plouffe has now searched up to tau(314747) and found no other examples. - njas, Mar 25 2007

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Index entries for "core" sequences
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B. Edixhoven et al., Computing the coefficients of a modular form
M. Z. Garaev, V. C. Garcia and S. V. Konyagin, Waring problem with the Ramanujan tau function

FORMULA G.f.: x Product_{k=1..infinity} (1 - x^k)^24.
|a(n)| = O(n^(11/2 + epsilon)), |a(p)| <= 2 p^(11/2) if p is prime. These were conjectured by Ramanujan and proved by Deligne.
Zagier says: The proof of these formulae, if written out from scratch, has been estimated at 2000 pages; in his book Manin cites this as a probable record for the ratio: `length of proof:length of statement' in the whole of mathematics.
G.f. A(x) satisfies 0=f(A(x),A(x^2),A(x^4)) where f(u,v,w)=uw(u+48v+4096w)-v^3. - Michael Somos, Jul 19 2004

EXAMPLE x Product (1 - x^k)^24 = x - 24*x^2 + 252*x^3 - 1472*x^4 + 4830*x^5 - 6048*x^6 - 16744*x^7 + 84480*x^8 - 113643*x^9 + ...

MAPLE
```
M := 50; t1 := series(x*mul((1-x^k)^24, k=1..M), x, M); A000594 := n->coeff(t1, x, n);
```

MATHEMATICA
```
CoefficientList[ Take[ Expand[ Product[ (1 - x^k)^24, {k, 1, 30} ]], 30], x] (* Or *)
(* first do *) Needs["NumberTheory`Ramanujan`"] (* then *) Table[ RamanujanTau[n], {n, 30}] (from Dean Hickerson (dean(AT)math.ucdavis.edu), Jan 03 2003)
```

PROGRAM
```
(MAGMA) M12:=ModularForms(Gamma0(1), 12); t1:=Basis(M12)[2]; PowerSeries(t1[1], 100); Coefficients($1);
(PARI) a(n)=if(n<1, 0, polcoeff(x*eta(x+x*O(x^n))^24, n))
(PARI) a(n)=if(n<1, 0, polcoeff(x*(sum(i=1, (sqrtint(8*n-7)+1)\2, (-1)^i*(2*i-1)*x^((i^2-i)/2), O(x^n)))^8, n))
```

CROSSREFS Cf. A076847 (tau(p)), A037955, A027364, A037945, A037946, A037947, A008408 (Leech).
For a(n) mod N for various values of N see A046694, A126811-...
Sequence in context: A052652 A052732 A086603 this_sequence A022716 A051828 A076847
Adjacent sequences: A000591 A000592 A000593 this_sequence A000595 A000596 A000597

KEYWORD sign,easy,core,mult,nice

AUTHOR njas

A121733 Numbers n such than two consecutive Ramanujan tau numbers are congruent mod 691, or **A000594**[n] == **A000594**[n+1] mod 691, or A046694[n] = A046694[n+1]. +20 7

184, 2103, 3421, 3638, 4342, 5181, 6029, 6233, 8323, 8628, 8721, 9658, 9905, 11322, 11774, 11888, 12410, 12774, 12811, 13063, 13484, 14744, 14906, 15065, 15247, 16581, 16610, 18248, 18396, 18703, 19514, 20476, 20479, 21657, 22089, 22984 (list; graph; listen)

OFFSET 1,1

COMMENT Corresponding Ramanujan tau numbers mod 691 are listed in A121734[n] = A046694[a(n)]. A121734[n] begins {483,209,21,632,650,541,546,281,666,440,397,576,18,251,356,207,532,361,121,642,288,167,348,505,561,0,108,166,97,492,58,255,632,151,679,185,141,587,0,...}.
There are instances of three consecutive equal terms in A046694, with A046694[n] = A046694[n+1] = A046694[n+2]. Equivalently there are consecutive equal terms a(n) such that a(n) = a(n+1). The first such set is A046694(290217) = A046694(290218) = A046694(290219) = 0. - Alexander Adamchuk (alex(AT)kolmogorov.com), Aug 18 2006

LINKS E. W. Weisstein, Link to a section of The World of Mathematics. Ramanujan's Tau Function.

EXAMPLE a(1) = 184 because the first pair of equal consecutive numbers in A046694[n] is A046694[184] = A046694[185] = 483 = A121734[1].

MATHEMATICA Select[Range[30000], Mod[DivisorSigma[11, #1], 691]==Mod[DivisorSigma[11, #1+1], 691]&]

CROSSREFS Cf. A121734, A046694, **A000594**.
Sequence in context: A028676 A030465 A061657 this_sequence A035831 A094631 A060491
Adjacent sequences: A121730 A121731 A121732 this_sequence A121734 A121735 A121736

KEYWORD nonn

AUTHOR Alexander Adamchuk (alex(AT)kolmogorov.com), Aug 18 2006

A121734 Ramanujan tau numbers such that **A000594**[n] == **A000594**[n+1] mod 691, or A046694[n] = A046694[n+1]. +20 7

483, 209, 21, 632, 650, 541, 546, 281, 666, 440, 397, 576, 18, 251, 356, 207, 532, 361, 121, 642, 288, 167, 348, 505, 561, 0, 108, 166, 97, 492, 58, 255, 632, 151, 679, 185, 141, 587, 0, 549, 459, 428, 549, 157, 559, 121, 605, 102 (list; graph; listen)

OFFSET 1,1

COMMENT Corresponding indices n are listed in A121733[n] = {184,2103,3421,3638,4342,5181,6029,6233,8323,8628,8721,9658,9905,...}.

LINKS E. W. Weisstein, Link to a section of The World of Mathematics. Ramanujan's Tau Function.

FORMULA a(n) = mod[**A000594**[A121733[n]], 691] = A046694[A121733[n]].

EXAMPLE a(1) = 483 because the first pair of equal consecutive numbers in A046694[n] is A046694[184] = A046694[185] = 483.

MATHEMATICA Do[f=Mod[DivisorSigma[11, n], 691]; g=Mod[DivisorSigma[11, n+1], 691]; If[f==g, Print[{n, f}]], {n, 1, 10000}]

CROSSREFS Cf. A121733, A046694, **A000594**.
Sequence in context: A020271 A124995 A033983 this_sequence A014803 A085120 A013771
Adjacent sequences: A121731 A121732 A121733 this_sequence A121735 A121736 A121737

KEYWORD nonn

AUTHOR Alexander Adamchuk (alex(AT)kolmogorov.com), Aug 18 2006

A128378 A000012^24 * **A000594**. +20 7

1, 0, -24, -24, 252, 552, -1196, -5496, -1218, 27808 (list; graph; listen)

OFFSET	1,3
COMMENT	Conjecture: Given the infinite set of sequences generated from using the partial sum operator on **A000594**, (i.e. A000012^k * **A000594**, k in succession =1,2,3,...), k=23 and k=24 are the only two sequences in the set with zeros. In A128379, k = 23: (1, -1, -24, 0, 276, 300, -1748, -4300,...).
FORMULA	A000012^24 * **A000594**; (partial sum operator performed 24 times on **A000594**).
CROSSREFS	Cf. A000012, **A000594**, A128379. Sequence in context: A004011 A056465 A056455 this_sequence A004513 A004465 A066024 Adjacent sequences: A128375 A128376 A128377 this_sequence A128379 A128380 A128381
KEYWORD	nonn
AUTHOR	Gary W. Adamson (qntmpkt(AT)yahoo.com), Feb 28 2007

A128379 A000012^23 * **A000594**. +20 7

1, -1, -24, 0, 276, 300, -1748, -4300, 4278, 29026 (list; graph; listen)

OFFSET	1,3
COMMENT	Conjecture: given A000012^k * **A000594**, k=23 and 24 are the only k's generating sequences with zeros. k=24 in A128378: (1, 0, -24, -24, 252, 552, -1196, -5496,...).
FORMULA	A000012 (partial sum operator) performed 23 times on **A000594**.
CROSSREFS	Cf. A128378, A000012, **A000594**. Sequence in context: A030464 A053558 A023923 this_sequence A111983 A040581 A040582 Adjacent sequences: A128376 A128377 A128378 this_sequence A128380 A128381 A128382
KEYWORD	nonn
AUTHOR	Gary W. Adamson (qntmpkt(AT)yahoo.com), Feb 28 2007

A063938 Numbers n such that n divides tau(n), where tau(n)=**A000594**(n) is Ramanujan's tau function. +20 6

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 24, 25, 27, 28, 30, 32, 35, 36, 40, 42, 45, 48, 49, 50, 54, 56, 60, 63, 64, 70, 72, 75, 80, 81, 84, 88, 90, 91, 92, 96, 98, 100, 105, 108, 112, 115, 120, 125, 126, 128, 135, 140, 144, 147, 150, 160, 161, 162, 168 (list; graph; listen)

OFFSET	1,2
COMMENT	Although most small numbers are in the sequence, it becomes sparser for larger values; e.g. only 504 numbers up to 10000 and only 184 numbers from 10001 to 20000 are in the sequence.
LINKS	E. W. Weisstein, Link to a section of The World of Mathematics.
MATHEMATICA	(* First do <<NumberTheory`Ramanujan` *) test[n_] := Mod[RamanujanTau[n], n]==0; Select[Range[200], test]
CROSSREFS	For the sequence when n is prime see A007659. Cf. A063940, **A000594**, A079334. Sequence in context: A080672 A056757 A079333 this_sequence A002473 A117296 A096503 Adjacent sequences: A063935 A063936 A063937 this_sequence A063939 A063940 A063941
KEYWORD	nonn,easy

AUTHOR Robert G. Wilson v (rgwv(AT)rgwv.com), Aug 31 2001

EXTENSIONS More terms from Dean Hickerson (dean(AT)math.ucdavis.edu), Jan 03 2003

A128380 A097806^24 * **A000594**. +20 5

1, 0, -48, -24, 1104, 1128, -15892, -25368, 156240, 360640 (list; graph; listen)

OFFSET 1,3

COMMENT Conjecture: Given the infinite set of sequences generated from the pairwise operation on **A000594** (A097806^k * **A000594**), k = 24, (A128380) is the only sequence in the set with a zero. The sequence generated from k=23 = (1, -1, -47, 23, 1081, 47, -15939,...). Analogous conjecture with the partial sum operator: (Cf. A128378, A128379); in which zeros are conjectured to occur only with k=23 and k=24. A128380 mod 24 = 1, 0, 0, 0, 0, 0, -4, 0, 0, 16,...

FORMULA Pairwise operation performed 24 times on **A000594**

CROSSREFS Cf. **A000594**, A097806, A128378, A128379.
Sequence in context: A124354 A033979 A033368 this_sequence A094658 A085517 A033694
Adjacent sequences: A128377 A128378 A128379 this_sequence A128381 A128382 A128383

KEYWORD nonn

AUTHOR Gary W. Adamson (qntmpkt(AT)yahoo.com), Feb 28 2007

A121742 Numbers n such that three consecutive Ramanujan tau numbers are congruent mod 691, or **A000594**[n] == **A000594**[n+1] == **A000594**[n+2] mod 691, or A046694[n] = A046694[n+1] = A046694[n+2]. +20 4

290217, 477155, 1051085, 1153412, 1409635, 1409636 (list; graph; listen)

OFFSET 1,1

COMMENT Corresponding Ramanujan tau numbers mod 691 are listed in A121743[n] = A046694[a(n)]. A121743[n] begins {0,276,91,79,0,0,...}. a(n) are the indices of the first number in the Ramanujan tau triplets mod 691. All a(n) belong to A121733[n] - indices of the first number in the Ramanujan tau twins mod 691. There are also quadruplets in the Ramanujan tau mod 691 such that A046694[n] = A046694[n+1] = A046694[n+2] = A046694[n+3]. The first such Ramanujan tau Quadruplet mod 691 starts with A046694[1409635] = 0.

LINKS E. W. Weisstein Link to a section of The World of Mathematics. Ramanujan's Tau Function.

MATHEMATICA
```
Do[f=Mod[DivisorSigma[11, n], 691]; g=Mod[DivisorSigma[11, n+1], 691]; h=Mod[DivisorSigma[11, n+2], 691]; If[f==g&&g==h, Print[{n, f}]], {n, 1, 1500000}]
```

CROSSREFS Cf. A121743, A121733, A121734, A046694, **A000594**.
Sequence in context: A131263 A096519 A043616 this_sequence A093792 A068241 A104328
Adjacent sequences: A121739 A121740 A121741 this_sequence A121743 A121744 A121745

KEYWORD nonn

AUTHOR Alexander Adamchuk (alex(AT)kolmogorov.com), Aug 19 2006

A128381 A007318^24 * **A000594**. +20 4

1, 0, -324, -10976, -260898, -4919184, -67536616, -212659776, 28757879829 (list; graph; listen)

OFFSET	1,3
COMMENT	Conjecture: given the bto performed any k times on **A000594** (k=1,2,3,...); k=6 and k=24 are the only members of the set with zeros. k=6 generates (1, -18, 0, 688, 4494, 5508,...).
FORMULA	Binomial transform operation performed 24 times on **A000594**
CROSSREFS	Cf. A007318, **A000594**, A128378, A128379, A128380, A128382. Sequence in context: A111278 A014792 A064197 this_sequence A128992 A088216 A121001 Adjacent sequences: A128378 A128379 A128380 this_sequence A128382 A128383 A128384
KEYWORD	nonn
AUTHOR	Gary W. Adamson (qntmpkt(AT)yahoo.com), Feb 28 2007

A126811 Ramanujan numbers (**A000594**) read mod 2. +20 3

1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 (list; graph; listen)

OFFSET	1,1
REFERENCES	H. P. F. Swinnerton-Dyer, On l-adic representations and congruences for coefficients, of modular forms, pp. 1-55 of Modular Functions of One Variable III (Antwerp 1972), Lect. Notes Math., 350, 1973.
CROSSREFS	Cf. **A000594**. Sequence in context: A014954 A015899 A015494 this_sequence A014057 A015689 A104124 Adjacent sequences: A126808 A126809 A126810 this_sequence A126812 A126813 A126814
KEYWORD	nonn
AUTHOR	njas, Feb 25 2007

Last modified November 14 18:26 EST 2007. Contains 134147 sequences.

AT&T Labs Research

Search completed in 0.082 seconds (0.000 user, 0.000 system, 0 memory pages accessed)

1 1
2 -24
3 252
4 -1472
5 4830
6 -6048
7 -16744
8 84480
9 -113643
10 -115920
11 534612
12 -370944
13 -577738
14 401856
15 1217160
16 987136
17 -6905934
18 2727432
19 10661420
20 -7109760
21 -4219488
22 -12830688
23 18643272
24 21288960
25 -25499225
26 13865712
27 -73279080
28 24647168
29 128406630
30 -29211840
31 -52843168
32 -196706304
33 134722224
34 165742416
35 -80873520
36 167282496
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39 -145589976
40 408038400
41 308120442
42 101267712

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186 319595480064
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209 5699723069040
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RAMANUJAN'S UNPUBLISHED MANUSCRIPT ON THE PARTITION AND TAU FUNCTIONS WITH PROOFS AND COMMENTARY

Bruce C. Berndt and Ken Ono

Dedicated to our good friend George Andrews on his 60th birthday

Introduction

When Ramanujan died in 1920, he left behind an incomplete, unpublished manuscript in two parts on the partition function $p(n)$ and, in contemporary terminology, Ramanujan's tau-function $\tau(n)$. The first part, beginning with the Roman numeral I, is written on 43 pages, with the last nine comprising material for insertion in the foregoing part of the manuscript. G. H. Hardy extracted a portion of Part I providing proofs of Ramanujan's congruences for $p(n)$ modulo 5, 7, and 11 and published it in 1921 [80], [82, pp. 232–238] under Ramanujan's name. In a footnote, Hardy remarks, "The manuscript contains a large number of further results. It is very incomplete, and will require very careful editing before it can be published in full. I have taken from it the three simplest and most striking results, ..." In 1952, J. M. Rushforth [89] published several further results, mostly on $\tau(n)$, from Part I. In 1977, R. A. Rankin [85] discussed several congruences for $\tau(n)$ found in Part I. Part II has not been discussed in the literature. Part I was not made available to the public until 1988 when it was photocopied in its original handwritten form and published with Ramanujan's lost notebook [83]. The existence of Part II was first pointed out by B. J. Birch [26] in 1975, but, like Part I, it also was hidden from the public until 1988, when a handwritten copy made by G. N. Watson was photocopied for [83]. Several theorems and proofs in this manuscript had not previously appeared before 1988.

The manuscript arises from the last three years of Ramanujan's life. It may have been written in nursing homes and sanitariums in 1917–1919, when we know, from letters that Ramanujan wrote to Hardy during this time [25, pp. 192–193], that Ramanujan was thinking deeply about partitions, or, more likely, it may have been written in India during the last year of his life. According to Rushforth [89], the manuscript was sent to Hardy by "Ramanujan a few months before the latter's death in 1920." If this is true, then it probably was enclosed with Ramanujan's last letter to Hardy, dated January 12, 1920 [25, pp. 220–223]. There is no mention of the manuscript in the extant portion of the letter; part of the letter has been lost. The manuscript was given by Hardy in 1928 to G. N. Watson, who had

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it in his possession until he died in 1965. At the suggestion of Rankin, Part I was sent shortly thereafter to the library of Trinity College, Cambridge, where it still resides. Watson's copy of Part II can be found in the library of Oxford's Mathematical Institute. We do not know if Ramanujan's orginal copy of Part II exists. For further historical information, see Rankin's two papers [85], [87].

Since many of the proofs in this manuscript had not been published before their appearance in handwritten form with the lost notebook [83], since many details were omitted by Ramanujan, since mathematicians have established results either proved or asserted in the manuscript since it was written, and since the manuscript contains many unproved claims, the purpose of this paper is to present the manuscript in its entirety, offer some additional details, and provide extensive commentary on it. Although many of the results in this manuscript have been proven or explained within a greater context in the works of P. Deligne, J.–P. Serre, H. P. F. Swinnerton–Dyer, and others, we were delighted to find a number of surprising new gems. For example, Ramanujan's claims (14.1)–(14.6) and many of the assertions in both Sections 15 and 16 were unexpected and entirely new to both authors. Moreover, in proving the claims in Section 14, the second author was led, by the "shape of Ramanujan's claims," to several new general results regarding the distribution of the partition function modulo every prime $m \geq 5$ [70]. Part II, beginning with Section 20, is also fascinating, for it contains Ramanujan's proof, albeit lacking in many details, of his conjectured congruences for $p(n)$ modulo arbitrary integral powers of 5.

Several editorial decisions needed to be made in our presentation of the manuscript.

1) The nine pages of insertions at the end of Part I were interposed at their intended positions.

2) None of Ramanujan's footnotes, such as "For a direct proof of this see," were completed in the manuscript. We have executed their completions, but we do not claim that they are what Ramanujan had in mind.

3) Due to Ramanujan's failure to tag certain equalities, the manuscript contains incomplete references, such as " ... deduce from ( ) and ( ) ... ." We have added the tags and inserted the equation numbers. Difficulties arose when tags needed to be inserted at places between already existing tags with consecutive numbers. We appended letters on such tags; e.g., (6.6a) lies between (6.6) and (6.7).

4) As with most of his mathematics, Ramanujan provided very few details in this manuscript. In Part I, Ramanujan indicates, at more than one place, that this is the first of two papers that he intends to write on $p(n)$ and $\tau(n)$. It is clear that as Ramanujan wrote the manuscript he continued to discover more and more theorems on the subject, and so he more and more frequently recorded his results with the promise that he would provide details in his next paper. Thus, details become more sparse as the manuscript progresses, so that in the last third of the manuscript there are hardly any details at all. However, rather than returning in Part II to the details omitted in Part I, Ramanujan sketched his proofs of the congruences for $p(n)$ modulo any power of 5 or 7. In Hardy's extraction [80], he considerably amplified Ramanujan's arguments. Similarly, Rushforth [89] provided many details omitted by Ramanujan. In his paper providing proofs of the general congruences modulo 5^n and $7^{[n/2]+1}$, Watson [104] had to supply most of the details omitted by Ramanujan. We have followed their leads and have supplied more details for some of Ramanujan's arguments. However, for those parts of the manuscript

examined by Hardy, Rushforth, and Watson, we have not added details here, as readers can find these in the aforementioned papers. So that readers remain clear about what was written by Ramanujan, we have placed our additions in square brackets.

5) We have taken the liberty of making minor editorial changes without comments. Such alterations include correcting misprints, adding punctuation, and introducing notation. In particular, Ramanujan generally wrote infinite series in expanded form without resorting to summation signs, which we have supplied.

Many unproved claims can be found in the manuscript. Since Ramanujan's death, some have been proved by others, often without realizing that Ramanujan had originally found them. Some claims are false, and others had not been proved. Because of the desire to make minimal additions within Ramanujan's manuscript, we have deferred discussions of most of Ramanujan's unproved claims to the end of this paper, where many references to the literature are cited.

PROPERTIES OF $p(n)$ AND $\tau(n)$ DEFINED BY THE FUNCTIONS $\sum_{n=0}^{\infty} p(n)q^n = (q;q)_\infty^{-1}$, $\sum_{n=1}^{\infty} \tau(n)q^n = q(q;q)_\infty^{24}$

S. RAMANUJAN

I

0. I have shown elsewhere by very simple arguments that

$$p(5n-1) \equiv 0 \pmod 5,$$
$$p(7n-2) \equiv 0 \pmod 7.$$

In the case of $\tau(n)$ such simple arguments give the following results.

Modulus 2

It is easy to see that the coefficients of q^n in the expansion of

$$q(q;q)_\infty^{24} \qquad \text{and} \qquad q(q^8;q^8)_\infty^3$$

are both odd or both even, [where here and in the sequel

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n),$$

where $|q|<1$.] But [by Jacobi's identity [48, p. 285, Thm. 357], [21, p. 39, Entry 24(ii)]],

$$q(q^8;q^8)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{(2n+1)^2}.$$

It follows that $\tau(n)$ is odd or even according as n is an odd square or not. Thus we see that the number of values of n not exceeding n for which $\tau(n)$ is odd is only

$$\left[\frac{1+\sqrt{n}}{2}\right].$$

Modulus 5

Further let J be any function of q with integral coefficients but not the same function throughout. It is easy to see that

$$q(q;q)_\infty^{24} = q(q;q)_\infty^4 (q^5;q^5)_\infty^4 + 5J.$$

But the coefficient of q^{5n} in

$$q(q;q)_\infty^4$$

is a multiple of 5.[1] It follows that

$$\tau(5n) \equiv 0 \pmod 5.$$

Modulus 7

This is the simplest of all cases. Here we have

$$q(q;q)_\infty^{24} = q(q;q)_\infty^3 (q^7;q^7)_\infty^3 + 7J.$$

But since

$$q(q;q)_\infty^3 = q\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2},$$

it is easy to see that the coefficients of $q^{7n}, q^{7n-1}, q^{7n-2}$ and q^{7n-4} are all multiples of 7. It follows that

$$\tau(7n), \tau(7n-1), \tau(7n-2), \tau(7n-4) \equiv 0 \pmod 7.$$

Modulus 23

We have

$$q(q;q)_\infty^{24} = q(q;q)_\infty (q^{23};q^{23})_\infty + 23J.$$

But [by Euler's pentagonal number theorem [48, p. 284, Thm. 353], [21, Entry 22(iii)]],

$$q(q;q)_\infty = \sum (-1)^\nu q^{1+\frac{1}{2}\nu(3\nu+1)}$$

where the summation extends over all values of ν from $-\infty$ to ∞. Now

$$1 + \frac{1}{2}\nu(3\nu+1) = (6\nu+1)^2 - \frac{23\nu(3\nu+1)}{2}.$$

[1] Recall that $p(5n+4) \equiv 0 \pmod 5$.

The residues of a square number for modulus 23 cannot be

$$5, 7, 10, 11, 14, 15, 17, 19, 20, 21, 22.$$

It follows from this that

$$\left\{\begin{array}{l}\tau(23n-1), \tau(23n-2), \tau(23n-3), \tau(23n-4), \\ \tau(23n+5), \tau(23n-6), \tau(23n+7), \tau(23n-8), \\ \tau(23n-9), \tau(23n+10), \tau(23n+11).\end{array}\right. \qquad \equiv 0 \pmod{23},$$

Modulus 5

1. Let

$$P := 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n},$$
$$Q := 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}$$

and

$$R := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n},$$

so that[2]

$$Q^3 - R^2 = 1728 q (q;q)_\infty^{24}. \tag{1.1}$$

Let $\sigma_s(n)$ denote the [sum of the] s^{th} powers of the divisors of n. Then it is easy to see that

$$Q = 1 + 5J; \qquad R = P + 5J. \tag{1.2}$$

Hence,

$$Q^3 - R^2 = Q - P^2 + 5J. \tag{1.3}$$

But[3]

$$Q - P^2 = 288 \sum_{n=1}^{\infty} n\sigma_1(n) q^n; \tag{1.4}$$

and it is obvious that

$$(q;q)_\infty^{24} = \frac{(q^{25};q^{25})_\infty}{(q;q)_\infty} + 5J. \tag{1.5}$$

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[2]For an elementary proof, see [77, eq. (44)].
[3]See [77, eq. (36)].

It follows from (1.1) and (1.3)–(1.5), that

$$q\frac{(q^{25};q^{25})_\infty}{(q;q)_\infty}=\sum_{n=1}^{\infty} n\sigma_1(n)q^n+5J. \tag{1.6}$$

In other words

$$(q^{25};q^{25})_\infty\sum_{n=0}^{\infty}p(n)q^{n+1}=\sum_{n=1}^{\infty} n\sigma_1(n)q^n+5J. \tag{1.7}$$

But the coefficient of q^{5n} in the right hand side is a multiple of 5. It follows that

$$p(5n-1)\equiv 0 \pmod 5. \tag{1.8}$$

It also follows from (1.7) that

$$\begin{aligned}&p(n-1)-p(n-26)-p(n-51)+p(n-126)\\+&p(n-176)-p(n-301)-\cdots-n\sigma_1(n)\equiv 0 \pmod 5,\end{aligned}$$

where 1, 26, 51, 126, ... are numbers of the form $\frac{1}{2}(5\nu+1)(15\nu+2)$ and $\frac{1}{2}(5\nu-1)(15\nu-2)$. The number of values of n not exceeding 200 for which $p(n)\equiv 0,1,2,3,4 \pmod 5$ is 69, 33, 34, 34, 30, respectively; and the least value of n for which $p(n)\equiv 4 \pmod 5$ is 30. These being so it appears that $p(n)\equiv 0 \pmod 5$ for about $\frac{1}{3}$ of the values of n while $p(n)\equiv 1,2,3$ or $4 \pmod 5$ for about $\frac{1}{6}$ of the values of n each. It seems extremely difficult to prove any result in this direction concerning $p(n)$, but the problem is much easier concerning $\tau(n)$.

2. It follows from (1.5) and (1.6) that

$$\begin{cases}\tau(n)-n\sigma_1(n)\equiv 0 \pmod 5,\\ \lambda(n)-n\sigma_1(n)\equiv 0 \pmod 5,\end{cases} \tag{2.1}$$

where

$$\sum_{n=1}^{\infty}\lambda(n)q^n=q\frac{(q^{25};q^{25})_\infty}{(q;q)_\infty},$$

so that $\lambda(n+1)$ is the number of partitions of n as the sum of integers which are not multiples of 25. But if n be written in the form

$$2^{a_2}\cdot 3^{a_3}\cdot 5^{a_5}\cdot 7^{a_7}\cdots,$$

where the a's are zeroes or positive integers, then

$$n\sigma_1(n)=\prod_p\frac{p^{a_p}(p^{1+a_p}-1)}{p-1}, \qquad p=2,3,5,\ldots. \tag{2.2}$$

But

$$\frac{p^{a_p}(p^{1+a_p}-1)}{p-1}\equiv 0 \pmod 5 \tag{2.3}$$

if

$$a_p \geq 1, \quad p = 5$$

or

$$a_p \equiv 1 \pmod 2, \qquad p \equiv 4 \pmod 5,$$

or

$$a_p \equiv 3 \pmod 4, \qquad p \equiv 2 \text{ or } 3 \pmod 5,$$

or

$$a_p \equiv 4 \pmod 5, \qquad p \equiv 1 \pmod 5,$$

and for no other values. Suppose now that

$$\begin{cases} t_n = 0, & \tau(n) \equiv 0 \pmod 5, \\ t_n = 1, & \tau(n) \not\equiv 0 \pmod 5. \end{cases} \tag{2.4}$$

Then it follows from (2.3) that

$$\sum_{n=1}^{\infty} \frac{t_n}{n^s} = \prod\nolimits_1 \prod\nolimits_2 \prod\nolimits_3, \tag{2.5}$$

where

$$\prod\nolimits_1 = \prod_p \frac{1}{1 - p^{-2s}},$$

p being a prime of the form $5k - 1$ and

$$\prod\nolimits_2 = \prod_p \frac{1 - p^{-3s}}{(1 - p^{-s})(1 - p^{-4s})},$$

p being a prime of the form $5k \pm 2$ and

$$\prod\nolimits_3 = \prod_p \frac{1 - p^{-4s}}{(1 - p^{-s})(1 - p^{-5s})},$$

p being a prime of the form $5k + 1$.

It is easy to prove from (2.5) that

$$\sum_{k=1}^{n} t_k = o(n). \tag{2.6}$$

It can be shown by transcendental methods that

$$\sum_{k=1}^{n} t_k \sim \frac{Cn}{(\log n)^{1/4}}, \tag{2.7}$$

and

$$\sum_{k=1}^{n} t_k = C\int_1^n \frac{dx}{(\log x)^{1/4}} + O\left(\frac{n}{(\log n)^r}\right), \tag{2.8}$$

where C is a constant and r is any positive number.

The proof of (2.6) is quite elementary and very similar to that for showing that $\pi(x) = o(x)$,[4] $\pi(x)$ being the number of primes not exceeding x. The result (2.6) can be stated roughly in other words that $\tau(n)$ and $\lambda(n)$ are divisible by 5 for almost all values of n, while (2.7) and (2.8) give a lot more information.

Modulus 25

3. It is easily seen from (1.2) that

$$\begin{aligned} Q^3 - R^2 &= 2(Q^2 - PR) - (Q - P^2) + Q(Q-1)^2 - (R-P)^2 \\ &= 2(Q^2 - PR) - (Q - P^2) + 25J. \end{aligned} \tag{3.1}$$

But[5]

$$Q^2 - PR = 1008\sum_{n=1}^{\infty} n\sigma_5(n)q^n; \tag{3.2}$$

and it is obvious that

$$(q;q)_\infty^{24} = \frac{(q^5;q^5)_\infty^5}{(q;q)_\infty} + 25J. \tag{3.3}$$

Now remembering that

$$\sigma_5(n) - \sigma_1(n) \equiv 0 \pmod 5, \tag{3.4}$$

it follows from (1.4) and (3.1)–(3.3) that

$$6q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty} = \sum_{n=1}^{\infty}\{2n\sigma_5(n) - n\sigma_1(n)\}q^n + 25J. \tag{3.5}$$

[By extracting those terms with exponents that are multiples of 5 and by employing the congruence $p(5n-1) \equiv 0 \pmod 5$,] we easily deduce that

$$(q;q)_\infty^5 \sum_{n=1}^{\infty} p(5n-1)q^n = \sum_{n=1}^{\infty}\{10n\sigma_5(n) - 5n\sigma_1(n)\}q^n + 25J,$$

and hence [by (3.4)] that

$$(q^5;q^5)_\infty \sum_{n=1}^{\infty} p(5n-1)q^n = 5\sum_{n=1}^{\infty} n\sigma_1(n)q^n + 25J. \tag{3.6}$$

[4]See Landau's *Primzahlen* [60, pp. 641–669].

[5]See [77, Table II].

Since the coefficient of q^{5n} is a multiple of 25 it follows that

$$p(25n-1) \equiv 0 \pmod{25}. \tag{3.7}$$

It also follows from (3.6) that

$$\begin{aligned} &p(5n-1) - p(5n-26) - p(5n-51) + p(5n-126) \\ &+p(5n-176) - \cdots - 5n\sigma_1(n) \equiv 0 \pmod{25}, \end{aligned}$$

where 1, 26, 51, 126, ... are the same as in (1.9).

4. It is easy to see [by Fermat's little theorem] that

$$n\sigma_9(n) - 2n\sigma_5(n) + n\sigma_1(n) \equiv 0 \pmod{25}. \tag{4.1}$$

It follows from this and (3.3) and (3.5) that

$$\tau(n) - n\sigma_9(n) \equiv 0 \pmod{25}. \tag{4.2}$$

It appears that, if k be any positive integer, it is possible to find two integers a and b such that

$$\tau(n) - n^a\sigma_b(n) \equiv 0 \pmod{5^k}, \tag{4.3}$$

if n is not a multiple of 5. Thus for instance

$$\tau(n) - n^{41}\sigma_{29}(n) \equiv 0 \pmod{125}, \tag{4.4}$$

if n is not a multiple of 5. I have not yet proved these results. If n is a multiple of 5, then

$$\tau(n) - 4830\tau\left(\frac{n}{5}\right) + 5^{11}\tau\left(\frac{n}{25}\right) = 0$$

in virtue of (7.6), $\tau(x)$ being considered as 0 if x is not an integer.

It also appears that the coefficient of q^n in the left hand side of (3.5) can be exactly determined in terms of the real divisors of n. Thus

$$q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty} = \sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2}, \tag{4.5}$$

[where $\left(\frac{n}{p}\right)$ denotes the Legendre symbol]. The allied function

$$\frac{(q;q)_\infty^5}{(q^5;q^5)_\infty} = 1 - 5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^n}. \tag{4.6}$$

It follows from (4.5) that

$$(q;q)_\infty^5\sum_{n=1}^{\infty}p(5n-1)q^n = 5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2}$$

and hence that[6]

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}. \tag{4.7}$$

Modulus 7

5. Since[7]

$$Q^2 = 1 + 480\sum_{n=1}^{\infty} \frac{n^7 q^n}{1-q^n}, \tag{5.1}$$

it is easy to see that

$$Q^2 = P + 7J; \qquad R = 1 + 7J; \tag{5.2}$$

and so

$$(Q^3 - R^2)^2 = P^3 - 2PQ + R + 7J. \tag{5.3}$$

But[8]

$$\begin{cases} PQ - R = 720\displaystyle\sum_{n=1}^{\infty} n\sigma_3(n)q^n, \\ P^3 - 3PQ + 2R = -1728\displaystyle\sum_{n=1}^{\infty} n^2\sigma_1(n)q^n; \end{cases} \tag{5.4}$$

and it is obvious that

$$(q;q)_\infty^{48} = \frac{(q^{49};q^{49})_\infty}{(q;q)_\infty} + 7J. \tag{5.5}$$

It follows from all these that

$$q^2\frac{(q^{49};q^{49})_\infty}{(q;q)_\infty} = \sum_{n=1}^{\infty}\left\{n^2\sigma_1(n) - n\sigma_3(n)\right\}q^n + 7J. \tag{5.6}$$

In other words

$$(q^{49};q^{49})_\infty \sum_{n=0}^{\infty} p(n)q^{n+2} = \sum_{n=1}^{\infty}\left\{n^2\sigma_1(n) - n\sigma_3(n)\right\}q^n + 7J. \tag{5.7}$$

It follows that

$$p(7n-2) \equiv 0 \pmod 7, \tag{5.8}$$

[6]For a direct proof of this result see [78].
[7]See [77, Table I].
[8]See [77, Tables II and III, resp.].

and

$$p(n-2) - p(n-51) - p(n-100) + p(n-247)$$
$$+p(n-345) - \cdots + n\sigma_3(n) - n^2\sigma_1(n) \equiv 0 \pmod 7, \tag{5.9}$$

where 2, 51, 100, 247, ... are the numbers of the form $\frac{1}{2}(7\nu+1)(21\nu+4)$ and $\frac{1}{2}(7\nu-1)(21\nu-4)$.

The number of values of n not exceeding 200 for which $p(n) \equiv 0,1,2,3,4,5,6 \pmod 7$ is 50, 33, 22, 28, 23, 23, 21, respectively, and the least value of n for which $p(n) \equiv 6 \pmod 7$ is 73. It appears that $p(n) \equiv 0 \pmod 7$ for about $\frac{1}{4}$ of the values of n while $p(n) \equiv 1,2,3,4,5,6 \pmod 7$ for about $\frac{1}{8}$ of the values of n each.

6. It follows from (5.2) that

$$Q^3 - R^2 = PQ - R + 7J. \tag{6.1}$$

It is easy to see from this and (5.4) that

$$\tau(n) - n\sigma_3(n) \equiv 0 \pmod 7. \tag{6.2}$$

Now if $n = 2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdot 7^{a_7} \cdots$, then

$$n\sigma_3(n) = \prod_p p^{a_p} \frac{p^{3(1+a_p)}-1}{p^3-1}, \qquad p = 2,3,5,7,\ldots. \tag{6.3}$$

But

$$p^{a_p} \frac{p^{3(1+a_p)}-1}{p^3-1} \equiv 0 \pmod 7, \tag{6.4}$$

if

$$a_p \equiv 6 \pmod 7, \qquad p \equiv 1, 2, \text{ or } 4 \pmod 7,$$

or

$$a_p \equiv 1 \pmod 2, \qquad p \equiv 3, 5, \text{ or } 6 \pmod 7,$$

or

$$a_p \geq 1, \qquad p = 7.$$

Suppose now that

$$t_n = 1, \qquad \tau(n) \not\equiv 0 \pmod 7,$$
$$t_n = 0, \qquad \tau(n) \equiv 0 \pmod 7.$$

Then it follows from (6.4) that

$$\sum_{n=1}^{\infty} \frac{t_n}{n^s} = \prod\nolimits_1 \prod\nolimits_2 \tag{6.5}$$

where

$$\prod\nolimits_1 = \prod_p \frac{1-p^{-6s}}{(1-p^{-s})(1-p^{-7s})},$$

p being a prime of the form $7k+1, 7k+2, 7k+4$, and

$$\prod\nolimits_2 = \prod_p \frac{1}{1-p^{-2s}},$$

p being a prime of the form $7k+3, 7k+5, 7k+6$. It is easy to prove from (6.5) by quite elementary methods that

$$\sum_{k=1}^{n} t_k = o(n). \tag{6.6}$$

It can be shown by transcendental methods that

$$\sum_{k=1}^{n} t_k \sim \frac{Cn}{(\log n)^{1/2}}; \tag{6.6a}$$

and

$$\sum_{k=1}^{n} t_k = C\int_1^n \frac{dx}{(\log x)^{1/2}} + O\left(\frac{n}{(\log n)^r}\right), \tag{6.7}$$

where r is any positive number and

$$\begin{aligned} C =& \frac{6^{1/2}}{7^{3/4}} \frac{1-2^{-6}}{1-2^{-7}} \frac{1-11^{-6}}{1-11^{-7}} \frac{1-23^{-6}}{1-23^{-7}} \frac{1-29^{-6}}{1-29^{-7}} \cdots \\ &\times \frac{1}{\{(1-3^{-2})(1-5^{-2})(1-13^{-2})(1-17^{-2})(1-19^{-2})\cdots\}^{1/2}}, \end{aligned}$$

2, 11, 23, ... being primes of the form $7k+1, 7k+2$, and $7k+4$ while 3, 5, 13, ... being primes of the form $7k+3, 7k+5$ and $7k+6$. Thus we see that $\tau(n)$ is divisible by 7 for almost all values of n; and at the same time the number of values of n for which $\tau(n)$ is divisible by 7 is far more numerous than that for which $\tau(n)$ is divisible by 5.

Now if

$$\sum_{n=1}^{\infty} \lambda(n) q^n = q^2 \frac{(q^{49}; q^{49})_\infty}{(q;q)_\infty},$$

so that $\lambda(n+2)$ is the number of partitions of n as the sum of integers which are not multiples of 49, it is clear from (5.6) that

$$\lambda(n) - n^2\sigma_1(n) + n\sigma_3(n) \equiv 0 \pmod 7. \tag{6.8}$$

But it is easy to show that $n^2\sigma_1(n)$ and $n\sigma_3(n)$ are divisible by 7 for almost all values of n. It follows that $\lambda(n)$ is divisible by 7 for almost all values of n. It can even be shown that the number of values of j not exceeding n for which $\lambda(j)$ is <u>not</u> divisible by j is

$$O\left(\frac{n}{(\log n)^{1/6}}\right). \tag{6.9}$$

The index $\frac{1}{6}$ in (6.9) is easily obtained by considering $n^2\sigma_1(n)$ and $n\sigma_3(n)$ separately; but whether this is the right index or not can be known only by considering

$$n^2\sigma_1(n) - n\sigma_3(n)$$

taken together, which seems rather complicated to deal with.

Modulus 49

7. We have

$$\begin{aligned}(Q^3-R^2)^2 =&(3P^2Q^2-4PQR-2Q^3+3R^2)\\ &-2(P^3-2PQ+R)+2P(Q^2-P)^2-(1+2PQ)(R-1)^2\\ &+\left\{Q(Q^2-P)-R^2+1\right\}^2\\ =&(3P^2Q^2-4PQR-2Q^3+3R^2)-2(P^3-2PQ+R)+49J\end{aligned}$$

in virtue of (5.2). But[9]

$$(7.1)\qquad \begin{cases} Q^3-R^2=1728\displaystyle\sum_{n=1}^{\infty}\tau(n)q^n,\\ 3Q^3+2R^2-5PQR=1584\displaystyle\sum_{n=1}^{\infty}n\sigma_9(n)q^n,\\ 5Q^3+4R^2-18PQR+9P^2Q^2=8640\displaystyle\sum_{n=1}^{\infty}n^2\sigma_7(n)q^n;\end{cases}$$

and it is obvious that

$$(7.2)\qquad (q;q)_\infty^{48}=\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty}+49J.$$

Now remembering that

$$(7.3)\qquad \begin{cases}\sigma_7(n)-\sigma_1(n)\equiv 0 \pmod 7,\\ \sigma_9(n)-\sigma_3(n)\equiv 0 \pmod 7,\end{cases}$$

it follows from the above equations and (5.4) that

$$(7.4)\qquad \begin{aligned} q^2\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty}=&\sum_{n=1}^{\infty}\left\{2n\sigma_9(n)-4n^2\sigma_7(n)\right.\\ &\left.+2n\sigma_3(n)-2n^2\sigma_1(n)+2\tau(n)\right\}q^n+49J.\end{aligned}$$

¿From this [and (6.2)] we deduce that

$$(7.5)\qquad (q;q)_\infty^7\sum_{n=0}^{\infty}p(7n+5)q^{n+1}=\sum_{n=1}^{\infty}\left\{28n\sigma_3(n)+2\tau(7n)\right\}q^n+49J.$$

[9] See [77, eq. (44), Table II, Table III, resp.].

I have stated in my previous paper that[10]

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_p \frac{1}{1-\tau(p)p^{-s}+p^{11-2s}}, \tag{7.6}$$

where p assumes all prime values. This has since been proved by Mr Mordell.[11] Now by actual calculation we find that

$$\tau(7) \equiv 14 \pmod{49}.$$

It follows from this and (7.6) that

$$\tau(7n) - 14\tau(n) \equiv 0 \pmod{49}.$$

It is easy to see from this and (7.5) that

$$(q^7;q^7)_\infty^7 \sum_{n=0}^{\infty} p(7n+5)q^{n+1} = 7\sum_{n=1}^{\infty} n\sigma_3(n)q^n + 49J. \tag{7.7}$$

Now if

$$n \equiv 3, 5, 6 \pmod{7},$$

then n must contain an odd power of a prime p of the form $7k+3, 7k+5$ or $7k+6$ as a divisor since all perfect squares are of the form $7k, 7k+1, 7k+2$ or $7k+4$; and so $\sigma_3(n)$ is divisible by p^3+1 which is divisible by 7. Also it is obvious that if n is a multiple of 7 then $n\sigma_3(n)$ is also divisible by 7. It follows that if

$$n \equiv 0, 3, 5, 6 \pmod{7},$$

then

$$n\sigma_3(n) \equiv 0 \pmod{7}.$$

It is easy to see from this and (7.7) that

$$p(49n-2), p(49n-9), p(49n-16), p(49n-30) \equiv 0 \pmod{49}. \tag{7.8}$$

It also follows from (7.7) that

$$\begin{aligned} &p(7n-2) - p(7n-51) - p(7n-100) + p(7n-247) \\ &+p(7n-345) - \cdots - 7n\sigma_3(n) \equiv 0 \pmod{49}, \end{aligned}$$

where 2, 51, 100, 247, ... are the same as in (5.9).

8. It appears that

$$q(q;q)_\infty^3(q^7;q^7)_\infty^3 + 8q^2\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty} = \sum_{n=1}^{\infty}\left(\frac{n}{7}\right)q^n\frac{1+q^n}{(1-q^n)^3} \tag{8.1}$$

[10][77, eq. (101)].

[11]On Mr Ramanujan's empirical expansions of modular functions, *Proc. Cambridge Philos. Soc.* 19 (1919), 117–124. A simpler proof is given in Hardy's lectures [47].

[where $\left(\frac{n}{7}\right)$ denotes the Legendre symbol], while the allied function

$$(8.2)\qquad 49q(q;q)_\infty^3(q^7;q^7)_\infty^3+8\frac{(q;q)_\infty^7}{(q^7;q^7)_\infty}=8-7\sum_{n=1}^{\infty}\left(\frac{n}{7}\right)\frac{n^2q^n}{1-q^n}.$$

Now remembering that

$$(q;q)_\infty^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}$$

and picking out the terms $q^7, q^{14}, q^{21}, \ldots$ from both sides in (8.1) we obtain

$$-7q(q;q)_\infty^3(q^7;q^7)_\infty^3+8(q;q)_\infty^7\sum_{n=1}^{\infty}p(7n-2)q^n=49\sum_{n=1}^{\infty}\left(\frac{n}{7}\right)q^n\frac{1+q^n}{(1-q^n)^3},$$

the series in the right hand side being the same as that in (8.1). It follows from this and (8.1) that[12]

$$(8.3)\qquad \sum_{n=0}^{\infty}p(7n+5)q^n=7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4}+49q^2\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}.$$

It also appears that if

$$\sum_{n=1}^{\infty}\lambda(n)q^n=q(q;q)_\infty^3(q^7;q^7)_\infty^3,$$

then

$$(8.4)\qquad \sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s}=\frac{1}{1+7^{1-s}}\prod\nolimits_1\prod\nolimits_2,$$

where

$$\prod\nolimits_1=\prod_p\frac{1}{1-p^{2-2s}},$$

p being a prime of the form $7k+3, 7k+5$, or $7k+6$, and

$$\prod\nolimits_2=\prod_p\frac{1}{1+(2p-a^2)p^{-s}+p^{2-2s}}$$

p being a prime of the form $7k+1, 7k+2$, or $7k+4$ and a and b being integers such that $4p=a^2+7b^2$. Thus $\lambda(n)$ can be completely ascertained. It follows from this and (8.1) and (8.2) that the coefficients of q^n in

$$\frac{(q;q)_\infty^7}{(q^7;q^7)_\infty},\qquad q^2\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty}$$

[12]For a direct proof of this see §. [Ramanujan evidently intended to give a proof of (8.3) elsewhere. In his paper [78], (8.3) is stated without proof. See the notes at the end of this paper for references to proofs of (8.3).]

can be completely ascertained.

Now it is easy to see that

$$3n^9 - 2n^3 \equiv 0, 1, \text{ or } -1 \pmod{49},$$

according as $n \equiv 0 \pmod 7, n \equiv 1, 2, 4 \pmod 7$, or $n \equiv 3, 5, 6 \pmod 7$. Also the coefficient of q^n in $q(1+q)/(1-q)^3$ is n^2. Hence the right side in (8.1) can be written as

$$\sum_{n=1}^{\infty} \left\{3n^2\sigma_7(n) - 2n^2\sigma_1(n)\right\} q^n + 49J. \tag{8.5}$$

It follows from this, (7.3), (7.4) and (8.1) that

$$\tau(n) - 3\lambda(n) + n\sigma_9(n) + n\sigma_3(n) \equiv 0 \pmod{49}, \tag{8.6}$$

where $\lambda(n)$ is the same as in (8.4). From the formulae (8.4) and (8.6) all the residues of $\tau(n)$ for modulus 49 can be completely ascertained.

Modulus 11

9. In this case we start with the series[13]

$$\begin{cases} 1 - 264 \displaystyle\sum_{n=1}^{\infty} \frac{n^9 q^n}{1-q^n} = QR, \\ 691 + 65520 \displaystyle\sum_{n=1}^{\infty} \frac{n^{11} q^n}{1-q^n} = 441Q^3 + 250R^2. \end{cases} \tag{9.1}$$

It follows that

$$QR = 1 + 11J; \qquad Q^3 - 3R^2 = -2P + 11J. \tag{9.2}$$

It is easy to see from this that

$$\begin{aligned}(Q^3 - R^2)^5 &= (Q^3 - 3R^2)^5 - Q(Q^3 - 3R^2)^3 - R(Q^3 - 3R^2)^2 - 5QR + 11J \\ &= P^5 - 3P^3Q - 4P^2R - 5QR + 11J.\end{aligned}$$

But[14]

$$\begin{cases} P^5 - 10P^3Q + 20P^2R - 15PQ^2 + 4QR = -20736 \displaystyle\sum_{n=1}^{\infty} n^4\sigma_1(n)q^n, \\ P^3Q - 3P^2R + 3PQ^2 - QR = 3456 \displaystyle\sum_{n=1}^{\infty} n^3\sigma_3(n)q^n, \\ P^2R - 2PQ^2 + QR = -1728 \displaystyle\sum_{n=1}^{\infty} n^2\sigma_5(n)q^n, \\ PQ^2 - QR = 720 \displaystyle\sum_{n=1}^{\infty} n\sigma_7(n)q^n; \end{cases} \tag{9.3}$$

[13] See [77, Table I].

[14] See [77, Table III, Table II].

and it is obvious that

$$(q;q)_\infty^{120} = \frac{(q^{121};q^{121})_\infty}{(q;q)_\infty} + 11J. \tag{9.4}$$

It is easy to see from all these that

$$q^5 \frac{(q^{121};q^{121})_\infty}{(q;q)_\infty} = \sum_{n=1}^{\infty} \left\{-n^4\sigma_1(n) + 3n^3\sigma_3(n) + 3n^2\sigma_5(n) - 5n\sigma_7(n)\right\} q^n + 11J. \tag{9.5}$$

It follows from this that

$$p(11n-5) \equiv 0 \pmod{11}; \tag{9.6}$$

and

$$\begin{aligned} &p(n-5) - p(n-126) - p(n-247) + p(n-610) + p(n-852) \\ &- \cdots + n^4\sigma_1(n) - 3n^3\sigma_3(n) - 3n^2\sigma_5(n) + 5n\sigma_7(n) \equiv 0 \pmod{11}, \end{aligned} \tag{9.7}$$

where 5, 126, 247, 610, ... are numbers of the form $\frac{1}{2}(11\nu+2)(33\nu+5)$ and $\frac{1}{2}(11\nu-2)(33\nu-5)$. It is only to prove the general result (9.7) we require all the details in (9.3). But we don't require all these details in order to prove (9.6) and the proof can be very much simplified as follows: we have[15]

$$q\frac{dP}{dq} = \frac{P^2-Q}{12}, \qquad q\frac{dQ}{dq} = \frac{PQ-R}{3}, \qquad q\frac{dR}{dq} = \frac{PR-Q^2}{2}. \tag{9.8}$$

Now using (9.2) and (9.8) we can show that[16]

$$(Q^3-R^2)^5 = q\frac{dJ}{dq} + 11J.$$

It follows from this and (9.4) that

$$q^5 \frac{(q^{121};q^{121})_\infty}{(q;q)_\infty} = q\frac{dJ}{dq} + 11J. \tag{9.9}$$

Since the coefficient of q^{11n} in the right hand side is a multiple of 11 it follows that

$$p(11n-5) \equiv 0 \pmod{11}.$$

The number of values of n not exceeding 200 for which $p(n) \equiv 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \pmod{11}$ is 77, 23, 24, 14, 15, 14, 5, 12, 8, 8, 0, respectively. Even though these values seem to be very irregular it appears from the residues of $p(n)$ for moduli 5 and 7 and also from the next section that $p(n) \equiv 0 \pmod{11}$ for about

[15] See [77, eq. (30)].

[16] As mentioned in the beginning, the J's are not the same functions.

$\frac{1}{6}$ of the values of n while $p(n) \equiv 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \pmod{11}$ for about $\frac{1}{12}$ of the values of n each.

10. Mr H. B. C. Darling observed the remarkable fact (before I began to write this paper) that $p(n)$ is divisible by 11 for 45 values of n not exceeding 100. This can be explained by the formula (9.7) and the congruency of

$$n^4\sigma_1(n) - 3n^3\sigma_3(n) - 3n^2\sigma_5(n) + 5n\sigma_7(n) \tag{10.1}$$

for modulus 11. It can be shown by quite elementary methods that (10.1) is divisible by 11 for almost all values of n. [A proof of this fact is sketched in Section 19.] It can even be shown that the number of values of n not exceeding n for which (10.1) is not divisible by 11 is

$$O\left(\frac{n}{(\log n)^{1/10}}\right) \tag{10.2}$$

by considering the divisibility of the four terms in (10.1) separately; but a better result can be found only by considering all the four terms in (10.1) taken together. The same remarks apply to the function $\lambda(n)$ defined by

$$\sum_{n=1}^{\infty} \lambda(n) q^n = q^5 \frac{(q^{121}; q^{121})_\infty}{(q;q)_\infty}; \tag{10.3}$$

so that $\lambda(n+5)$ is the number of partitions of n as the sum of integers which are not multiples of 121; that is to say $\lambda(n)$ is divisible by 11 for almost all values of n; and the number of values of $\lambda(n)$ not divisible by 11 is of the form (10.2). It appears from (10.3) that the number of values of n for which $p(n) \equiv 0 \pmod{11}$ cannot be so high as 45% if n exceeds 120. Thus the number of values of p divisible by 11 is

$$\begin{aligned} 45\%, &\qquad 0 < n \le 40\\ 45\%, &\qquad 40 < n \le 80\\ 45\%, &\qquad 80 < n \le 120\\ 35\%, &\qquad 120 < n \le 160\\ 22\tfrac{1}{2}\%, &\qquad 160 < n \le 200. \end{aligned}$$

It is also very remarkable that, in the table of the first 200 values of $p(n)$, there is not a single value of $p(n)$ of the form $11k-1$. This is probably due to such a high percentage of the values of $p(n)$ divisible by 11 in the beginning.

I have not yet investigated completely the residues of $\tau(n)$ for modulus 11. But it appears that if

$$\sum_{n=1}^{\infty} \lambda(n) q^n = q(q;q)_\infty^2 (q^{11}; q^{11})_\infty^2,$$

then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{1}{1-11^{-s}} \prod_p \frac{1}{1-\lambda(p)p^{-s}+p^{1-2s}}, \tag{10.4}$$

p assuming all prime values except 11, and that $\lambda(p)$ can be determined also. If that is so then the residues of $\tau(n)$ for modulus 11 can also be ascertained since it is easily seen that

$$\tau(n) - \lambda(n) \equiv 0 \pmod{11}. \tag{10.5}$$

Again it is easy to show by using (7.6) [and the values $\tau(2) = -24, \tau(3) = 252, \tau(5) = 4830, \tau(7) = -16744$, and $\tau(11) = 534612$, which can be found in a table in Ramanujan's paper [77], [82, p. 153]] that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} =& \frac{1}{1+2^{1-s}+2^{1-2s}} \frac{1}{(1-3^{3-s})^2} \frac{1}{(1-5^{2-s})(1-5^{4-s})} \\ & \times \frac{1}{(1+7^{2-s})(1-7^{4-s})} \frac{1}{1-11^{-s}} \cdots + 11j,\end{aligned} \tag{10.5a}$$

where j is a Dirichlet series of the form

$$\sum \frac{a_n}{n^s},$$

a_n being an integer.

¿From this we can deduce a number of results such as

$$\tau(2^{4\lambda-1}n) \equiv 0 \pmod{11} \tag{10.61}$$

if n is an odd integer;

$$\tau(3^{11\lambda-1}n) \equiv 0 \pmod{11} \tag{10.62}$$

if n is not a multiple of 3;

$$\tau(5^{5\lambda-1}n) \equiv 0 \pmod{11} \tag{10.63}$$

if n is not a multiple of 5;

$$\tau(7^{10\lambda-1}n) \equiv 0 \pmod{11} \tag{10.64}$$

if n is not a multiple of 7;

$$\tau(11^{\lambda}n) - \tau(n) \equiv 0 \pmod{11} \tag{10.7}$$

and so on. [The five congruences above can be established by expanding the appropriate factors in (10.5a) in geometric series. For example, consider

$$\begin{aligned}\frac{1}{1+2^{1-s}+2^{1-2s}} &= -\frac{i}{2^{1-s}+1-i} + \frac{i}{2^{1-s}+1+i} \\ &= -\frac{i}{1-i}\sum_{n=0}^{\infty}\left(\frac{2^{1-s}}{i-1}\right)^n + \frac{i}{1+i}\sum_{n=0}^{\infty}\left(\frac{2^{1-s}}{-i-1}\right)^n \\ &= i\sum_{n=0}^{\infty} 2^{n(1-s)}e^{-3\pi i(n+1)/4} - i\sum_{n=0}^{\infty} 2^{n(1-s)}e^{3\pi i(n+1)/4}.\end{aligned}$$

Since $\sin\{3\pi(n+1)/4\} = 0$ if and only if $n \equiv -1 \pmod 4$, the assertion (10.61) follows from (10.5a).]

Even though (10.61)–(10.64) are very analogous to one another further equations are not necessarily quite similar to these; sometimes there are more than one equation and sometimes there are equations of the form

$$\tau(19n) \equiv 0 \pmod{11} \tag{10.8}$$

if n is not a multiple of 19, and

$$\tau(29n) \equiv 0 \pmod{11} \tag{10.9}$$

if n is not a multiple of 29.

It is very likely that the primes 19, 29, ... occurring in equations like (10.8) and (10.9) are such that the sum of their reciprocals is a divergent series. If this assertion is true then $\tau(n)$ is divisible by 11 for almost all values of n which is easily seen from (10.2).

Moduli 2 and 3

11. [It will be convenient to introduce Ramanujan's theta-functions $\varphi(q)$ and $\psi(q)$, defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty} \tag{11.1a}$$

and

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{11.1b}$$

where the product representations are easy consequences of Jacobi's triple product identity.]

Before we proceed to consider higher moduli we shall see what the analogous formulae are in the cases of moduli 2 and 3. It is easy to see that [by (11.1b)]

$$\frac{(q^4;q^4)_\infty}{(q;q)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} + 2J = \psi(q) + 2J. \tag{11.1}$$

It follows that

$$p(n) - p(n-4) - p(n-8) + p(n-20) + p(n-28) - \cdots \tag{11.2}$$

is odd or even according as n is a triangular number or not, 4, 8, 20, ... being numbers of the form $2\nu(3\nu+1)$ and $2\nu(3\nu-1)$.

$p(n)$ is odd for 110 values of n not exceeding 200 and even for 90 values of n in the same range. Thus $p(n)$ seems to be odd for more values of n than those for which $p(n)$ is even.

If

$$\sum_{n=0}^{\infty} \lambda(n) q^n = \frac{(q^4;q^4)_\infty}{(q;q)_\infty}$$

so that $\lambda(n)$ is the number of partitions of n as the sum of integers which are not multiples of 4 then [by (11.1) and (11.1b)] $\lambda(n)$ is odd or even according as n is a triangular number or not.

Again we have

$$\frac{(q^9;q^9)_\infty}{(q;q)_\infty} = \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} + 3J. \tag{11.3}$$

But it can be shown [23] that

$$q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty} = \sum_{n=1}^{\infty} \chi_0(n) \frac{q^n}{1+q^n+q^{2n}} \tag{11.4}$$

[where $\chi_0(n)$ is the principal character modulo 3]. But the right hand side in (11.4) is of the form

$$\sum_{n=1}^{\infty} \chi_0(n) \frac{q^n}{(1-q^n)^2} + 3J;$$

and the coefficient of q^{3n+1} in the above series is $\sigma_1(3n+1)$. It follows from this and (11.3) and (11.4) that

$$\frac{(q^9;q^9)_\infty}{(q;q)_\infty} = \sum_{n=0}^{\infty} \sigma_1(3n+1) q^n + 3J. \tag{11.5}$$

¿From this we easily deduce that

$$\begin{aligned} &p(n) - p(n-9) - p(n-18) + p(n-45) + p(n-63) \\ &-p(n-108) - \cdots - \sigma_1(3n+1) \equiv 0 \pmod 3, \end{aligned} \tag{11.6}$$

where 9, 18, 45, ... are numbers of the form $\frac{9}{2}\nu(3\nu+1)$ and $\frac{9}{2}\nu(3\nu-1)$.

The number of values of n not exceeding 200 for which $p(n) \equiv 0, 1, 2 \pmod 3$ is 66, 68, 66 respectively. Thus it appears that $p(n) \equiv 0, 1, 2 \pmod 3$ for about $\frac{1}{3}$ of the number of values of n each.

It follows from (11.5) that if

$$\sum_{n=0}^{\infty} \lambda(n) q^n = \frac{(q^9;q^9)_\infty}{(q;q)_\infty}$$

so that $\lambda(n)$ is the number of partitions of n as the sum of integers which are not multiples of 9, then

$$\lambda(n) - \sigma_1(3n+1) \equiv 0 \pmod 3.$$

Again the left hand side of (11.4) is of the form

$$q(q;q)_\infty^{24} + 3J \tag{11.7}$$

while the right hand side of (11.4) is of the form

$$\sum_{n=1}^{\infty} \frac{n^2 q^n}{(1-q^n)^2} + 3J.$$

It follows that

$$\tau(n) - n\sigma_1(n) \equiv 0 \pmod 3. \tag{11.8}$$

Suppose now that

$$\begin{cases} t_n = 0, & \lambda(n) \equiv 0 \pmod 3, \\ t_n = 1, & \lambda(n) \not\equiv 0 \pmod 3, \end{cases}$$

and that

$$\begin{cases} T_n = 0, & \tau(n) \equiv 0 \pmod 3, \\ T_n = 1, & \tau(n) \not\equiv 0 \pmod 3. \end{cases}$$

Then we can easily deduce from (11.7), (11.8), and (2.2) that

$$\sum_{n=0}^{\infty} \frac{t_n}{(3n+1)^s} = \sum_{n=0}^{\infty} \frac{T_n}{n^s} = \prod_1 \prod_2$$

where

$$\prod_1 = \prod_p \frac{1}{1-p^{-2s}}$$

p assuming prime values of the form $3k-1$ and

$$\prod_2 = \prod_p \frac{1+p^{-s}}{1-p^{-3s}}$$

p assuming prime values of the form $3k+1$. We easily deduce from this that

$$\begin{cases} \displaystyle\sum_{k=1}^{n} t_k = o(n), \\ \displaystyle\sum_{k=1}^{n} T_k = o(n). \end{cases}$$

In other words $\lambda(n)$ and $\tau(n)$ are divisible by 3 for almost all values of n. We can show by transcendental methods that

$$\begin{cases} \displaystyle\sum_{k=1}^{n} t_k = C\int_1^n \frac{dx}{(\log x)^{1/2}} + O\left(\frac{n}{(\log n)^r}\right), \\ \displaystyle\sum_{k=1}^{n} T_k = \frac{C}{3}\int_1^n \frac{dx}{(\log x)^{1/2}} + O\left(\frac{n}{(\log n)^r}\right) \end{cases} \tag{11.8a}$$

where r is any positive number and

$$C = \frac{2^{1/2}}{3^{1/4}} \frac{1-7^{-2}}{1-7^{-3}} \frac{1-13^{-2}}{1-13^{-3}} \frac{1-19^{-2}}{1-19^{-3}} \cdots \frac{1}{\{(1-2^{-2})(1-5^{-2})(1-11^{-2})\cdots\}^{1/2}}$$

in both cases, 2, 5, 11, ... being primes of the form $3k-1$ and 7, 13, 19, ... being primes of the form $3k+1$.

Further properties of $\tau(n)$

12. It is easy to see [from (11.1b)] that

$$(q;q)_\infty^{24} = \frac{(q^2;q^2)_\infty^8}{(q;q^2)_\infty^8} + 32J = \psi^8(q) + 32J.$$

But [21, p. 139, Ex. (ii)]

$$q\psi^8(q) = \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^{2n}},$$

and

$$\sum_{n=1}^{\infty} n^4 q^n = \frac{q}{1-q^2} + 16J,$$

and

$$\sum_{n=1}^{\infty} n^8 q^n = \frac{q}{1-q^2} + 32J,$$

[since

$$\sum_{n=1}^{\infty} n^4 q^n \equiv 1\cdot q + 0\cdot q^2 + 1\cdot q^3 + 0\cdot q^4 + \cdots = \frac{q}{1-q^2} \pmod{16},$$

as $n^4 \equiv 0, 1 \pmod{16}$, according as n is even or odd, and

$$\sum_{n=1}^{\infty} n^8 q^n \equiv 1\cdot q + 0\cdot q^2 + 1\cdot q^3 + 0\cdot q^4 + \cdots = \frac{q}{1-q^2} \pmod{32},$$

as $n^8 \equiv 0, 1 \pmod{32}$, according as n is even or odd.] It is easy to see from all these that

$$\begin{cases} \tau(n) - n^3\sigma_1(n) \equiv 0 \pmod{16}; \\ \tau(n) - n^3\sigma_5(n) \equiv 0 \pmod{32}. \end{cases} \tag{12.1}$$

Again we have

$$(q;q)_\infty^{24} = \frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^3} + 27J.$$

But it can be shown that [22, p. 143, Thm. 8.7]

$$q\frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^3} = \sum_{n=1}^{\infty} \frac{n^2 q^n}{1+q^n+q^{2n}}. \tag{12.2}$$

Now it is easy to see that

$$\sum_{n=1}^{\infty} n^3 q^n = \frac{q}{1+q+q^2} + 9J$$

and

$$\sum_{n=1}^{\infty} n^9 q^n = \frac{q}{1+q+q^2} + 27J,$$

[since

$$\sum_{n=1}^{\infty} n^3 q^n \equiv 1\cdot q - 1\cdot q^2 + 0\cdot q^3 + 1\cdot q^4 - 1\cdot q^5 + 0\cdot q^6 + \cdots = \frac{q-q^2}{1-q^3} = \frac{q}{1+q+q^2} \pmod{9},$$

as $n^3 \equiv 0, 1, -1 \pmod{9}$, according as $n \equiv 0, 1, -1 \pmod{3}$, and

$$\sum_{n=1}^{\infty} n^9 q^n \equiv 1\cdot q - 1\cdot q^2 + 0\cdot q^3 + 1\cdot q^4 - 1\cdot q^5 + 0\cdot q^6 + \cdots = \frac{q}{1+q+q^2} \pmod{27},$$

as $n^9 \equiv 0, 1, -1 \pmod{27}$, according as $n \equiv 0, 1, -1 \pmod{3}$.] It follows that

$$\begin{cases} \tau(n) - n^2\sigma_1(n) \equiv 0 \pmod{9}, \\ \tau(n) - n^2\sigma_7(n) \equiv 0 \pmod{27}. \end{cases} \tag{12.3}$$

It is easy to deduce from (2.1), (4.2), (12.1) and (12.3) that

$$\begin{cases} \tau(n) - n\sigma_1(n) \equiv 0 \pmod{30}, \\ \tau(n) - n^2\sigma_1(n) \equiv 0 \pmod{36}, \\ \tau(n) - n^3\sigma_1(n) \equiv 0 \pmod{48}, \\ \tau(n) - n^5\sigma_1(n) \equiv 0 \pmod{120}, \end{cases} \tag{12.4}$$

$$\begin{cases} \tau(n) - n\sigma_3(n) \equiv 0 \pmod{42}, \\ \tau(n) - n^2\sigma_3(n) \equiv 0 \pmod{60}, \\ \tau(n) - n^4\sigma_3(n) \equiv 0 \pmod{168}, \end{cases} \tag{12.5}$$

$$\begin{cases} \tau(n) - n^3\sigma_5(n) \equiv 0 \pmod{288}, \\ \tau(n) - n^2\sigma_7(n) \equiv 0 \pmod{540}, \\ \tau(n) - n\sigma_9(n) \equiv 0 \pmod{1050}. \end{cases} \tag{12.6}$$

Again it easily follows from the second equation in (9.1) that

$$\tau(n) - \sigma_{11}(n) \equiv 0 \pmod{691}. \tag{12.7}$$

It is easy to deduce from this that $\tau(n)$ is divisible by 691 for almost all values of n, and by transcendental methods that the number of values of n not exceeding n for which $\tau(n)$ is not divisible by 691 is of the form

$$C\int_1^n \frac{dx}{(\log x)^{1/690}} + O\left(\frac{n}{(\log n)^r}\right) \tag{12.7a}$$

where C is a constant and r is any positive number.

It is easy to prove that

$$q(-q;-q)_\infty^{24} = q(q;q)_\infty^{24} + 48q^2(q^2;q^2)_\infty^{24} + 2^{12}q^4(q^4;q^4)_\infty^{24}. \tag{12.7b}$$

[To prove (12.7b), set, after Ramanujan,

$$f(-q) := (q;q)_\infty.$$

Thus, (12.7b) can be written in the equivalent formulation

$$qf^{24}(q) = qf^{24}(-q) + 48q^2f^{24}(-q^2) + 2^{12}q^4f(-q^4). \tag{12.7c}$$

To prove (12.7c), we use the catalogue of evaluations for f found in Entry 12 of Chapter 17 in Ramanujan's second notebook [21, p. 124], in particular,

(12.7c)

$$\begin{aligned} f(q) =&\sqrt{z}2^{-1/6}\{x(1-x)/q\}^{1/24}, \qquad & f(-q) = \sqrt{z}2^{-1/6}(1-x)^{1/6}(x/q)^{1/24},\\ f(-q^2) =&\sqrt{z}2^{-1/3}\{x(1-x)/q\}^{1/12}, \qquad & f(-q^4) = \sqrt{z}2^{-2/3}(1-x)^{1/24}(x/q)^{1/6}, \end{aligned}$$

where $x = k^2$, with k being the modulus, and $z = (2/\pi)K$, with K being the complete elliptic integral of the first kind. Using these evaluations in (12.7c), we easily verify its truth.] ¿From this it is easy to deduce that

$$\tau(2n) + 24\tau(n) + 2^{11}\tau(\tfrac{1}{2}n) = 0 \tag{12.8}$$

where n is any integer and $\tau(x) = 0$ if x is not an integer.

[Recall that φ and ψ are defined in (11.1a) and (11.1b), respectively.] Again it is easy to prove that

$$q\psi^8(q)\varphi^{16}(-q) = qf^{24}(-q).$$

[To prove this identity, use (12.7c) and the evaluations [21, Entry 11(i), p. 123; Entry 10(ii), p. 122]

$$\psi(q) = \sqrt{\tfrac{1}{2}z}(x/q)^{1/8} \qquad \text{and} \qquad \varphi(-q) = \sqrt{z}(1-x)^{1/4}.\Big] \tag{12.8a}$$

But [by the binomial theorem],

$$\varphi^{16}(-q) = -4\varphi^4(-q) + 16\varphi^2(-q) - 11 + 256J.$$

Hence

$$\begin{aligned} qf^{24}(-q) =&4\left\{1-\varphi^4(-q)\right\}q\psi^8(q) - 16\left\{1-\varphi^2(-q)\right\}q\psi^8(q) + q\psi^8(q) + 256J\\ =&4\left\{1-\varphi^4(-q)\right\}q\psi^4(q^2) - 16\left\{1-\varphi^2(-q)\right\}q\psi^4(q^2) + q\psi^8(q) + 256J. \end{aligned}$$

But

$$q\psi^8(q) = \sum_{n=0}^{\infty} \frac{n^3 q^n}{1-q^{2n}}, \tag{12.8b}$$

$$q\psi^4(q^2) = \sum_{n=0}^{\infty} \frac{(2n+1)q^{2n+1}}{1-q^{4n+2}}, \tag{12.8c}$$

(12.8d)

$$q\psi^4(q^2)\varphi^4(-q) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^3q^n}{1-q^{2n}},$$

$$\begin{aligned} q\psi^4(q^2)\varphi^2(-q) &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2q^n}{1+q^{2n}} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)^2q^{2n+1}}{1-q^{4n+2}} - \sum_{n=1}^{\infty}\frac{(2n)^2q^{2n}}{1+q^{4n}} + 16J. \end{aligned} \tag{12.8e}$$

[The identities (12.8b) and (12.8c) are, respectively, Examples (ii) and (iii) in Section 17 of Chapter 17 in Ramanujan's second notebook [21, p. 139].

By Entry 11(iii) in Chapter 17 of Ramanujan's second notebook [21, p. 123],

$$\psi(q^2) = \tfrac{1}{2}\sqrt{z}(x/q)^{1/4}. \tag{12.8f}$$

It follows from (12.8a) and (12.8f) that

$$q\psi(q^2)\varphi^4(-q) = \tfrac{1}{16}z^4x(1-x). \tag{12.8g}$$

On the other hand by Entry 14(ii), (ix) in Chapter 17 of the second notebook [21, p. 130],

$$\begin{aligned} \sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^3q^n}{1-q^{2n}} &= \sum_{n=1}^{\infty}(-1)^{n-1}n^3\left(\frac{q^n}{1+q^n}+\frac{q^{2n}}{1-q^{2n}}\right) \\ &= \frac{1}{16}\left(1+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^3q^n}{1+q^n} - 1 + 16\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^3q^{2n}}{1-q^{2n}}\right) \\ &= \tfrac{1}{16}z^4x(1-x). \end{aligned} \tag{12.8h}$$

The equality (12.8d) is now a trivial consequence of (12.8g) and (12.8h).

To prove (12.8e), first observe, by (12.8a) and (12.8f), that

$$q\psi^4(q^2)\varphi^2(-q) = \tfrac{1}{16}z^3x\sqrt{1-x}. \tag{12.8i}$$

Next,

$$\begin{aligned} \sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2q^n}{1+q^{2n}} &= -\sum_{n=1}^{\infty}\frac{4n^2q^{2n}}{1+q^{4n}} + \sum_{n=1}^{\infty}\frac{(2n+1)^2q^{2n+1}}{1+q^{4n+2}} \\ &= -8\sum_{n=1}^{\infty}\frac{n^2q^{2n}}{1+q^{4n}} + \sum_{n=1}^{\infty}\frac{n^2q^n}{1+q^{2n}} \\ &= -8\sum_{n=1}^{\infty}\frac{n^2q^{2n}}{1+q^{4n}} + \tfrac{1}{16}z^3x, \end{aligned} \tag{12.8j}$$

by Entry 17(ii) in Chapter 17 of Ramanujan's second notebook [21, p. 138]. To evaluate the sum on the far right side of (12.8j), we apply the process of duplication [21, p. 125] to Entry 17(ii) cited above. Accordingly,

$$\begin{aligned}-8\sum_{n=1}^{\infty}\frac{n^2q^{2n}}{1+q^{4n}} &= -\frac{1}{2}\left(\frac{1}{2}z(1+\sqrt{1-x})\right)^3\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2\\ &= -\tfrac{1}{16}z^3x(1-\sqrt{1-x}),\end{aligned} \tag{12.8k}$$

after simplification. Putting (12.8k) into (12.8j), we readily find that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2q^n}{1+q^{2n}} = \tfrac{1}{16}z^3x\sqrt{1-x}. \tag{12.8m}$$

Combining (12.8i) and (12.8k), we complete the proof of the first part of (12.8e).

To prove the second part of (12.8e), it clearly suffices to prove that

$$S := \sum_{n=0}^{\infty}\frac{(2n+1)^2q^{2n+1}}{1+q^{4n+2}} \equiv \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)^2q^{2n+1}}{1-q^{4n+2}} =: T \pmod{16} \tag{12.8n}$$

Now,

$$\begin{aligned}S =& \sum_{n=0}^{\infty}\frac{(2n+1)^2q^{2n+1}}{1-q^{4n+2}} - 2\sum_{n=0}^{\infty}\frac{(2n+1)^2q^{6n+3}}{1-q^{8n+4}}\\ =& T + 2\sum_{n=0}^{\infty}\frac{(4n+3)^2q^{4n+3}}{1-q^{8n+6}} - 2\sum_{n=0}^{\infty}\frac{(2n+1)^2q^{6n+3}}{1-q^{8n+4}}\\ \equiv& T + 2\sum_{n=0}^{\infty}\frac{q^{4n+3}}{1-q^{8n+6}} - 2\sum_{n=0}^{\infty}\frac{q^{6n+3}}{1-q^{8n+4}} \pmod{16}\\ =& T + 2\sum_{n=0}^{\infty}\frac{q^{6n+3}}{1-q^{8n+4}} \pmod{16} - 2\sum_{n=0}^{\infty}\frac{q^{6n+3}}{1-q^{8n+4}} \pmod{16}\\ =& T \pmod{16},\end{aligned}$$

where in the antepenultimate line above we expanded the summands of the first series in geometric series and then reversed the order of summation. This completes the proof of (12.8n), and hence the proof of the second equality of (12.8e).]

It follows from all these that

$$\begin{aligned}q(q;q)_\infty^{24} =& -3\sum_{n=0}^{\infty}\frac{(2n+1)^3q^{2n+1}}{1-q^{4n+2}} + 5\sum_{n=1}^{\infty}\frac{(2n)^3q^{2n}}{1-q^{4n}} - 12\sum_{n=0}^{\infty}\frac{(2n+1)q^{2n+1}}{1-q^{4n+2}}\\ &+ 16\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)^2q^{2n+1}}{1-q^{4n+2}} - 16\sum_{n=1}^{\infty}\frac{(2n)^2q^{2n}}{1+q^{4n}} + 256J.\end{aligned}$$

Now equating only the odd powers of q we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\tau(2n+1)q^{2n+1} =& -3\sum_{n=0}^{\infty}\frac{(2n+1)^3q^{2n+1}}{1-q^{4n+2}} + 16\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)^2q^{2n+1}}{1-q^{4n+2}}\\ &- 12\sum_{n=0}^{\infty}\frac{(2n+1)q^{2n+1}}{1-q^{4n+2}} + 256J.\end{aligned}$$

But if n be of the form $4k+1$ then it is easy to see that

$$n^{11}+3n^3-16n^2+12n\equiv 0 \pmod{256}.$$

Changing n to $-n$ in this formula we see that if n be of the form $4k-1$ then

$$n^{11}+3n^3+16n^2+12n\equiv 0 \pmod{256}.$$

It follows that

$$\sum_{n=0}^{\infty}\tau(2n+1)q^{2n+1}=\sum_{n=0}^{\infty}\frac{(2n+1)^{11}q^{2n+1}}{1-q^{4n+2}}+256J.$$

In other words,

$$\tau(n)-\sigma_{11}(n)\equiv 0 \pmod{256} \tag{12.9}$$

for all <u>odd</u> values of n, while the formula (12.8) combined with this enable us to find the residues of $\tau(n)$ for modulus 2^{11} for <u>even</u> values of n. Thus

$$\tau(n)+24\sigma_{11}(n)\equiv 0 \pmod{2048}$$

for all values of n.

It follows from (12.7) and (12.9) that

$$\tau(n)-\sigma_{11}(n)\equiv 0 \pmod{176896}$$

for all odd values of n.

Modulus 13

13. In this case we start with the second series in (9.1) and the series

$$1-24\sum_{n=1}^{\infty}\frac{n^{13}q^n}{1-q^n}=Q^2R. \tag{13.1}$$

It follows from these that

$$Q^3-3R^2=-2+13J; \qquad Q^2R=P+13J. \tag{13.2}$$

Hence we have

$$\begin{aligned}(Q^3-R^2)^7 =& -2(R^2-1)^7+13J\\ =& -5R^6(3R^2-2)^4-2R^4(3R^2-2)^3+6R^4(3R^2-2)^2\\ & -6R^2(3R^2-2)^2-6R^2(3R^2-2)-2(R^2-1)+13J\\ =& -5P^6-2P^4Q+6P^3R-6P^2Q^2-6PQR-(Q^3-R^2)+13J.\end{aligned}$$

But[17]

(13.3)

$$\begin{cases} 5(P^6 - 15P^4Q + 40P^3R - 45P^2Q^2 + 24PQR) \\ \quad - (9Q^3 + 16R^2) = -248832 \sum_{n=1}^{\infty} n^5\sigma_1(n)q^n, \\ 7(P^4Q - 4P^3R + 6P^2Q^2 - 4PQR) + (3Q^3 + 4R^2) = 41472 \sum_{n=1}^{\infty} n^4\sigma_3(n)q^n, \\ 2(P^3R - 3P^2Q^2 + 3PQR) - (Q^3 + R^2) = -5184 \sum_{n=1}^{\infty} n^3\sigma_5(n)q^n, \\ 9(PQ - R)^2 + 5(Q^3 - R^2) = 8640 \sum_{n=1}^{\infty} n^3\sigma_7(n)q^n, \\ 5PQR - (3Q^3 + 2R^2) = -1584 \sum_{n=1}^{\infty} n\sigma_9(n)q^n, \\ Q^3 - R^2 = 1728 \sum_{n=1}^{\infty} \tau(n)q^n; \end{cases}$$

and it is obvious that

$$(q;q)_\infty^{168} = \frac{(q^{169};q^{169})_\infty}{(q;q)_\infty} + 13J. \tag{13.4}$$

It is easy to see from all these that

$$q^7 \frac{(q^{169};q^{169})_\infty}{(q;q)_\infty} = (q^{169};q^{169})_\infty \sum_{n=0}^{\infty} p(n)q^{n+7}$$

(13.5)

$$= \sum_{n=1}^{\infty} \left\{n^5\sigma_1(n) - 4n^4\sigma_3(n) - 3n^3\sigma_5(n) + 6n^2\sigma_7(n) - 3n\sigma_9(n) + 3\tau(n)\right\} q^n + 13J.$$

It is easy to see by actual calculation that $\tau(13) \equiv 8 \pmod{13}$ in virtue of (7.6) and hence $\tau(13n) - 8\tau(n) \equiv 0 \pmod{13}$. It follows from this and (13.5) that

$$\sum_{n=1}^{\infty} p(13n-7)q^n \ (q^{13};q^{13})_\infty = 11 \sum_{n=1}^{\infty} \tau(n)q^n + 13J. \tag{13.6}$$

It is not necessary to know all the details above in order to prove (13.6). The proof can be very much simplified as follows; using (9.8) and (13.2) we can show that

$$(Q^3 - R^2)^7 = q\frac{dJ}{dq} + 3(Q^3 - R^2) + 13J. \tag{13.7}$$

[17] See [77], where not all these equalities are given, but where the same methods can be employed to provide proofs.

It follows from this that

$$q^7\frac{(q^{169};q^{169})_\infty}{(q;q)_\infty} = q\frac{dJ}{dq} + 3\sum_{n=1}^{\infty}\tau(n)q^n + 13J. \tag{13.8}$$

¿From this we easily deduce (13.6).

Again picking out the terms $q^{13}, q^{26}, q^{39}, \ldots$ in (13.6) we obtain [using the congruence $\tau(13n) \equiv 8\tau(n) \pmod{13}$]

$$\sum_{n=1}^{\infty} p(13^2n-7)q^n \ (q;q)_\infty = 10\sum_{n=1}^{\infty}\tau(n)q^n + 13J. \tag{13.9}$$

It follows from (13.5) that if

$$\sum_{n=1}^{\infty}\lambda(n)q^n = q^7\frac{(q^{169};q^{169})_\infty}{(q;q)_\infty}$$

so that $\lambda(n+7)$ is the number of partitions of n as the sum of integers which are not multiples of 169, then

$$\begin{aligned}&\lambda(n) - n^5\sigma_1(n) + 4n^4\sigma_3(n) + 3n^3\sigma_5(n)\\ &- 6n^2\sigma_7(n) + 3n\sigma_9(n) - 3\tau(n) \equiv 0 \pmod{13}.\end{aligned}$$

The results analogous to (10.61)–(10.9) in the case of modulus 13 are

$$\tau(5^{12\lambda-1}n) \equiv 0 \pmod{13}$$

if n is not a multiple of 5;

$$\tau(7n) \equiv 0 \pmod{13}$$

if n is not a multiple of 7;

$$\tau(11n) \equiv 0 \pmod{13}$$

if n is not a multiple of 11;

$$\tau(13n) - 8\tau(n) \equiv 0 \pmod{13}$$

if n is any integer;

$$\tau(19^{4\lambda-1}n) \equiv 0 \pmod{13}$$

if n is not a multiple of 19;

$$\tau(23^{3\lambda-1}n) \equiv 0 \pmod{13}$$

if n is not a multiple of 23;

$$\tau(29^{6\lambda-1}n) \equiv 0 \pmod{13}$$

if n is not a multiple of 29; and so on.

14. The formulae (13.6) and (13.9) can be written as

$$\sum_{n=0}^{\infty} p(13n+6)q^n = 11(q;q)_\infty^{11} + 13J; \tag{14.1}$$

and

$$\sum_{n=0}^{\infty} p(13^2 n+162)q^n = 23(q;q)_\infty^{23} + 13J. \tag{14.2}$$

Since I began to write this paper I have found by a different method that if λ be any positive odd integer then

$$\sum_{n=0}^{\infty} p\left(13^\lambda n + \frac{11\cdot 13^\lambda+1}{24}\right) q^n = -2^{(5\lambda-3)/2}(q;q)_\infty^{11} + 13J; \tag{14.3}$$

and if λ be any positive even integer then

$$\sum_{n=0}^{\infty} p\left(13^\lambda n + \frac{23\cdot 13^\lambda+1}{24}\right) q^n = -2^{(5\lambda-2)/2}(q;q)_\infty^{23} + 13J. \tag{14.4}$$

I shall reserve the discussion of these results to another paper.

A number of results such as the following can be deduced from (14.3) and (14.4). [Note that

$$(q;q)_\infty^{11} = 1 - 11q + 44q^2 - 55q^3 - 110q^4 + 374q^5 - 143q^6 + \cdots$$

and

$$(q;q)_\infty^{23} = 1 - 23q + 230q^2 - 1265q^3 + 3795q^4 - 3519q^5 - 16445q^6 + \cdots .]$$

If λ be any positive odd integer then

$$\begin{cases} p\left(\frac{11\cdot 13^\lambda+1}{24}\right) + 2^{(5\lambda-3)/2}, & p\left(\frac{35\cdot 13^\lambda+1}{24}\right) + 2^{(5\lambda-1)/2}, \\ p\left(\frac{59\cdot 13^\lambda+1}{24}\right) - 2^{(5\lambda+3)/2}, & p\left(\frac{83\cdot 13^\lambda+1}{24}\right) - 2^{5(\lambda+1)/2}, \\ p\left(\frac{107\cdot 13^\lambda+1}{24}\right) - 2^{(5\lambda+7)/2}, & p\left(\frac{131\cdot 13^\lambda+1}{24}\right) - 2^{5(\lambda+1)/2}, \\ p\left(\frac{155\cdot 13^\lambda+1}{24}\right), & \end{cases} \tag{14.5}$$

and so on are all divisible by 13; and if λ be any positive even integer then

$$\begin{cases} p\left(\frac{23\cdot 13^\lambda+1}{24}\right) + 2^{(5\lambda-2)/2}, & p\left(\frac{47\cdot 13^\lambda+1}{24}\right) + 2^{(5\lambda+6)/2}, \\ p\left(\frac{71\cdot 13^\lambda+1}{24}\right) - 2^{(5\lambda+2)/2}, & p\left(\frac{95\cdot 13^\lambda+1}{24}\right) - 2^{(5\lambda+2)/2}, \\ p\left(\frac{119\cdot 13^\lambda+1}{24}\right) - 2^{(5\lambda-2)/2}, & p\left(\frac{143\cdot 13^\lambda+1}{24}\right) + 2^{(5\lambda+2)/2}, \\ p\left(\frac{167\cdot 13^\lambda+1}{24}\right), & \end{cases} \tag{14.6}$$

and so on are all divisible by 13. In other words if n is fixed and $\lambda + n$ is an even integer then the residue of

$$p\left(\frac{13^{\lambda}(12n-1)+1}{24}\right) \tag{14.7}$$

for modulus 13 can be completely ascertained.

General Theory

Modulus ϖ
where ϖ is a prime greater than 3

15. We start with the two series

$$v_{\varpi-1} + (-1)^{(\varpi-1)/2} 2(\varpi-1)\delta_{\varpi-1} \sum_{n=1}^{\infty} \frac{n^{\varpi-2}q^n}{1-q^n} = \sum K'_{\ell,m} Q^{\ell} R^m, \tag{15.1}$$

where $K'_{\ell,m}$ is a constant integer and the summation extends over all positive integral values of ℓ and m (including zero) such that

$$4\ell + 6m = \varpi - 1;$$

and

$$v_{\varpi+1} + (-1)^{(\varpi+1)/2} 2(\varpi+1)\delta_{\varpi+1} \sum_{n=1}^{\infty} \frac{n^{\varpi}q^n}{1-q^n} = \sum K_{\ell,m} Q^{\ell} R^m, \tag{15.2}$$

where $K_{\ell,m}$ is a constant integer and the summation extends over all positive integral values of ℓ and m (including zero) such that

$$4\ell + 6m = \varpi + 1.$$

In both the series v_s and δ_s are the numerator and the denominator of B_s in its lowest terms where

$$B_2 = \frac{1}{6}, \quad B_4 = \frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = \frac{1}{30}, \quad B_{10} = \frac{5}{66}, \ldots$$

are the Bernoulli numbers. Now by von Staudt's Theorem

$$\delta_{\varpi-1} \equiv 0 \pmod{\varpi},$$

and also we have

$$n^{\varpi} - n \equiv 0 \pmod{\varpi}.$$

And so the left hand side in (15.1) is of the form

$$c' + \varpi J \tag{15.3}$$

where c' is a constant integer while that in (15.2) is of the form

$$k + cP + \varpi J \tag{15.31}$$

where c and k are constant integers.

It appears that k can be taken as zero always. This involves the assertion that

$$6v_{\varpi+1} + (-1)^{(\varpi+1)/2}\frac{\varpi+1}{2}\delta_{\varpi+1} \equiv 0 \pmod{\varpi}. \tag{15.4}$$

I have not yet proved this result but in every particular case this can actually found to be true. Thus (15.31) can be replaced by

$$cP + \varpi J. \tag{15.5}$$

Now using (15.3), (15.5) and (9.8) we can show in particular cases that

$$(Q^3 - R^2)^{(\varpi^2-1)/24} = q\frac{dJ}{dq} + (Q^3 - R^2)\sum k_{\ell,m}Q^\ell R^m + \varpi J \tag{15.6}$$

where $k_{\ell,m}$ is a constant integer and the summation extends over all positive integral values of ℓ and m (including zero) such that

$$4\ell + 6m = \varpi - 13.$$

But it is obvious that

$$(q;q)_\infty^{\varpi^2-1} = \frac{(q^{\varpi^2};q^{\varpi^2})_\infty}{(q;q)_\infty} + \varpi J. \tag{15.7}$$

It follows from (15.6) and (15.7) that

$$q^{(\varpi^2-1)/24}\frac{(q^{\varpi^2};q^{\varpi^2})_\infty}{(q;q)_\infty} = q\frac{dJ}{dq} + (Q^3 - R^2)\sum k_{\ell,m}Q^\ell R^m + \varpi J \tag{15.8}$$

where the remark about the summation in (15.6) applies here also. From this we can always deduce in every particular case that

$$\begin{aligned}&\sum_{n=1}^{\infty} p\left(n\varpi + \varpi\left[\frac{\varpi}{24}\right] - \frac{\varpi^2-1}{24}\right) q^{n+[\varpi/24]}\ (q^{\varpi^2};q^{\varpi^2})_\infty \\ &=(Q^3 - R^2)^{1+[\varpi/24]}\sum k_{\ell,m}Q^\ell R^m + \varpi J\end{aligned} \tag{15.9}$$

where $k_{\ell,m}$ is a constant integer and the summation extends over all positive integral values of ℓ and m (including zero) such that

$$4\ell + 6m = \varpi - 13$$

and $[t]$ denotes as usual the greatest integer in t.

Even though all these results are very difficult to prove in general they can be easily proved when $\varpi \le 23$.

Moduli 17, 19 and 23

16. In these cases we can easily prove that

$$\sum_{n=1}^{\infty} p(17n-12)q^n \ (q^{17};q^{17})_\infty = 7\sum_{n=1}^{\infty} \tau_2(n)q^n + 17J, \tag{16.1}$$

where

$$\sum_{n=1}^{\infty} \tau_2(n)q^n = Qq(q;q)_\infty^{24};$$

$$\sum_{n=1}^{\infty} p(19n-15)q^n \ (q^{19};q^{19})_\infty = 5\sum_{n=1}^{\infty} \tau_3(n)q^n + 19J, \tag{16.2}$$

where

$$\sum_{n=1}^{\infty} \tau_3(n)q^n = Rq(q;q)_\infty^{24};$$

and

$$\sum_{n=1}^{\infty} p(23n-22)q^n \ (q^{23};q^{23})_\infty = \sum_{n=1}^{\infty} \tau_5(n)q^n + 23J, \tag{16.3}$$

where

$$\sum_{n=1}^{\infty} \tau_5(n)q^n = QRq(q;q)_\infty^{24}.$$

I have stated without proof in my previous paper[18] that

$$\begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{\tau_2(n)}{n^s} = \prod_p \frac{1}{1-\tau_2(p)p^{-s}+p^{15-2s}}, \\ \displaystyle\sum_{n=1}^{\infty} \frac{\tau_3(n)}{n^s} = \prod_p \frac{1}{1-\tau_3(p)p^{-s}+p^{17-2s}}, \\ \displaystyle\sum_{n=1}^{\infty} \frac{\tau_4(n)}{n^s} = \prod_p \frac{1}{1-\tau_4(p)p^{-s}+p^{19-2s}}, \\ \displaystyle\sum_{n=1}^{\infty} \frac{\tau_5(n)}{n^s} = \prod_p \frac{1}{1-\tau_5(p)p^{-s}+p^{21-2s}}, \\ \displaystyle\sum_{n=1}^{\infty} \frac{\tau_7(n)}{n^s} = \prod_p \frac{1}{1-\tau_7(p)p^{-s}+p^{25-2s}}, \end{cases} \tag{16.4}$$

where

$$\sum_{n=1}^{\infty} \tau_4(n)q^n = Q^2q(q;q)_\infty^{24}$$

[18] See [77, eq. (108)].

and

$$\sum_{n=1}^{\infty} \tau_7(n) q^n = Q^2 R q (q;q)_\infty^{24},$$

and p assumes all prime values. All these seem to be capable of proof as the case of $\tau(n)$ by Mordell's method.[19]

Now using (16.4) we can deduce from (16.1), (16.2) and (16.3) that

$$\sum_{n=1}^{\infty} p(n17^2 - 12) q^n \ (q;q)_\infty = c_2 \sum_{n=1}^{\infty} \tau_2(n) q^n + 17J, \tag{16.5}$$

$$\sum_{n=1}^{\infty} p(n19^2 - 15) q^n \ (q;q)_\infty = c_3 \sum_{n=1}^{\infty} \tau_3(n) q^n + 19J, \tag{16.6}$$

and

$$\sum_{n=1}^{\infty} p(n23^2 - 22) q^n \ (q;q)_\infty = c_5 \sum_{n=1}^{\infty} \tau_5(n) q^n + 23J, \tag{16.7}$$

where c_2, c_3 and c_5 are constants.

I have found that there are formulae quite analogous to those for modulus 13 even in these cases. I shall reserve the discussion of these as well as those for higher primes to another paper; but I shall consider in the II part of this paper the analogous formulae for the smaller primes 5, 7, and 11.

The corresponding formulae for primes greater than 23 are not quite analogous. For instance in the cases of

moduli 29 and 31

we have

$$\sum_{n=1}^{\infty} p(29n - 6) q^{n+1} \ (q^{29};q^{29})_\infty = 8 \sum_{n=1}^{\infty} \Omega_2(n) q^n + 29J, \tag{16.8}$$

where

$$\sum_{n=1}^{\infty} \Omega_2(n) q^n = Q q^2 (q;q)_\infty^{48};$$

and

$$\sum_{n=1}^{\infty} p(31n - 9) q^{n+1} \ (q^{31};q^{31})_\infty = 10 \sum_{n=1}^{\infty} \Omega_3(n) q^n + 31J, \tag{16.9}$$

where

$$\sum_{n=1}^{\infty} \Omega_3(n) q^n = R q^2 (q;q)_\infty^{48}.$$

[19] loc. cit.

The functions

$$\sum_{n=1}^{\infty} \frac{\Omega_2(n)}{n^s}, \qquad \sum_{n=1}^{\infty} \frac{\Omega_3(n)}{n^s}$$

are obviously not capable of a single product as in (16.4); but they are, as a matter of fact, the differences of two such products.

17. I have not yet investigated the residues of $\tau(n)$ for other moduli besides what was stated before but the case 23 seems to be (comparatively) simple. For it appears that if

$$\sum_{n=1}^{\infty} \lambda(n) q^n = q(q;q)_\infty (q^{23};q^{23})_\infty$$

so that

$$\tau(n) - \lambda(n) \equiv 0 \pmod{23} \tag{17.1}$$

then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{1}{1-23^{-s}} \prod\nolimits_1 \prod\nolimits_2 \prod\nolimits_3, \tag{17.2}$$

where

$$\prod\nolimits_1 = \prod_p \frac{1}{1-p^{-2s}},$$

p assuming all prime values of the form[20]

$$p \equiv 5, 7, 10, 11, 14, 15, 17, 19, 20, 21, 22 \pmod{23} \tag{17.3}$$

and

$$\prod\nolimits_2 = \prod_p \frac{1}{1+p^{-s}+p^{-2s}}$$

p assuming all prime values of the form[21]

$$p \equiv 1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18 \pmod{23} \tag{17.4}$$

except of the form $23a^2+b^2$, and

$$\prod\nolimits_3 = \prod_p \frac{1}{(1-p^{-s})^2}$$

p assuming all primes of the form $23a^2+b^2$. Thus $\lambda(n)$ can be completely determined and consequently the residues of $\tau(n)$ for modulus 23 can be completely ascertained.

Suppose now that

$$\begin{cases} t_n = 0, & \tau(n) \equiv 0 \pmod{23}; \\ t_n = 1, & \tau(n) \not\equiv 0 \pmod{23}. \end{cases} \tag{17.5}$$

[20] This can be written as $p^{11} \equiv -1 \pmod{23}$.

[21] This can be written as $p^{11} \equiv 1 \pmod{23}$.

Then it is easy to see from ( ) that

(17.6)
$$\sum_{n=1}^{\infty} \frac{t_n}{n^s} = \prod\nolimits_1 \prod\nolimits_2 \prod\nolimits_3,$$

where
$$\prod\nolimits_1 = \prod_p \frac{1}{1-p^{-2s}},$$

p assuming all primes of the form (17.3),
$$\prod\nolimits_2 = \prod_p \frac{1+p^{-s}}{1-p^{-3s}},$$

p assuming all primes of the form (17.4) except those of the form $23a^2 + b^2$, and
$$\prod\nolimits_3 = \prod_p \frac{1-p^{-22s}}{(1-p^{-s})(1-p^{-23s})}$$

p assuming all primes of the form $23a^2 + b^2$.

It is easy to prove from (17.6) by quite elementary methods that

(17.7)
$$\sum_{k=1}^{n} t_k = o(n);$$

and by transcendental methods that

(17.8)
$$\sum_{k=1}^{n} t_k = C \int_1^n \frac{dx}{(\log x)^{1/2}} + O\left(\frac{n}{(\log n)^r}\right),$$

where r is any positive number and
$$\begin{aligned} C =& \frac{66^{1/2}}{23^{3/4}} \frac{1-2^{-2}}{1-2^{-3}} \frac{1-3^{-2}}{1-3^{-3}} \frac{1-13^{-2}}{1-13^{-3}} \frac{1-29^{-2}}{1-29^{-3}} \cdots \\ &\times \frac{1}{\{(1-5^{-2})(1-7^{-2})(1-11^{-2})(1-17^{-2})\cdots\}^{1/2}} \\ &\times \frac{1-59^{-22}}{1-59^{-23}} \frac{1-101^{-22}}{1-101^{-23}} \frac{1-167^{-22}}{1-167^{-23}} \cdots, \end{aligned}$$

2, 3, 13, ... being primes of the form (17.4) except those of the form $23a^2+b^2$, and 5, 7, 11, 17, ... being primes of the form (17.3) and 59, 101, 167, ... are those of the form $23a^2 + b^2$. Thus we see that $\tau(n)$ is almost always divisible by 23.

We have also shown that among the values of $\tau(n)$, multiples of 3, 7 and 23 are more or less equally numerous while the multiples of 5 are less numerous than these and multiples of 2 are the most numerous.

Since
$$\begin{aligned} (1-p^{-s})(1-p^{11-s}) =& (1-p^{-2s}) - (p^{11}+1)(p^{-s}-p^{-2s}) \\ =& (1-p^{-s})^2 - (p^{11}-1)(p^{-s}-p^{-2s}) \end{aligned}$$

it is easy to see from (17.2) and (12.7) that if the prime divisors of n are of the form (17.3) or of the form $23a^2 + b^2$[22] then

$$\tau(n) - \sigma_{11}(n) \equiv 0 \pmod{15893}, \tag{17.9}$$

15893 being $23 \cdot 691$. If, in addition to the restrictions on the values of n in (17.9), we impose the restriction that n is odd also then if follows from (12.9) that

$$\tau(n) - \sigma_{11}(n) \equiv 0 \pmod{4068608},$$

4068608 being $23 \cdot 256 \cdot 691$.

Modulus 121

18. The case of modulus ϖ^2 seems to be much more complicated than the case of modulus ϖ even though the method is practically the same as may be seen from the case of modulus 49. I shall now consider the case of modulus 121.

It is easy to show by using (9.2) that

$$\begin{aligned}(Q^3-R^2)^5 =& P(Q^3-3R^2)(3P^3-PQ+4R)+4QR(4P^3Q-3P^2R+2QR)\\ &-26P^5+23P^3Q+16P^2R-22PQ^2+9QR+121J.\end{aligned} \tag{18.1}$$

¿From this we can deduce that

$$\begin{aligned}q^5\frac{(q^{11};q^{11})_\infty^{11}}{(q;q)_\infty} =& \sum_{n=1}^{\infty}\big[n^4\{a_1\sigma_1(n)+b_1\sigma_{11}(n)\}+n^3\{a_2\sigma_3(n)+b_2\sigma_{13}(n)\}\\ &+n^2\{a_3\sigma_5(n)+b_3\sigma_{15}(n)\}+n\{a_4\sigma_7(n)+b_4\sigma_{17}(n)\}\\ &+c_1n^2\tau_2(n)+c_2n\tau_3(n)+c_3\tau_4(n)\big]q^n+121J\end{aligned} \tag{18.2}$$

where the a's, b's and c's are constant integers and $\tau_2(n), \tau_3(n)$ and $\tau_4(n)$ are the same as in (16.4). But it is easy to show that

$$\begin{cases}\tau_2(n)-n\sigma_3(n),\\ \tau_3(n)-n\sigma_5(n),\\ \tau_4(n)-n\sigma_7(n),\end{cases} \equiv 0 \pmod{11}. \tag{18.3}$$

It is easy to see from (16.4) that

$$\tau_4(11n) - \tau_4(11)\tau_4(n) \equiv 0 \pmod{121}, \tag{18.4}$$

and by actual calculation we find that

$$\tau_4(11) \equiv 0 \pmod{11}. \tag{18.5}$$

It is also obvious that

$$\sigma_{17}(n) - \sigma_7(n) \equiv 0 \pmod{11}. \tag{18.6}$$

[22] Some may be of one form and some may be of the other form.

Now remembering (18.3)–(18.6) and picking out the terms $q^{11}, q^{22}, q^{33}, \ldots$ in ( ) we obtain

$$\sum_{n=1}^{\infty} p(11n-5)q^n \ (q^{11};q^{11})_\infty = 11\sum_{n=1}^{\infty} n\sigma_7(n)q^n + 121J. \tag{18.7}$$

It follows from this that

$$p(121n-5) \equiv 0 \pmod{121}, \tag{18.8}$$

and

$$\begin{aligned} &p(11n-5) - p(11n-126) - p(11n-247) \\ &+p(11n-610) + \cdots - 11n\sigma_7(n) \equiv 0 \pmod{121}. \end{aligned} \tag{18.9}$$

19. In concluding the first part of this paper I shall consider the numbers which are the divisors of $\tau(n)$ for almost all values of n.

Suppose that $\varpi_1, \varpi_2, \varpi_3, \ldots$ are an infinity of primes such that

$$\sum_{n=1}^{\infty} \frac{1}{\varpi_n} \tag{19.1}$$

is a <u>divergent</u> series and also suppose that $a_2, a_3, a_5, a_7, \ldots$ assume some or all of the positive integers (including zero) but that $a_{\varpi_1}, a_{\varpi_2}, a_{\varpi_3}, \ldots$ <u>never</u> assume the value unity. Then it is easy to show that the number of numbers of the form

$$2^{a_2} \cdot 3^{a_3} \cdot 5^{a_5} \cdot 7^{a_7} \cdots \tag{19.2}$$

not exceeding n is of the form

$$o(n). \tag{19.3}$$

In particular if a_ϖ <u>never</u> assumes the value unity for all prime values of ϖ of the form

$$\varpi \equiv c \pmod{k}, \tag{19.4}$$

where c and k are any two integers which are prime to each other, then the number of numbers of the form (19.2) is of the form

$$o(n) \tag{19.5}$$

and more accurately is of the form

$$O\left(\frac{n}{(\log n)^{1/(k-1)}}\right) \tag{19.6}$$

where k is the same as in (19.4).

Thus for example if s be an odd positive integer, the number of values of n not exceeding n for which $\sigma_s(n)$ is <u>not</u> divisible by k, where k is any positive integer, is of the form

$$o(n) \tag{19.7}$$

and more accurately is of the form

$$O\left(\frac{n}{(\log n)^{1/(k-1)}}\right). \tag{19.8}$$

For if n be written in the form

$$2^{a_2}\cdot 3^{a_3}\cdot 5^{a_5}\cdot 7^{a_7}\cdots$$

then we have

$$\sigma_s(n)=\prod_p \frac{p^{s(1+a_p)}-1}{p^s-1}, \qquad p=2,3,5,7,11,\dots.$$

Since s is odd, $\sigma_s(n)$ is divisible by k at any rate when $a_p=1$ for all values of p of the form

$$p\equiv -1 \pmod k$$

and hence the results stated follow. Thus we see that, if s is odd, $\sigma_s(n)$ is divisible by any given integer for almost all values of n.

It follows from all these and the formulae in Sections 4, 8, 12, and 17, that

$$\tau(n)\equiv 0 \pmod{2^5\cdot 3^3\cdot 5^2\cdot 7^2\cdot 23\cdot 691} \tag{19.9}$$

for almost all values of n.

It appears that $\tau(n)$ is almost always divisible by any power of 2, 3, and 5. It also appears from Section 9 that there are reasons to suppose that $\tau(n)$ is almost always divisible by 11 also. But I have no evidence at present to say anything about the other powers of 7 and other primes one way or the other.

Among the values of $\tau(n)$ multiples of 2, 3, 5, 7 and 23 are very numerous from the beginning but multiples of 691 begin at a very late stage. For instance $\tau(n)$ is divisible by 23 for 132 values of n not exceeding 200 while the first value of n for which $\tau(n)$ is divisible by 691 is 1381 and this is the only such value of n among the first 5000 values.

II

Moduli 5 and 25

20. In this second part we shall use J_1, J_2, J_3 and G_1, G_2, G_3 to denote functions of q with integral powers of q as well as integral coefficients. These are the same functions in the same section unlike J. We shall also use J in the same sense as in the first part.

We start with Euler's identities

$$(q;q)_\infty=\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} \tag{20.1}$$

and Jacobi's identity

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}. \tag{20.11}$$

It is easy to see from (20.1) that

$$\frac{(q^{1/5};q^{1/5})_\infty}{(q^5;q^5)_\infty} = J_1 - q^{1/5} + q^{2/5}J_2. \tag{20.2}$$

Now cubing both sides we obtain

$$\begin{aligned}\frac{\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/10}}{\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{5n(n+1)/2}} =&(J_1^3 - 3J_2^2q) - q^{1/5}(3J_1^2 - J_2^3q) + 3J_1q^{2/5}(1+J_1J_2)\\ &- q^{3/5}(1+6J_1J_2) + 3J_2q^{4/5}(1+J_1J_2).\end{aligned} \tag{20.3}$$

But it is easy to see that

$$\frac{\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/10}}{\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{5n(n+1)/2}} = G_1 + q^{1/5}G_2 + 5q^{3/5}. \tag{20.31}$$

Hence

$$J_1(1+J_1J_2) = 0, \qquad 1+6J_1J_2 = -5, \qquad J_2(1+J_1J_2) = 0. \tag{20.4}$$

These three equations give one and the same relation between J_1 and J_2, viz.

$$J_1J_2 = -1.$$

Using this we obtain

$$\frac{(q^5;q^5)_\infty}{(q^{1/5};q^{1/5})_\infty} = \frac{1}{J_1 - q^{1/5} + q^{2/5}J_2} \tag{20.5}$$

$$= \frac{(J_1^4 + 3J_2q) + q^{1/5}(J_1^3 + 2J_2^2q) + q^{2/5}(2J_1^2 + J_2^3q) + q^{3/5}(3J_1 + J_2^4q) + 5q^{4/5}}{J_1^5 - 11q + q^2J_2^5}$$

by rationalizing the denominator $J_1 - q^{1/5} + q^{2/5}J_2$. It follows from (20.5) that

$$\sum_{n=0}^{\infty} p(5n+4)q^n \ (q^5;q^5)_\infty = \frac{5}{J_1^5 - 11q + q^2J_2^5}. \tag{20.6}$$

But we see from (20.2) that

$$\frac{(\omega q^{1/5};\omega q^{1/5})_\infty}{(q^5;q^5)_\infty} = J_1 - \omega q^{1/5} + \omega^2q^{2/5}J_2, \tag{20.21}$$

where $\omega^5 = 1$. Now writing the five values of ω in (20.21) and multiplying them together we obtain

$$\frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6} = J_1^5 - 11q + q^2J_2^5. \tag{20.7}$$

It follows from this and (20.6) that

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}. \tag{20.8}$$

It follows that

$$p(5n-1) \equiv 0 \pmod{5}. \tag{20.81}$$

Again the right hand side in (20.8) is of the form

$$5\frac{(q^5;q^5)_\infty^4}{(q;q)_\infty} + 25J.$$

It follows from this and (20.81) that the coefficients of $q^4, q^9, q^{14}, \ldots$ in this are all multiples of 25 and consequently the coefficient of q^{5n-1} in the left hand side of (20.8) is a multiple of 25. In other words

$$p(25n-1) \equiv 0 \pmod{25}. \tag{20.82}$$

It follows also from (20.8) that

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5(q;q)_\infty^{19} + 125J.$$

Modulus 125

21. Changing q to $q^{1/5}$ in (20.8) and arguing as before, using (20.5) and (20.7) we find that

$$\begin{aligned}\sum_{n=0}^{\infty} p(25n+24)q^n =& 5^2\cdot 63\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty^7} + 5^5\cdot 52q\frac{(q^5;q^5)_\infty^{12}}{(q;q)_\infty^{13}} + 5^7\cdot 63q^2\frac{(q^5;q^5)_\infty^{18}}{(q;q)_\infty^{19}}\\ &+ 5^{10}\cdot 6q^3\frac{(q^5;q^5)_\infty^{24}}{(q;q)_\infty^{25}} + 5^{12}q^4\frac{(q^5;q^5)_\infty^{30}}{(q;q)_\infty^{31}}.\end{aligned} \tag{21.1}$$

Now

$$\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty^7} = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}(q^5;q^5)_\infty^4 + 5J \quad \text{etc.} \tag{21.2}$$

and the coefficients of $q^{5n-1}, q^{5n-2}, q^{5n-3}$ in $\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}$ are easily seen to be zero or multiples of 5. It follows that the coefficients of $q^{5n-1}, q^{5n-2}, q^{5n-3}$ in the left hand side of (21.1) are multiples of 125. In other words

$$\begin{cases} p(125n-1) \\ p(125n-26) \equiv 0 \pmod{125} \\ p(125n-51). \end{cases} \tag{21.3}$$

It is also easy to see from (21.1) that

$$\sum_{n=0}^{\infty} p(25n+24)q^n = 75(q;q)_\infty^{23} + 125J. \tag{21.4}$$

The right hand side in (21.4) can be written in the form

$$75\frac{(q;q)_\infty^{48}}{(q^{25};q^{25})_\infty} + 125J. \tag{21.5}$$

But it is easy to show that

$$(Q^3-R^2)^2 = -2\sum_{n=1}^{\infty}(n^3-n)\sigma_1(n)q^n + 5J. \tag{21.6}$$

[To prove (21.6), we need Ramanujan's formula [77, Table III], [82, p. 142]

$$6912\sum_{n=1}^{\infty} n^3\sigma_1(n)q^n = 6P^2Q - 8PR + 3Q^2 - P^4.$$

Using this formula together with (1.4) and (1.2), we can readily prove that

$$2\sum_{n=1}^{\infty}(n^3-n)\sigma_1(n)q^n = -1 + 2P^2 - P^4 + 5J.$$

On the other hand, from (1.2) and (1.3),

$$(Q^3-R^2)^2 = 1 - 2P^2 + P^4 + 5J.$$

The last two equalities yield (21.6).] It follows that

$$\sum_{n=0}^{\infty} p(25n+24)q^{n+2}\ (q^{25};q^{25})_\infty = 25\sum_{n=1}^{\infty}(n^3-n)\sigma_1(n)q^n + 125J. \tag{21.7}$$

In other words

$$\begin{aligned} &p(25n-26) - p(25n-651) - p(25n-1276)\\ &+ p(25n-3151) + \cdots - 25(n^3-n)\sigma_1(n) \equiv 0 \pmod{125}. \end{aligned} \tag{21.8}$$

$p(199)$ is the coefficient of q^7 in (21.2).

$$\begin{aligned} p(199) =& 5^2\cdot 63\cdot 12195 + 5^2\cdot 52\cdot 60541 + 5^7\cdot 63\cdot 66862 + 5^{10}\cdot 6\cdot 29575 + 5^{12}\cdot 6448\\ =& 3646072432125. \end{aligned}$$

Moduli $5^4, 5^5, \ldots$

22. Changing again q to $q^{1/5}$ in (21.1) and arguing as before using (20.5) and (20.7) we can show that

$$\sum_{n=0}^{\infty} p(125n+99)q^n = \sum_{r=1}^{25} a_r \frac{(q^5;q^5)_\infty^{6r-1}}{(q;q)_\infty^{6r}}, \tag{22.1}$$

where the a's are positive integers such that $a_1 = p(99) = 5^3 \cdot 1353839$ and $a_2, a_3, a_4, \ldots$ contain higher powers of 5 than a_1 as factors. It is easy to see from this that

$$\sum_{n=0}^{\infty} p(125n+99)q^n = 4 \cdot 5^3 (q;q)_\infty^{19} + 5^4 J. \tag{22.2}$$

In this way arguing as before, we can show that if λ be any positive odd integer, then

$$\sum_{n=0}^{\infty} p\left(\frac{19 \cdot 5^\lambda + 1}{24} + 5^\lambda n\right) q^n = \sum_{\nu=1}^{5^{\lambda-1}} a_\nu \frac{(q^5;q^5)_\infty^{6\nu-1}}{(q;q)_\infty^{6\nu}}, \tag{22.3}$$

where the a's are positive integers such that $a_2, a_3, a_4, \ldots$ contain higher powers of 5 than a_1 as factors; and if λ be a positive even integer then

$$\sum_{n=0}^{\infty} p\left(\frac{23 \cdot 5^\lambda + 1}{24} + 5^\lambda n\right) q^n = \sum_{\nu=1}^{5^{\lambda-1}} a_\nu \frac{(q^5;q^5)_\infty^{6\nu}}{(q;q)_\infty^{6\nu+1}}, \tag{22.4}$$

where the a's have the same properties as before. We deduce from (22.3) and (22.4) that if λ is a positive odd integer then

$$\sum_{n=0}^{\infty} p\left(\frac{19 \cdot 5^\lambda + 1}{24} + 5^\lambda n\right) q^n = c_\lambda \cdot 5^\lambda (q;q)_\infty^{19} + 5^{\lambda+1} J, \tag{22.5}$$

and if λ is a positive even integer then

$$\sum_{n=0}^{\infty} p\left(\frac{23 \cdot 5^\lambda + 1}{24} + 5^\lambda n\right) q^n = c_\lambda \cdot 5^\lambda (q;q)_\infty^{23} + 5^{\lambda+1} J, \tag{22.6}$$

where c_λ in both cases is a constant.

We easily deduce from these that if λ is an odd integer greater than 1, then

$$\left\{\begin{aligned} &p\left(5^\lambda n - \frac{5^{\lambda-1}-1}{24}\right) \\ &p\left(5^\lambda n - \frac{5^{\lambda+1}-1}{24}\right)^{23} \\ &p\left(5^\lambda n - \frac{49 \cdot 5^{\lambda-1}-1}{24}\right), \end{aligned}\right. \equiv 0 \pmod{5^\lambda} \tag{22.7}$$

and if λ is a positive even integer, then

$$p\left(5^\lambda n - \frac{5^\lambda - 1}{24}\right) \equiv 0 \pmod{5^\lambda}. \tag{22.8}$$

23. We have seen that we can take $c_1 = 1, c_2 = -2, c_3 = 4$ in (22.5) and (22.6). It appears from Section 22 that c_λ may probably be some simple function such as $(-2)^\lambda$. If we calculate a few more values of c_λ, we can definitely know what it is. Then we can make use of the formulae (22.5) and (22.6) to determine completely the residues of

$$p\left(5^\lambda n - \frac{5^{\lambda+1} - 1}{24}\right)$$

for odd values of λ and those of

$$p\left(5^\lambda n - \frac{5^\lambda - 1}{24}\right)$$

for even values of λ for modulus $5^{\lambda+1}$. [To determine these residues, we need the expansions

$$\begin{aligned}(q;q)_\infty^{19} =& 1 - 19q + 152q^2 - 627q^3 + 1140q^4 + 988q^5 - 9063q^6 \\ &+ 14212q^7 + 7410q^8 - 44270q^9 + 22781q^{10} + 38114q^{11} \\ &+ 36176q^{12} - 137256q^{13} - 154850q^{14} + 480605q^{15} + \cdots\end{aligned}$$

and

$$\begin{aligned}(q;q)_\infty^{23} =& 1 - 23q + 230q^2 - 1265q^3 + 3795q^4 - 3519q^5 - 16445q^6 \\ &+ 64285q^7 - 64515q^8 - 120175q^9 + 354706q^{10} - 123763q^{11} \\ &- 407560q^{12} - 48530q^{13} + 817190q^{14} + 1464341q^{15} + \cdots\end{aligned}$$

in, respectively, (22.5) and (22.6).] Thus for instance if follows immediately from (22.5) and (22.6) that if λ is an odd integer then

$$p\left(5^\lambda - \frac{5^{\lambda+1} - 1}{24}\right) - 5^\lambda c_\lambda, \qquad p\left(2 \cdot 5^\lambda - \frac{5^{\lambda+1} - 1}{24}\right) - 5^\lambda c_\lambda,$$

$$p\left(3 \cdot 5^\lambda - \frac{5^{\lambda+1} - 1}{24}\right) - 2 \cdot 5^\lambda c_\lambda, \qquad p\left(4 \cdot 5^\lambda - \frac{5^{\lambda+1} - 1}{24}\right) + 2 \cdot 5^\lambda c_\lambda,$$

$$p\left(5 \cdot 5^\lambda - \frac{5^{\lambda+1} - 1}{24}\right), \qquad p\left(6 \cdot 5^\lambda - \frac{5^{\lambda+1} - 1}{24}\right) - 2 \cdot 5^\lambda c_\lambda,$$

$$p\left(7 \cdot 5^\lambda - \frac{5^{\lambda+1} - 1}{24}\right) - 2 \cdot 5^\lambda c_\lambda, \qquad p\left(8 \cdot 5^\lambda - \frac{5^{\lambda+1} - 1}{24}\right) - 2 \cdot 5^\lambda c_\lambda,$$

$$p\left(9 \cdot 5^\lambda - \frac{5^{\lambda+1} - 1}{24}\right), \qquad p\left(10 \cdot 5^\lambda - \frac{5^{\lambda+1} - 1}{24}\right),$$

$$p\left(11 \cdot 5^\lambda - \frac{5^{\lambda+1} - 1}{24}\right) - 5^\lambda c_\lambda, \qquad p\left(12 \cdot 5^\lambda - \frac{5^{\lambda+1} - 1}{24}\right) + 5^\lambda c_\lambda,$$

$$p\left(13 \cdot 5^\lambda - \frac{5^{\lambda+1} - 1}{24}\right) - 5^\lambda c_\lambda, \qquad p\left(14 \cdot 5^\lambda - \frac{5^{\lambda+1} - 1}{24}\right) + 5^\lambda c_\lambda,$$

$$p\left(15 \cdot 5^\lambda - \frac{5^{\lambda+1} - 1}{24}\right), \qquad p\left(16 \cdot 5^\lambda - \frac{5^{\lambda+1} - 1}{24}\right),$$

[23] λ may also be 1 in this formula.

and so on are all multiples of $5^{\lambda+1}$; and if λ is an even integer, then

$$
\begin{aligned}
&p\left(5^\lambda - \frac{5^\lambda-1}{24}\right) - 5^\lambda c_\lambda, && p\left(2\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right) - 2\cdot 5^\lambda c_\lambda,\\
&p\left(3\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right), && p\left(4\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right),\\
&p\left(5\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right), && p\left(6\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right) - 5^\lambda c_\lambda,\\
&p\left(7\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right), && p\left(8\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right),\\
&p\left(9\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right), && p\left(10\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right),\\
&p\left(11\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right) - 5^\lambda c_\lambda, && p\left(12\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right) - 2\cdot 5^\lambda c_\lambda,\\
&p\left(13\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right), && p\left(14\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right),\\
&p\left(15\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right), && p\left(16\cdot 5^\lambda - \frac{5^\lambda-1}{24}\right) - 5^\lambda c_\lambda,
\end{aligned}
$$

and so on are all multiples of $5^{\lambda+1}$.

Moduli 7 and 49

24. It is easy to see from (20.1) that

$$\frac{(q^{1/7};q^{1/7})_\infty}{(q^7;q^7)_\infty} = J_1 + q^{1/7}J_2 - q^{2/7} + q^{5/7}J_3. \tag{24.1}$$

Now cubing both sides we obtain

$$
\begin{aligned}
&\frac{\sum_{n=0}^\infty (-1)^n(2n+1)q^{n(n+1)/14}}{\sum_{n=0}^\infty (-1)^n(2n+1)q^{7n(n+1)/2}}\\
=&(J_1^3 + 3J_2^2J_3q - 6J_1J_3q) + q^{1/7}(3J_1^2J_2 - 6J_2J_3q + J_3^2q^2) + 3q^{2/7}(J_1J_2^2 - J_1^2 + J_3q)\\
&+ q^{3/7}(J_2^3 - 6J_1J_2 + 3J_1J_3^2q) + 3q^{4/7}(J_1 - J_2^2 + J_2J_3^2q)\\
&+ 3q^{5/7}(J_2 + J_1^2J_3 - J_3^2q) + q^{6/7}(6J_1J_2J_3 - 1).
\end{aligned}
$$

But it is easy to see that

$$\frac{\sum_{n=0}^\infty (-1)^n(2n+1)q^{n(n+1)/14}}{\sum_{n=0}^\infty (-1)^n(2n+1)q^{7n(n+1)/2}} = G_1 + q^{1/7}G_2 + q^{3/7}G_3 - 7q^{6/7}.$$

Hence

$$
\begin{cases}
J_1J_2^2 - J_1^2 + J_3q &= 0,\\
J_1 - J_2^2 + J_2J_3^2q &= 0,\\
J_2 + J_1^2J_3 - J_3^2q &= 0,\\
6J_1J_2J_3 - 1 &= -7.
\end{cases} \tag{24.2}
$$

All these four equations give the two independent relations

(24.2a) $$J_1J_2J_3 = -1, \qquad \frac{J_1^2}{J_3} + \frac{J_2}{J_3^2} = q.$$

Now write (24.1) in the form

(24.3) $$\frac{(\omega q^{1/7}; \omega q^{1/7})_\infty}{(q^7;q^7)_\infty} = J_1 + \omega q^{1/7} J_2 - \omega^2 q^{2/7} + \omega^5 q^{5/7} J_3,$$

where $\omega^7 = 1$. Again writing the seven values of ω in (24.3) and multiplying them together and using (24.2a) we can show that

(24.4) $$J_1^7 + J_2^7 q + J_3^7 q^5 = \frac{(q;q)_\infty^8}{(q^7;q^7)_\infty^8} + 14q\frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4} + 57q^3,$$

(24.5) $$J_1^3 J_2 + J_2^3 J_3 q + J_3^3 J_1 q^2 = -\frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4} - 8q,$$

(24.6)
$$J_1^2 J_2^3 + J_3^2 J_1^3 q + J_2^2 J_3^3 q^2 = -\frac{(q;q)_\infty^4}{(q^7;q^7)_\infty^4} - 5q.$$

Again taking the reciprocals of both sides in (24.1) and rationalizing the denominator by using as in Section 20, we can show that

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}.$$

$$7^2 \cdot 2546, 7^4 \cdot 48 \cdot 934, 7^5 \cdot 1418989, 7^8 \cdot 335400.$$

$$\{p(47)q^3 + \cdots\}(q^{49};q^{49})_\infty = 7\sum_{n=1}^{\infty}\left\{22n^4\sigma_0(n) - 21n^2\sigma_1(n) - \tau(n)\right\}q^n + 7^3 J.$$

COMMENTARY

0. The designation, Section 0, for the first batch of Ramanujan's insertions is due to the present authors.

K. G. Ramanathan [75] also observed that $\tau(n)$ is even unless n is an odd square.

The congruences $\tau(7n-r) \equiv 0 \pmod 7, r = 0, 1, 2, 4$, were evidently first proved by J. R. Wilton [107]. G. H. Hardy, in his book *Ramanujan* [47, pp. 165–166] also gives a proof, as does Ramanathan [76].

The congruences $\tau(23n-r) \equiv 0 \pmod{23}$, where r is a quadratic residue modulo 23, were also first established by Wilton [107].

1. Without the insertions, the beginning of the paper actually begins with the definitions of the Eisenstein series P, Q, and R, which are denoted by L, M, and N, respectively, in Ramanujan's notebooks [81]. Since the remainder of this section was extracted for [80] with additional details supplied by Hardy, we have not added more details here. However, it seems appropriate here to provide an introduction

to congruences for the partition function in arithmetic progressions, since a large portion of the manuscript focuses on this topic.

In this manuscript Ramanujan proves his well known congruences for $p(n)$, namely,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

These congruences are the first cases of the infinite families,

$$\begin{aligned} p(5^k n+\delta_{5,k}) &\equiv 0 \pmod{5^k}, \\ p(7^k n+\delta_{7,k}) &\equiv 0 \pmod{7^{[k/2]+1}}, \\ p(11^k n+\delta_{11,k}) &\equiv 0 \pmod{11^k}, \end{aligned} \tag{C1.1}$$

where $\delta_{p,k} := 1/24 \pmod{p^k}$. The literature on these congruences is extensive, and there are now many proofs and approaches to them, e.g., [3], [5], [30], [38], [39], [40], [41], [42], [49], [50], [51], [53], [64], [68], [73], and [104].

These congruences are indeed surprising for they appear to be examples of a very rare and isolated phenomenon. In fact, Ramanujan [79], [82, p. 230] remarked that "It appears that there are no equally simple properties for any moduli involving primes other than these three."

In view of Ramanujan's claim, it is natural to ask about the frequency of congruences for $p(n)$ and the possibility of finding new ones. In this direction, the second author has made some progress [67], [69] towards quantifying the rarity of such congruences, and A. O. L. Atkin and J. N. O'Brien [6], [7] have found other congruences for $p(n)$. For instance, Atkin has proved that

$$p(17303n+237) \equiv 0 \pmod{13}.$$

It is reasonable to conclude that such congruences are quite rare, but not so rare that one cannot find infinitely many such congruences.

2. The congruence $\tau(n) \equiv n\sigma(n) \pmod{5}$ was established by Wilton [106], and is also proved in Hardy's book [47, pp. 166–167]. This congruence was generalized by R. P. Bambah and S. Chowla [15], who proved that, if n is not a multiple of 5, then

$$\tau(n) \equiv 5n^2\sigma_7(n) - 4n\sigma_9(n) \pmod{5^3}.$$

The asymptotic formula (2.7) can be proved by using the method devised by E. Landau in his book [60, Sect. 183] to determine an asymptotic formula for the number of integers $\leq x$ that can be represented as a sum of two squares. Alternatively, one can appeal to a general Tauberian theorem, such as that proved by H. Delange [35]. However, as first pointed out by G. K. Stanley [98], the claim (2.8) is false. Indeed, by using the ideas of Landau [60, Sects. 176–183], one can establish an asymptotic formula of the shape

$$\sum_{n\leq x} t_n = C\frac{x}{(\log x)^{1/4}}\left(1+\sum_{n=1}^{r-1}\frac{c_n}{(\log x)^n}+O\left(\frac{1}{(\log x)^r}\right)\right),$$

for certain constants $c_n, 1 \leq n \leq r-1$. However, generally, these constants are not equal to those which would be obtained by successive integrations by parts in (2.8). Ramanujan made a similar error in his first letter to Hardy [82, p. xxiv], [25, p. 24] when he claimed that the number of integers $\leq x$ that can be represented as a sum of two squares is asymptotic to a constant times a similar integral. See either the sections of Landau's book cited above or Hardy's book [47, pp. 60–63]. In Sections 6, 11, and 17, Ramanujan records similar asymptotic formulas, and, in contrast to the asymptotic formula in this section, calculates the leading coefficients in each case. R. A. Rankin [87] has verified that the leading terms, including the coefficients, are correct in each of the instances cited by Ramanujan.

4. The congruence (4.2) was first proved in print by Wilton [106] and later by Bambah [10].

Rankin [85, p. 5] has pointed out that Ramanujan's conjecture (4.3) is false for $k \geq 4$. Observe that 443 is prime and that its powers are congruent to $\pm 1, \pm 443$ (mod 5^4). From Watson's [105] table of values for $\tau(n)$, $\tau(443) \equiv -58 \pmod{5^4}$. Hence, no integers a and b exist for which (4.3) holds with $n = 443$ and $k \geq 4$.

However, the congruence (4.4) is true; a congruence equivalent to (4.4) was first proved by J.–P. Serre [91], [101].

The equality below (4.4) is a special instance of the relation

$$\tau(p^{n+1}) = \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1}), \qquad n > 1, \tag{C4.1}$$

where p is a prime, which along with (7.6), were first proved by L. J. Mordell [62], after Ramanujan had made these conjectures in his paper [77, Sect. 18], [82, p. 153].

Proofs of either of the famous equalities (4.5) or (4.6) (or both) have been given by, in chronological order, Ramanujan [78], [82, pp. 210–213], H. B. C. Darling [34], L. J. Mordell [62], H. Rademacher and H. S. Zuckerman [73], [72, pp. 186–202], S. D. Chowla [31], D. Kruswijk [58], W. N. Bailey [8], [9], J. M. Dobbie [37], N. J. Fine [39], S. Raghavan [74], H. H. Chan [30], and M. D. Hirschhorn [49], [50]. These proofs are quite varied. Some authors use q–series; some, such as Rademacher, Zuckerman, and Raghavan, use the the theory of modular forms; Chan's proof uses a variant of one of Ramanujan's trigonometric series identities in [77].

As indicated by Ramanujan, (4.6) is a companion to (4.5). Bailey, Chan, Darling, Mordell, and Raghavan in the aforementioned papers have also given proofs of (4.6). In contrast to (4.5), equality (4.6) can be found in Ramanujan's notebooks [21, p. 257, Entry 9(i)].

5. Since this section was also extracted by Hardy for [80], we have not added details here.

6. The congruence (6.2) was established by Ramanathan [76], Gupta [46], and Bambah [11].

The comments made in Section 2 about Ramanujan's asymptotic formulas have analogues here. Although the asymptotic formula (6.6a) is correct, Ramanujan's stronger claim (6.7) is false, since the constants obtained by integrating by parts in (6.7) do not generally match those obtained in a proper asymptotic expansion of $\sum_{k=1}^{n} t_k$.

7. The content of this section can be found with more detail in Rushforth's paper [89]. (In the second equality of (7.1), Rushforth [89, eq. (7.3)] wrote $-2R^2$ for $2R^2$,

and in (7.4) [89, penultimate equality on p. 407] Rushforth wrote $-2n\sigma_3(n)$ for $2n\sigma_3(n)$.)

8. Ramanujan's proof of (8.1) can be found in his paper [78], [82], while other proofs of (8.1) have been given by Mordell [62], Rademacher and Zuckerman [73], N. J. Fine [39], O. Kolberg [55], Raghavan [74], and Chan [30]. Further identities akin to (8.1) and (8.2) have been established by Rademacher [71], [72, pp. 252–279]. These authors then continue to prove (8.3).

Equality (8.4) is true, and its truth is equivalent to the assertion that $\eta^3(z)\eta^3(7z)$ is a Hecke eigenform with complex multiplication in $S_3(\Gamma_0(7), \chi_{-7})$, where $S_k(\Gamma_0(N), \chi)$ denotes the complex vector space of cusp forms of weight k with respect to the congruence subgroup $\Gamma_0(N)$ with Nebentypus character χ [88]. (The notation $S_k(\Gamma_0(N))$, with χ absent, simply means that the character χ is trivial.) Here the character χ_{-7} denotes the usual Kronecker character for the field $\mathbb{Q}(\sqrt{-7})$. That this form is an eigenform follows immediately from the fact that this space is one dimensional [33]. To deduce (8.4) in a more elementary fashion, first notice that Jacobi's identity

$$q\prod_{n=1}^{\infty}(1-q^{8n})^3 = \eta^3(8z) = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{(2n+1)^2}$$

implies that

$$\eta^3(8z)\eta^3(56z) = \sum_{x,y\geq 0}(-1)^{x+y}(2x+1)(2y+1)q^{(2x+1)^2+7(2y+1)^2}.$$

Proving Ramanujan's claim now follows after a straightforward computation.

The claims regarding the Euler product expansions Π_1 and Π_2 follow easily from the theory of complex multiplication.

In regard to the congruence (8.6), we remark that O. Kolberg [57] proved the beautiful congruence

$$\tau(n) \equiv n\sigma_9(n) \pmod{49}, \qquad \text{if } \left(\frac{n}{7}\right) = -1.$$

9. This proof is given in more detail in [80].

10. Equality (10.4) is true, and its truth is equivalent to the assertion that $\eta^2(z)\eta^2(11z)$ is an eigenform of the Hecke operators acting on $S_2(\Gamma_0(11))$. That this form is an eigenform follows immediately from the fact that this space is one dimensional.

Some of Ramanujan's congruences for $\tau(n)$ are immediate consequences of its multiplicative properties. For instance, Ramanujan [77], [82, p. 153, eq. (103)] conjectured and Mordell [62] proved that, if m and n are relatively prime integers, then

$$\tau(mn) = \tau(m)\tau(n). \tag{C10.1}$$

For example, the congruences (10.8) and (10.9) follow easily since $\tau(19) = 10661420 \equiv 0 \pmod{11}$ and $\tau(29) = 128406630 \equiv 0 \pmod{11}$. Other congruences follow from (C4.1) or from (C4.1) and (C10.1) together.

In 1969, P. Deligne [36] proved Serre's conjecture [92] on the existence of ℓ-adic Galois representations ρ_ℓ attached to modular forms on $\Gamma_0(N)$. Then, in 1972, Swinnerton–Dyer [99] determined the possible images of $\tilde{\rho}_\ell$, the reduction $\mod \ell$ of ρ_ℓ, and showed that 'small' images imply certain congruences for the coefficients of modular forms.

The existence of these representations and their study has been at the forefront of arithmetic geometry ever since Serre formulated his original conjectures. Every congruence for $\tau(n)$ involving divisor functions, and the congruence

$$\tau(n) \equiv 0 \pmod{23}$$

for $\left(\frac{n}{23}\right) = -1$, follows from this theory. For more details, readers should consult [36], [92], [99], [100], [101], [102].

11. Ramanujan's speculation that $p(n)$ is odd more often than it is even is not substantiated by more extensive calculations. Indeed, it is a long outstanding conjecture that asymptotically $p(n)$ is equally often even and odd. In Sections 1, 5, 9, and 11, based on a table of values for $p(n), 1 \le n \le 200$, computed by P. A. MacMahon, Ramanujan offers conjectures on the distribution of $p(n)$ modulo 5, 7, 11, and 3, respectively. We examine these conjectures in detail.

If $D(r, M)$ denotes the proportion of integers n for which $p(n) \equiv r \pmod{M}$ (assuming that such densities exist), Ramanujan conjectured (in Sections 11, 11, 1, 5, and 9, respectively) that

$$D(0,2) < D(1,2),$$

$$D(i,3) = \tfrac{1}{3}, \qquad \text{for } 0 \le i \le 2,$$

$$D(i,5) = \begin{cases} \frac{1}{3}, & \text{if } i = 0, \\ \frac{1}{6}, & \text{if } 1 \le i \le 4, \end{cases}$$

$$D(i,7) = \begin{cases} \frac{1}{4}, & \text{if } i = 0, \\ \frac{1}{8}, & \text{if } 1 \le i \le 6, \end{cases}$$

$$D(i,11) = \begin{cases} \frac{1}{6}, & \text{if } i = 0, \\ \frac{1}{12}, & \text{if } 1 \le i \le 10. \end{cases}$$

¿From elementary considerations, we show that Ramanujan's conjectures for $D(i, M), M = 5, 7, 11$, are unreasonable. Remove the values $n = 5k+4, 7k+5, 11k+6$, from consideration when $M = 5, 7, 11$, respectively. Assuming that the remaining values of $p(n)$ are distributed randomly among the M residue classes in each of these three cases, we would expect that

$$D(i,M) = \begin{cases} \frac{2M-1}{M^2}, & \text{if } i = 0, \\ \frac{M-1}{M^2}, & \text{if } 1 \le i \le M. \end{cases}$$

In particular, we expect that $D(0,5) = \frac{9}{25}$ and $D(i,5) = \frac{4}{25}, 1 \le i \le 4$, in contrast to Ramanujan's conjectures. Similar discrepancies exist for $M = 7, 11$.

Let $\delta(r, M)$ denote the proportion of integers $n \leq 100000$ for which $p(n) \equiv r \pmod{M}$. Here are some values of $\delta(r, M)$ for $M \in \{2, 3, 5, 7, 11, 13\}$.

r	$\delta(r,2)$	$\delta(r,3)$	$\delta(r,5)$	$\delta(r,7)$	$\delta(r,11)$	$\delta(r,13)$
0	0.498	0.333	0.362	0.272	0.174	0.080
1	0.502	0.332	0.158	0.121	0.083	0.078
2	*	0.334	0.161	0.122	0.083	0.076
3	*	*	0.160	0.122	0.082	0.077
4	*	*	0.158	0.122	0.084	0.077
5	*	*	*	0.120	0.083	0.076
6	*	*	*	0.120	0.083	0.075
7	*	*	*	*	0.081	0.077
8	*	*	*	*	0.082	0.076
9	*	*	*	*	0.081	0.078
10	*	*	*	*	0.082	0.075
11	*	*	*	*	*	0.076
12	*	*	*	*	*	0.077

As this data suggests, if the densities $\delta(r, M)$ are well defined, then Ramanujan's conjectures are mostly incorrect. The data suggests that he may be correct when $M = 3$, but not for any other values. At present, very little is known about the densities $\delta(r, M)$ apart from lower bounds for $\delta(0, M)$ for those M possessing congruences of the sort discussed in the commentary for Section 1. At present, by the work of S. Ahlgren [1], [2], and J.–L. Nicolas, I. Z. Ruzsa, A. Sárközy, and J.–P. Serre [66], it is known that

$$\begin{aligned}
\#\{n \leq X \ : \ p(n) \equiv 0 \pmod{2}\} &\gg \sqrt{X}, \\
\#\{n \leq X \ : \ p(n) \equiv 1 \pmod{2}\} &\gg \frac{\sqrt{X}}{\log X}, \\
\#\{n \leq X \ : \ p(n) \not\equiv 0 \pmod{M}\} &\gg \frac{\sqrt{X}}{\log X}.
\end{aligned}$$

On the other hand, the second author [70] has shown that if $M \geq 5$ is prime, then

$$\#\{n \leq X \ : \ p(n) \equiv 0 \pmod{M}\} \gg_M X.$$

In a similar direction, M. Newman [65] conjectured that every positive integer M has the property that each residue class $m \pmod{M}$ has infinitely many integers n for which $p(n) \equiv m \pmod{M}$. A. O. L. Atkin [6], O. Kolberg [56], and M. Newman [65] have verified this conjecture for each $M \in \{2, 5, 7, 13\}$. Because of the validity of (14.3) and (14.4), Ramanujan had also proved this conjecture when $M = 13$. Motivated by Ramanujan's work, the second author [70] has proved Newman's conjecture for every prime $M < 1000$, with the exception of $M = 3$. He also has found a simple criterion for verifying Newman's conjecture for any prime $M \geq 5$.

The equality (11.4) can be found in a fragment published with Ramanujan's lost notebook [83, p. 354, eq. (1.42)]. A proof may be found in Berndt's paper [23, Entry 21].

The congruence (11.8) has been proved several times in the literature. Most frequently, it is given in the equivalent formulation

$$\tau(n) \equiv \begin{cases} \sigma(n) \pmod 3, & \text{if } (3,n)=1, \\ 0 \pmod 3, & \text{if } 3|n. \end{cases}$$

For proofs, see papers by D. P. Banerji [19], Bambah and Chowla [12], Gupta [45], and Bambah, Chowla, Gupta, and Lahiri [18]. Bambah and Chowla [15] proved the generalization

$$\tau(n) \equiv (n^2+k)\sigma_7(n) \pmod{3^4}, \qquad (3,n)=1,$$

where $k=0$, if $n \equiv 1 \pmod 3$, and $k=9$, if $n \equiv 2 \pmod 3$.

The asymptotic formulas in (11.8a) need to be corrected in the same manner that the asymptotic formulas in Sections 2 and 6 needed to be recast.

12. The sums $\sum_{n=1}^{\infty} n^a q^n$, where a is a positive integer, can be explicitly evaluated in terms of Eulerian polynomials [20, p. 113, Entry 4].

Bambah, Chowla, and Gupta [17] and Bambah, Chowla, Gupta, and Lahiri [18] proved the congruence

$$\tau(n) \equiv \sigma(n) \pmod 8, \qquad \text{if } n \text{ is odd},$$

which, in fact, is implied by the first congruence in (12.1).

We have been unable to find the identity (12.2) in the literature prior to the work of Ramanujan. On page 257 in his second notebook [81], Ramanujan actually offers a general formula for

$$S_{2r} := \sum_{n=1}^{\infty} \frac{n^{2r} q^n}{1+q^n+q^{2n}},$$

which was first proved by Berndt, S. Bhargava, and F. G. Garvan [24], [22, p. 143]. The values of $S_{2r}, 1 \le r \le 4$, are explicitly given by Ramanujan. The formula for S_2 is given without proof in an equivalent form in a paper by J. M. and P. B. Borwein [28], and this equivalent formula is proved in [29] by the Borweins and Garvan. A particularly simple proof of (12.2), based on an identity of N. J. Fine, has been given by S. H. Son [97, Lemma 2.6].

A proof of the first congruence in (12.3) was given by Bambah and Chowla [14]. The second congruence in (12.3) was established in another paper by the same authors [15].

Bambah and Chowla [14] proved the second congruence in (12.4).

The first proof in print of (12.7) was evidently given by J. R. Wilton [106]. Later proofs were found by G. N. Watson [103] and D. H. Lehmer [61].

As with corresponding results in Sections 2, 6, and 11, the asymptotic formula (12.7a) needs to be corrected.

Bambah and Chowla [16] gave the first published proof of (12.9).

The congruence below (12.9) is false, in general. For example, it is false for $n = 1, 3, 4, 5$.

13. In Sections 2, 6, 10, 11, and 13 Ramanujan considers the t-regular partition functions $\lambda(n)$ whose generating function are given by

$$\sum_{n=0}^{\infty} \lambda(n)q^n = \sum_{n=0}^{\infty} b_t(n)q^n := \prod_{n=1}^{\infty} \frac{(1-q^{tn})}{(1-q^n)}.$$

The dependence of λ on t is always clear from the context. For instance, in Section 2, he considers the case where $t = 25$. In this case he shows that $\lambda(n)$ is almost always a multiple of 5. A recent paper by B. Gordon and the second author [43] makes considerable progress in describing this phenomenon for all t. Let $p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ be the prime factorization of t. By [43, Thm. 1], if p_i is a prime for which $p_i^{a_i} \geq \sqrt{t}$, then, for every positive integer k, almost every integer n has the property that $b_t(n)$ is a multiple of p_i^k. This theorem immediately implies all of Ramanujan's claims of this sort for the functions $\lambda(n)$.

Equality (13.6) has been proved by H. S. Zuckerman [108] and W. H. Simons [96].

14. The claims (14.1)–(14-6) are among the most fascinating results in the unpublished manuscript. For example, these results indicate that

$$\sum_{n=0}^{\infty} p(13n+6)q^{24n+11} \equiv 11\eta^{11}(24z) \pmod{13}.$$

M. Newman [64] has proved some of these claims. However, the second author [70] has shown that this phenomenon also holds with respect to other moduli. In particular, if $m \geq 5$ is prime and k is a positive integer, then

$$\sum_{n=0}^{\infty} p\left(\frac{m^k n+1}{24}\right) q^n$$

is the reduction modulo m of a holomorphic cusp form of weight $\frac{m^2-m-1}{2}$. This implies that results like (14.1)–(14.6) exist for every prime $m \geq 5$, not just $m = 13$. Moreover, using the theory of Hecke operators of half-integral weight, the Shimura correspondence, and the theory of Galois representations, the second author [70] has proved that for every prime $m \geq 5$ that there are integers $0 \leq b < a$ for which

$$p(an+b) \equiv 0 \pmod{m}$$

for every non-negative integer n.

15. In Section 15 Ramanujan gives a brief description of the method he employs to obtain generating functions of the type

$$\sum p(\varpi n + b_{\varpi})q^n \pmod{\varpi},$$

where $\varpi > 3$ is prime. Let B_k denote the kth Bernoulli number in contemporary notation. Note that Ramanujan's convention for Bernoulli numbers is different from the contemporary one in which the Bernoulli numbers B_k are defined by

$$\text{(C15.1)} \qquad \frac{x}{e^x-1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k, \qquad |x| < 2\pi.$$

For every positive even integer k, define positive coprime integers v_k and δ_k by

$$\frac{v_k}{\delta_k} := |B_k|.$$

If $k \geq 4$ is an even integer, let

$$E_{2k}(z) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

denote the normalized Eisenstein series of weight $2k$ with respect to the full modular group $\Gamma_0(1)$, where B_k is defined by (C15.1).

Now assume that $\varpi > 3$ is prime. It is easy to check that the left hand side of (15.1) is $v_{\varpi-1} \cdot E_{\varpi-1}(z)$ and that the left hand side of (15.2) represents $v_{\varpi+1} \cdot E_{\varpi+1}(z)$. That both of these q-series have integer coefficients is obvious, and that they have the desired representation as sums of $Q^{\ell} R^m$ is well known [86, p. 199, Thm. 6.1.3]. In fact, every holomorphic modular form with respect to the full modular group has such a representation.

As Ramanujan claims, (15.3) and (15.31) are easy deductions from Fermat's Little Theorem and the von Staudt and Claussen theorem. However, the claim that $k = 0$ in (15.31) is not entirely clear. In fact, this is one of the questions in the manuscript that Ramanujan admits still requires proof.

Proposition. *Ramanujan's claim that* $k \equiv 0 \pmod{\varpi}$ *in* (15.31) *is true.*

Proof. A simple calculation verifies Ramanujan's assertion that the truth of (15.4) implies that k is indeed 0, or more precisely $0 \pmod{\varpi}$. In particular, it suffices to prove that

$$\text{(C15.2)} \qquad 12 v_{\varpi+1} + (-1)^{\frac{\varpi+1}{2}} \delta_{\varpi+1} \equiv 0 \pmod{\varpi}.$$

Using the well known Voronoi congruences [52, p. 237, Prop. 15.2.3], we find, for every integer a coprime to ϖ, that

$$(a^2 - 1) v_{\varpi+1} \equiv a \cdot (-1)^{\frac{\varpi-1}{2}} \delta_{\varpi+1} \sum_{j=1}^{\varpi-1} j \left[\frac{ja}{\varpi}\right] \pmod{\varpi},$$

since the sign of B_{2k} is $(-1)^{k+1}$ for every positive integer k. Therefore we find that

$$\text{(C15.3)} \qquad \left(\frac{a^2-1}{a}\right) v_{\varpi+1} + (-1)^{\frac{\varpi+1}{2}} \delta_{\varpi+1} \sum_{j=1}^{\varpi-1} j \left[\frac{ja}{\varpi}\right] \equiv 0 \pmod{\varpi}.$$

In view of (C15.2) and (C15.3), it suffices to prove that, for each integer a coprime to ϖ,

$$\text{(C15.4)} \qquad \sum_{j=1}^{\varpi-1} j \left[\frac{ja}{\varpi}\right] \equiv \frac{a^2-1}{12a} \pmod{\varpi}.$$

We now prove (C15.4) by examining Dedekind sums. If k is a positive integer, and h is coprime to k, then the Dedekind sum $s(h,k)$ is defined by

$$s(h,k) := \sum_{j=1}^{k-1} \frac{j}{k}\left(\frac{hj}{k} - \left[\frac{hj}{k}\right] - \frac{1}{2}\right).$$

It is easy to verify that

$$12a\varpi s(a,\varpi) = \frac{12a^2}{\varpi}\cdot\frac{\varpi(\varpi-1)(2\varpi-1)}{6} - 12a\sum_{j=1}^{\varpi-1} j\left[\frac{ja}{\varpi}\right] - 6a\cdot\frac{\varpi(\varpi-1)}{2}.$$

However, by [4, p. 64, Th. 3.8], it is known that

$$12a\varpi s(a,\varpi) \equiv a^2+1 \pmod{\varpi},$$

and so we find that

$$a^2+1 \equiv 2a^2 - 12a\sum_{j=1}^{\varpi-1} j\left[\frac{ja}{\varpi}\right] \pmod{\varpi}.$$

This is (C15.4), and this completes the proof of Ramanujan's claim.

The remainder of Section 15 is straightforward and follows from Ramanujan's collection of formulas involving the operator $q\frac{d}{dq}$.

16. The equalities (16.3) and (16.7) are proved in Rushforth's paper [89]. As with the key results in Section 14, the claims (16.1)–(16.3) and (16.5)–(16.9) follow from the work of Ono [70].

In (16.4), Ramanujan claims that the Dirichlet series

$$\sum_{n=1}^{\infty}\frac{\tau_2(n)}{n^s},\quad \sum_{n=1}^{\infty}\frac{\tau_3(n)}{n^s},$$
$$\sum_{n=1}^{\infty}\frac{\tau_4(n)}{n^s},\quad \sum_{n=1}^{\infty}\frac{\tau_5(n)}{n^s},\quad \sum_{n=1}^{\infty}\frac{\tau_7(n)}{n^s}$$

have Euler products. This is easily verified since all corresponding modular forms are eigenforms of Hecke operators, for they each lie in a one dimensional space of cusp forms [33].

At the end of Section 16, Ramanujan claims that the two Dirichlet series

$$\sum_{n=1}^{\infty}\frac{\Omega_2(n)}{n^s} \quad\text{and}\quad \sum_{n=1}^{\infty}\frac{\Omega_3(n)}{n^s}$$

are both differences of two series with Euler products. In terms of classical modular forms, $\sum_{n=1}^{\infty}\Omega_2(n)q^n \in S_{28}(\Gamma_0(1))$ and $\sum_{n=1}^{\infty}\Omega_3(n)q^n \in S_{30}(\Gamma_0(1))$. Both of these two spaces are two dimensional [33], and one can easily check that

$$S_{28}(\Gamma_0(1)) = \mathbb{C}Q\Delta^2 \oplus \mathbb{C}Q^4\Delta,$$
$$S_{30}(\Gamma_0(1)) = \mathbb{C}R\Delta^2 \oplus \mathbb{C}R^3\Delta,$$

where $\Delta := \Delta(q) = q(q;q)_\infty^{24}$. It is easy to show that the space $S_{28}(\Gamma_0(1))$ is spanned by the eigenforms

$$f_1 := (-5076 + 108\sqrt{18209})Q\Delta^2 + Q^4\Delta,$$
$$f_2 := (-5076 - 108\sqrt{18209})Q\Delta^2 + Q^4\Delta.$$

(For calculations of this sort, see N. Koblitz's text [54, p. 173, Prop. 51].) Since $\sum_{n=1}^\infty a_1(n)q^n$ and $\sum_{n=1}^\infty a_2(n)q^n$ are eigenforms, the two Dirichlet series

$$\sum_{n=1}^\infty \frac{a_1(n)}{n^s} \quad \text{and} \quad \sum_{n=1}^\infty \frac{a_2(n)}{n^s}$$

have Euler products as in (16.4) with weight 28. The "difference" to which Ramanujan alludes is the identity

$$Q\Delta^2 = \sum_{n=1}^\infty \Omega_2(n)q^n = \frac{1}{216\sqrt{18209}}(f_1 - f_2).$$

Similarly, one can easily verify that the space $S_{30}(\Gamma_0(1))$ is spanned by the eigenforms

$$g_1 := (5856 + 2208\sqrt{-83})R\Delta^2 + R^3\Delta,$$
$$g_2 := (5856 - 2208\sqrt{-83})R\Delta^2 + R^3\Delta.$$

Since $\sum_{n=1}^\infty b_1(n)q^n$ and $\sum_{n=1}^\infty b_2(n)q^n$ are eigenforms, it easily follows that the two Dirichlet series

$$\sum_{n=1}^\infty \frac{b_1(n)}{n^s} \quad \text{and} \quad \sum_{n=1}^\infty \frac{b_2(n)}{n^s}$$

have Euler products as in (16.4) with weight 30. The difference to which Ramanujan alludes is the identity

$$R\Delta^2 = \sum_{n=1}^\infty \Omega_3(n)q^n = \frac{1}{4416\sqrt{-83}}(g_1 - g_2).$$

17. The claim (17.2) is equivalent to the assertion that $\eta(z)\eta(23z)$ is an eigenform with complex multiplication in the space $S_1(\Gamma_0(23), \chi_{-23})$ [44], [88], [33]. Here χ_{-23} is the usual Kronecker character for the quadratic field $\mathbb{Q}(\sqrt{-23})$. Although the claims regarding the Euler products Π_1, Π_2 and Π_3 follow easily from the theory of complex multiplication, one can more easily obtain them from Euler's pentagonal number theorem,

$$(q;q)_\infty = \sum_{n=-\infty}^\infty (-1)^n q^{n(3n+1)/2}.$$

Here one would use an argument similar to that briefly outlined in the commentary for Section 8. In analogy with our comments in Sections 2 and 6, the claim (17.8) is false, but the leading term in the asymptotic expansion is indeed $Cx/\sqrt{\log x}$.

18. The proof of (18.1) is quite difficult, but it is given in Rushforth's paper [89]. Ramanujan has omitted many details in his assertion (18.2). For the remainder of the proof of Ramanujan's congruence modulo 11^2 to be completed, it is necessary to explicitly determine the constants a_4, b_4, and c_3 in (18.2). Rushforth does not prove (18.2) but proceeds by a different route to (18.7). The third congruence in (18.3) is proved in Rushforth's paper [89]. The equation to which Ramanujan refers before (18.7) is not given in the manuscript, but would arise from (18.2) by using (18.3)–(18.6).

19. This manuscript contains many results on the divisibility of $\tau(n)$. In several sections Ramanujan concludes that $\tau(n)$ is a multiple of a given integer M for almost all n. In other words, for such M,

$$\lim_{X\to\infty} \frac{\#\{1 \le n \le X \ : \ \tau(n) \equiv 0 \pmod{M}\}}{X} = 1.$$

Specifically, Ramanujan finds in (19.9) that $\tau(n)$ is a multiple of $2^5 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 23 \cdot 691$ for almost all n. Various authors have proved versions of (19.9) with varying exponents on the six primes. It was first proved by Chowla [32] that, in fact, the conclusion still holds if the powers of 2, 3, 5, 7, 23, and 691 are replaced by any set of six positive integral powers.

Ramanujan obtains his results by employing the congruences for $\tau(n)$ with modulus $M \in \{2^5, 3^3, 5^2, 7^2, 23, 691\}$. In each case, he finds that a positive density of primes p has the property that $\tau(p) \equiv 0 \pmod{M}$. A Tauberian argument based on the multiplicativity of $\tau(n)$ then leads to his conclusion [94, Sect. 2].

Results of this type depend upon the divisibility of divisor functions. Improving on Watson's theorem [103], Rankin [84] found an asymptotic formula for the number of positive integers $\le x$ for which $\sigma_\nu(n)$ is not divisible by the prime number k. These results were generalized by E. J. Scourfield [90].

J.–P. Serre [93], [94], [95] has obtained a substantial generalization of Ramanujan's claims for all modular forms of integral weight with respect to congruence subgroups of the full modular group. In particular, if $\sum_{n=1}^{\infty} a(n)q^n$ $(q := e^{2\pi i z})$ is the Fourier expansion of a modular form of integral weight with integral coefficients, then for every positive integer M, $a(n) \equiv 0 \pmod{M}$ for almost all n. M. R. Murty and V. K. Murty [63] have obtained an interesting improvement on the original formulation of Serre's result.

Serre's theorem is based on the existence of ℓ-adic Galois representations associated to modular forms (see the comments on Section 10). In addition to providing an arithmetic and group theoretic description of congruences for Fourier coefficients $a(n)$ of the types found by Ramanujan for $\tau(n)$, their mere existence implies, by the Chebotarev Density Theorem, that a positive proportion of primes p have the property that $a(p) \equiv 0 \pmod{M}$.

Bambah and Chowla [13] state without proofs several congruences for $\tau(n)$. Lahiri [59] gives an enormous number of congruences involving $\tau(n)$. Van der Blij's beautiful paper [27], giving congruences and other properties of $\tau(n)$, is particularly recommended. Except for those employing the theory of ℓ-adic Galois representations, almost all the authors giving proofs of congruences for $\tau(n)$ whom we have cited use ideas similar to those employed by Ramanujan in this manuscript.

20.–23. These sections contain Ramanujan's proof of his congruence for $p(n)$ modulo any positive integral power of 5, with (22.5)–(22.8) being the principal congruences. Observe that (22.7) and (22.8) include (C1.1). As mentioned earlier, the ideas here were expanded into a more detailed proof given in 1938 by Watson [104], who does not mention Part II of Ramanujan's unpublished manuscript in his paper, although according to Rushforth [89], Watson received a copy from Hardy in 1928.

The details in Section 20 are reasonably ample, but beginning with Section 21, the details are sparse. In particular, (21.1) is difficult to prove. The proof given by Watson may follow along somewhat different lines than those indicated by Ramanujan. Readers can likely follow the details for the remainder of Section 21. We have added some details for (21.6), which is not used in Watson's work. The heart of Ramanujan's proof lies in (22.1)–(22.6), for which Ramanujan provides no details. These are developed in Watson's paper [104].

24. Clearly, Ramanujan intended to follow the same lines of attack for powers of 7 as he did for powers of 5 in Sections 20–23. If he had completed his argument, he would have undoubtedly seen that his original conjecture modulo powers of 7 needed to be corrected. Most likely, his declining health prevented him from working out the remaining details, which were completed by Watson [104]. To verify the equations (24.4)–(24.6), it suffices to notice that all three equations are essentially claims about the presentation of modular functions with respect to $\Gamma_0(7)$. In each case, one may multiply both sides of the claimed identity by $(q^7;q^7)^8$. After doing so, one needs to compare, up to a shifted power of q, the Fourier expansions of two cusp forms of weight 4. One can then easily deduce these claims from the results in [33].

At the end of Part II are two detatched fragments. They actually appear at the end of Section 21 in Watson's copy of the manuscript, but it seems to us that they are better placed at the end of the section pertaining to the moduli 7 and 49.

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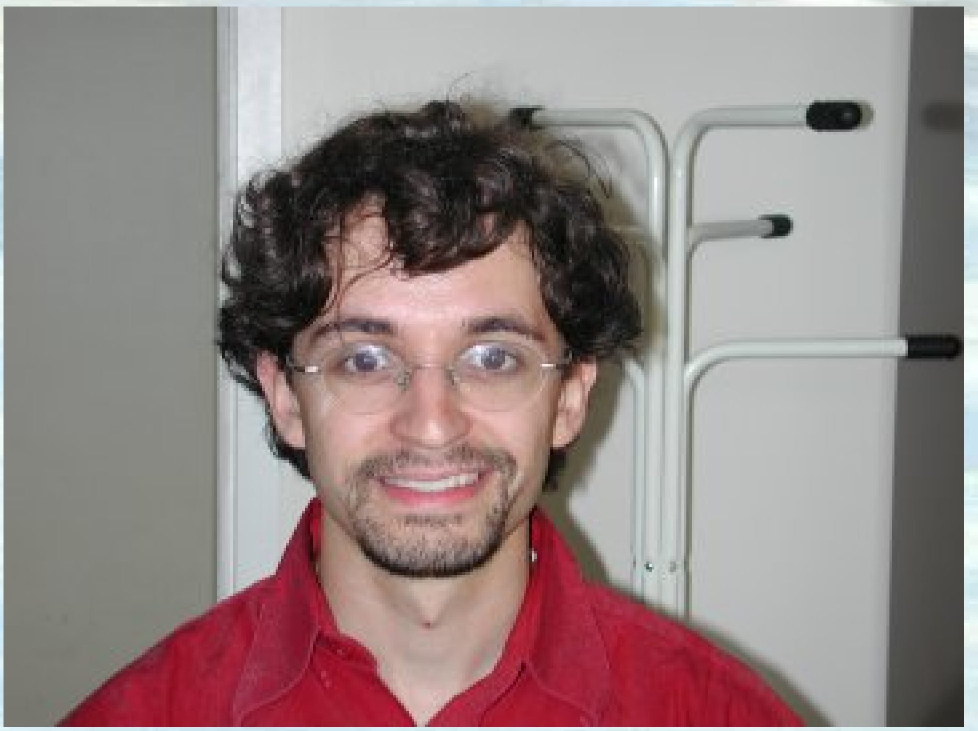

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COMPUTING THE RAMANUJAN TAU FUNCTION

DENIS XAVIER CHARLES

ABSTRACT. We show that the Ramanujan Tau function $\tau(n)$ can be computed by a randomized algorithm in time $O(n^{\frac{1}{2}+\epsilon})$ for every $\epsilon > 0$. The same method also yields a deterministic algorithm that runs in time $O(n^{\frac{3}{4}+\epsilon})$ for every $\epsilon > 0$ to compute $\tau(n)$. Previous algorithms to compute $\tau(n)$ require $\Omega(n)$ time.

1. INTRODUCTION

Let $\tau(n)$ be the coefficient of q^n in the formal expansion $q\prod_{1\leq n}(1-q^n)^{24} = \sum_{1\leq n}\tau(n)q^n$. The following properties of the τ-function are well known:

1. If $n, m \in \mathbb{Z}_{>0}$ such that $\gcd(n, m) = 1$ then $\tau(nm) = \tau(n)\tau(m)$.
2. If $r \geq 1$ and p is a prime then $\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$.

Thus $\tau(n)$ is completely determined by $\tau(p)$ for primes $p|n$. Here is a table of $\tau(p)$ for small prime numbers p.

p	2	3	5	7	11	13
$\tau(p)$	-24	252	4830	-16744	534612	-577738

The importance of the τ-function comes from the fact that it gives the fourier coefficients of a modular form. Namely, the function $\Delta(z) = q\prod_{1\leq n}(1-q^n)^{24}$ where $q = e^{2\pi i z}$ is a cusp form of weight 12 for the full modular group (see [La76]). A famous conjecture of D. H. Lehmer says that $\tau(n)$ is never zero. This conjecture has been verified for all $n \leq 22689242781695999$ [JorKe99]. The function $\tau(n)$ seems to be a hard function to compute. Methods to compute $\tau(n)$ based on recurrence relations that it satisfies or its relations to other arithmetic functions such as $\sigma_k(n)$ require $\Omega(n)$ time steps. Since the number n requires $\log_2 n$ bits these algorithms require exponential time in the length of the input. In this article we show that $\tau(n)$ can be computed in time $O(n^{\frac{1}{2}+\epsilon})$ by a randomized algorithm for every $\epsilon > 0$. Though this algorithm is still an exponential time algorithm it is significantly faster than the other methods. Moreover, algorithms based on recurrences compute values of $\tau(m)$ for $m < n$ when computing $\tau(n)$. Our algorithm has the feature that it does not compute any of the previous values of the τ-function. On the other hand, this algorithm is not well suited to building a table of $\tau(m)$ for all $m < n$ since the table can be built in roughly $O(n)$ time by the other methods, whereas this method would require $O(n^{\frac{3}{2}+\epsilon})$ time. Our algorithm is more suited to computing "spot" values of $\tau(n)$. In the next section we will give the details of the algorithm and prove its running time.

2. THE ALGORITHM

Since we can compute $\tau(n)$ in $O(\log^3 n)$ time provided we know the factorization of the integer n *and* the values of $\tau(p)$ for primes $p|n$, we will concentrate on computing $\tau(p)$ for primes p. There are deterministic algorithms that can factor n in $O(n^{\frac{1}{4}+\epsilon})$ time ([Co93]). We use such an algorithm to find the primes $p|n$. The main idea of the algorithm is to make use of the Selberg Trace formula to compute $\tau(p)$.

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Theorem 2.1. [Sel56] *Let $k \geq 4$ be an even integer and let m be an integer > 0. Then the trace of the Hecke operator $T(m)$ on the space of cusp forms $S_k(\Gamma)$ is given by*

$$\text{Tr } T(m) = -\frac{1}{2} \sum_{-\infty < t < \infty} P_k(t.m) H(4m - t^2) - \frac{1}{2} \sum_{dd' = m} \min\{d, d'\}^{k-1}.$$

In the above sum $H(D)$ refers to the Hurwitz class number of D, and $P_k(t, N) = \frac{\rho^{k-1} - \overline{\rho}^{k-1}}{\rho - \overline{\rho}}$ where ρ is a complex number satisfying $\rho + \overline{\rho} = t$ and $\rho\overline{\rho} = N$.

Note that the sum is actually finite since $H(D) = 0$ if $D < 0$ and so if $t > 2\sqrt{m}$, $H(4m - t^2) = 0$.

In our case $\Delta \in S_{12}(\Gamma)$ and it is a one dimensional vector space. The Hecke operators are a family of linear operators $T(n) : S_k(\Gamma) \to S_k(\Gamma)$ for $n \geq 1$ an integer. Since $\dim S_{12}(\Gamma) = 1$, Δ is a simultaneous eigenform for every $T(n)$. It is known (see [La76]) that $T(n)\Delta(z) = \tau(n)\Delta(z)$ where $\Delta(z) \in S_{12}(\Gamma)$ is the function defined earlier. Thus the eigenvalue of the n-th Hecke operator is $\tau(n)$. Since $\dim S_{12}(\Gamma) = 1$, we have $\text{Tr } T(n) = \tau(n)$ and specializing Theorem 2.1 to our case we get the following result:

Theorem 2.2. *Let p be a prime. Then*

$$\tau(p) = -\sum_{0 < t \leq \sqrt{4p}} P(t, p) H(4p - t^2) \; + \frac{1}{2} p^5 H(4p) \; - 1$$

where

$$P(t, p) = t^{10} - 9t^8 p + 28t^6 p^2 - 35t^4 p^3 + 15t^2 p^4 - p^5$$

and $H(D)$ is the Hurwitz class number.

We will use the above theorem to compute $\tau(p)$. In fact, we only need to show how the Hurwitz class numbers can be computed, since it is easy to compute the above sum. For this task we need the following lemma (see [Co93] Lemma 5.3.7):

Lemma 2.3. *Let $w(-3) = 3, w(-4) = 2$ and $w(D) = 1$ for $D < -4$, and set $h'(D) = \frac{h(D)}{w(D)}$, where $h(D)$ is defined to be the class number of the field $\mathbb{Q}(\sqrt{D})$ if $D \equiv 0, 1 \mod 4$ otherwise we define $h(D)$ to be zero. Then for $N > 0$ we have*

$$H(N) = \sum_{d^2 | N} h'\left(-\frac{N}{d^2}\right).$$

There are randomized sub-exponential time algorithms to compute the class number (see [Co93]).

Theorem 2.4. *The class number $h(D)$ can be computed deterministically in time $|D|^{\frac{1}{4}+\epsilon}$ for every $\epsilon > 0$, or by a randomized algorithm with expected running time $e^{O(\sqrt{\ln |D| \ln\ln |D|})}$.*

Proposition 2.5. *The Hurwitz class number $H(N)$ can be computed by a deterministic algorithm in time $O(N^{\frac{1}{4}+\epsilon})$ or a randomized algorithm with an expected running time $O(N^{\epsilon})$ for every $\epsilon > 0$.*

Proof : By Lemma 2.3 we have

$$H(N) = \sum_{d^2 | N} h'\left(-\frac{N}{d^2}\right).$$

The function $h'(D)$ is essentially just the class number of $\mathbb{Q}(\sqrt{D})$ and so can be computed in time $O(|D|^{\epsilon})$ if we use the randomized algorithm or in time $O(|D|^{\frac{1}{4}+\epsilon})$ if we use the deterministic algorithm. The number of terms in the sum is at most the number of divisors of N. It is known (see [Ten95]) that the number of divisors $d(N) \ll_{\epsilon} N^{\epsilon}$ for every $\epsilon > 0$. Thus the sum can be evaluated by computing each of the terms in the stated time bound. □

Thus putting all these results together we get the following:

Theorem 2.6. *There is a randomized algorithm to compute* $\tau(p)$ *with expected running time* $O(p^{\frac{1}{2}+\epsilon})$ *for every* $\epsilon > 0$.

Theorem 2.7. *There is a deterministic algorithm to compute* $\tau(p)$ *in time* $O(p^{\frac{3}{4}+\epsilon})$ *for every* $\epsilon > 0$.

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New representations of Ramanujan's tau function

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$$x\prod_{1}^{\infty}(1-x^n)^{24}=\sum_{1}^{\infty}\tau(n)x^n \quad (|x|<1),$$

are presented. We also present a congruence modulo 3 for some of the function values.

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NEW REPRESENTATIONS OF RAMANUJAN'S TAU FUNCTION

JOHN A. EWELL

(Communicated by Dennis A. Hejhal)

Abstract. Several formulas for Ramanujan's function τ, defined by

$$x\prod_{1}^{\infty}(1-x^n)^{24}=\sum_{1}^{\infty}\tau(n)x^n \quad (|x|<1),$$

are presented. We also present a congruence modulo 3 for some of the function values.

1. Introduction

Ramanujan's function τ is defined by the expansion

$$x\prod_{1}^{\infty}(1-x^n)^{24}=\sum_{1}^{\infty}\tau(n)x^n, \tag{1.1}$$

which is valid for each complex number x such that $|x|<1$. In this paper we present several formulas for τ, including one congruence involving function values modulo 3. Since these formulas involve several additional functions, we collect these in the following definition.

Definition 1.1. For $\mathbb{N}:=\{0,1,2,\cdots\}$, put $\mathbb{P}:=\mathbb{N}-\{0\}$. Then, for each $k\in\mathbb{P}$ and each $n\in\mathbb{N}$,

$$r_k(n):=|\{(x_1,x_2,\cdots,x_k)\in\mathbb{Z}^k|n=x_1^2+x_2^2+\cdots+x_k^2\}|.$$

(Of course, $\mathbb{Z}:=\{0,\pm1,\pm2,\cdots\}$.)

For each $n\in\mathbb{P}, b(n)$ is the exponent of the exact power of 2 dividing n, and then $Od(n):=n2^{-b(n)}$ is the odd part of n.

For each $k\in\mathbb{N}$ and each $n\in\mathbb{P}, \sigma_k(n)$ is the sum of the kth powers of all of the positive divisors of n. For simplicity, $\sigma(n):=\sigma_1(n)$.

We are now prepared to state our main result.

Theorem 1.2. *For each $n\in\mathbb{N}$ and each $m\in\mathbb{P}$,*

(1.2)
$$\tau(4n+2)=-3\sum_{k=1}^{2n+1}2^{3b(2k)}\sigma_3(Od(2k))\sum_{j=0}(-1)^j r_8(4n+2-2k-j)r_8(j);$$

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$$\sum_{k=1}^{n} 2^{3b(2k)}\sigma_3(Od(2k))\sum_{j=0}(-1)^j r_8(2n+1-2k-j)r_8(j)=0; \tag{1.3}$$

(1.4)

$$\tau(4m)=-2^{11}\tau(m)-3\sum_{k=1}^{2m} 2^{3b(2k)}\sigma_3(Od(2k))\sum_{j=0}(-1)^j r_8(4m-2k-j)r_8(j).$$

[*Here, the second sums of* (1.2), (1.3) *and* (1.4) *have respectively upper limits of summation* $4n+2-2k,\ 2n+1-2k$ *and* $4m-2k$.]

In section 2 we prove this theorem, and also present two immediate corollaries of the theorem.

In [6, pp.275-278] there is a good, though not exhaustive, list of references for the function τ. Here cited are almost all of the known properties of the function, including formulas, recurrences and congruences which some values satisfy. Perhaps the best known recursive determination of τ is that of Ramanujan [5, p. 152]. For certain analytic properties of τ, not cited in [6], the reader might consult the paper of Moreno [4].

2. Proof of Theorem 1.2

First of all, we state four identities which we require in our development.

$$\prod_1^\infty (1-x^{2n})(1+tx^{2n-1})(1+t^{-1}x^{2n-1})=\sum_{-\infty}^{\infty} x^{n^2}t^n; \tag{2.1}$$

$$\prod_1^\infty (1+x^n)(1-x^{2n-1})=1; \tag{2.2}$$

$$x\prod_1^\infty \frac{(1-x^{2n})^8}{(1-x^{2n-1})^8}=\sum_1^\infty 2^{3b(n)}\sigma_3(Od(n))x^n; \tag{2.3}$$

$$\prod_1^\infty (1+x^{2n-1})^8=\prod_1^\infty (1-x^{2n-1})^8+16x\prod_1^\infty (1+x^{2n})^8. \tag{2.4}$$

Identity (2.1), the celebrated triple-product identity, is valid for each pair of complex numbers t, x such that $t\neq 0$ and $|x|<1$. Each of the identities (2.2), (2.3) and (2.4) is valid for each complex number x such that $|x|<1$. For proofs of (2.1) and (2.2) see [3, pp. 282-283 and p. 277]. For a proof of (2.3) see [1, pp. 1291-1292]; and, for a proof of (2.4) see [2, pp. 421-422]. Actually, we need only two special cases of (2.1) corresponding to the substitutions $t\to 1$ and $t\to -1$. Under the former substitution we observe that the kth power of the right side of the resulting identity generates the sequence $r_k(n), n\in\mathbb{N}$. Similarly, under the latter substitution the sequence $(-1)^n r_k(n), n\in\mathbb{N}$, is generated. We begin our argument by multiplying both sides of (2.4) by the infinite product
$\prod(1-x^{2n})^8$ to get

$$-16x\prod_1^\infty (1-x^{4n})^8=\prod_1^\infty (1-x^n)^8-\prod_1^\infty (1-(-x)^n)^8.$$

Then, we raise each side of this identity to the third power, and multiply the resulting identity by x to get

(2.5)

$$-2^{12}\sum_{1}^{\infty}\tau(n)x^{4n} = 2\sum_{1}^{\infty}\tau(2n)x^{2n} - 3x\prod_{1}^{\infty}(1-x^n)^{16}(1-(-x)^n)^8 + 3x\prod_{1}^{\infty}(1-x^n)^8(1-(-x)^n)^{16}.$$

Next,

$$\begin{aligned} & -3x\prod_{1}^{\infty}(1-x^n)^{16}(1-(-x)^n)^8 \\ &= 3(-x)\prod_{1}^{\infty}\frac{(1-x^{2n})^{16}(1-x^{2n-1})^{16}(1-x^{2n})^8(1+x^{2n-1})^{16}}{(1+x^{2n-1})^8} \\ &= 3(-x)\prod_{1}^{\infty}\frac{(1-x^{2n})^8}{(1+x^{2n-1})^8}\prod_{1}^{\infty}(1-x^{2n})^8(1-x^{2n-1})^{16}\prod_{1}^{\infty}(1-x^{2n})^8(1+x^{2n-1})^{16} \\ &= 3\sum_{1}^{\infty}(-1)^h 2^{3b(h)}\sigma_3(Od(h))x^h\cdot\sum_{0}^{\infty}(-1)^j r_8(j)x^j\cdot\sum_{0}^{\infty}r_8(k)x^k. \end{aligned}$$

Similarly,

$$3x\prod_{1}^{\infty}(1-x^n)^8(1-(-x)^n)^{16} = 3\sum_{1}^{\infty}2^{3b(h)}\sigma_3(Od(h))x^h\cdot\sum_{0}^{\infty}(-1)^j r_8(j)x^j\cdot\sum_{0}^{\infty}r_8(k)x^k.$$

Expanding the two products of three series, substituting the resulting expansions into (2.5), and cancelling a factor of 2, we get

$$\begin{aligned} & -2^{11}\sum_{1}^{\infty}\tau(n)x^{4n} \\ = \quad & \sum_{0}^{\infty}\tau(4n+2)x^{4n+2} + \sum_{1}^{\infty}\tau(4n)x^{4n} \\ & + \sum_{n=0}^{\infty}x^{2n+1}3\sum_{k=1}^{n}2^{3b(2k)}\sigma_3(Od(2k))\sum_{j=0}(-1)^j r_8(2n+1-2k-j)r_8(j) \\ & + \sum_{n=0}^{\infty}x^{4n+2}3\sum_{k+1}^{2n+1}2^{3b(2k)}\sigma_3(Od(2k))\sum_{j=0}(-1)^j r_8(4n+2-2k-j)r_8(j) \\ & + \sum_{n=1}^{\infty}x^{4n}3\sum_{k=1}^{2n}2^{3b(2k)}\sigma_3(Od(2k))\sum_{j=0}(-1)^j r_8(4n-2k-j)r_8(j). \end{aligned}$$

Equating coefficients of like powers of x in the foregoing identity we thus prove our theorem.

Corollary 2.1. *For each $n \in \mathbb{N}$,*

$$\tau(2n+1) = \sum_{k=1}^{2n+1} 2^{3\{b(2k)-1\}}\sigma_3(Od(2k)) \sum_{j=0} (-1)^j r_8(4n+2-2k-j) r_8(j),$$

where the upper limit of summation of the second sum is $4n+2-2k$.

Proof. By the multiplicativity of $\tau, \tau(4n+2) = \tau(2(2n+1)) = \tau(2)\tau(2n+1)$. But, for $n = 0$, (1.2) yields $\tau(2) = -24$. Then, cancellation of -24 in (1.2) proves the corollary. □

Corollary 2.2. *For each $m \in \mathbb{P}$,*

$$\tau(4m) \equiv \tau(m) \pmod 3.$$

Proof. Part (1.4) of the theorem, and the obvious result $2 \equiv -1$ (mod 3). □

Concluding remarks. How could one possibly use Theorem 1.2 to compute the values $\tau(n), n \in \mathbb{P}$? Well, first of all, we'd realize that for $n \in \mathbb{P}, r_8(n)$, like $\sigma_3(n)$, can also be expressed in terms of the positive divisors of n. As a matter of fact, for each $n \in \mathbb{P}$,

$$r_8(n) = 16(-1)^n \sum_{d|n} (-1)^d d^3.$$

For example, see [3, p. 314]. Then, we'd use Corollary 2.1 to compute $\tau(n)$ for odd values of n. For $n \equiv 2$ (mod 4) we'd compute $\tau(n)$ by (1.2). And, for $n \equiv 0$ (mod 4) we'd appeal to (1.4) and induction.

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ON RAMANUJAN'S TAU FUNCTION

J.A. EWELL

Abstract:

A formula for Ramanujan's function τ, defined by the expansion

$$x\prod_1^\infty(1-x^n)^{24}=\sum_1^\infty\tau(n)x^n,\quad |x|<1,$$

is presented.

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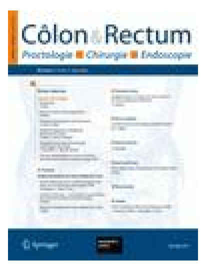

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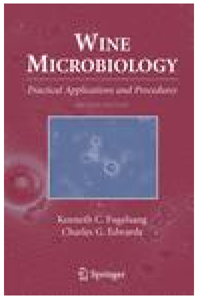

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$$756\tau(n) = 65\sigma_{11}(n) + 691\sigma_5(n) - 691 \cdot 252 \cdot \sum_{k=1}^{n-1} \sigma_5(k)\sigma_5(n-k)$$

when n is prime and the formula

$$\tau(p^n) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n-j}{n-2j} p^{11j} \tau(p)^{n-2j}$$

Here $\sigma_k(n)$ is the sum of the k-th powers of the divisors of n.
(See T.M. Apostol, *Modular functions and Dirichlet series in number theory*, 20-22, 140 and the MathWorld account.)

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Mathematics

New infinite families of exact sums of squares formulas, Jacobi elliptic functions, and Ramanujan's tau function

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Abstract

In this paper, we give two infinite families of explicit exact formulas that generalize Jacobi's (1829) 4 and 8 squares identities to $4n^2$ or $4n(n + 1)$ squares, respectively, without using cusp forms. Our 24 squares identity leads to a different formula for Ramanujan's tau function $\tau(n)$, when n is odd. These results arise in the setting of Jacobi elliptic functions, Jacobi continued fractions, Hankel or Turánian determinants, Fourier series, Lambert series, inclusion/exclusion, Laplace expansion formula for determinants, and Schur functions. We have also obtained many additional infinite families of identities in this same setting that are analogous to the η-function identities in appendix I of Macdonald's work [Macdonald, I. G. (1972) *Invent. Math.* 15, 91–143]. A special case of our methods yields a proof of the two conjectured [Kac, V. G. and Wakimoto, M. (1994) in *Progress in Mathematics*, eds. Brylinski, J.-L., Brylinski, R., Guillemin, V. & Kac, V. (Birkhäuser Boston, Boston, MA), Vol. 123, pp. 415–456] identities involving representing a positive integer by sums of $4n^2$ or $4n(n + 1)$ triangular numbers, respectively. Our 16 and 24 squares identities were originally obtained via multiple basic hypergeometric series, Gustafson's C_ℓ nonterminating ${}_6\psi_5$ summation theorem, and Andrews' basic hypergeometric series proof of Jacobi's 4 and 8 squares identities. We have (elsewhere) applied symmetry and Schur function techniques to this original approach to prove the existence of similar infinite families of sums of squares identities for n^2 or $n(n + 1)$ squares, respectively. Our sums of more than 8 squares identities are not the same as the formulas of Mathews (1895), Glaisher (1907), Ramanujan (1916), Mordell (1917, 1919), Hardy (1918, 1920), Kac and Wakimoto, and many others.

Keywords: Jacobi continued fractions, Hankel or Turánian determinants, Fourier series, Lambert series, Schur functions

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1. Introduction

In this paper, we announce two infinite families of explicit exact formulas that generalize Jacobi's (1) 4 and 8 squares identities to $4n^2$ or $4n(n + 1)$ squares, respectively, without using cusp forms. Our 24 squares identity leads to a different formula for Ramanujan's (2) tau function $\tau(n)$, when n is odd. These results arise in the setting of Jacobi elliptic functions, Jacobi continued fractions, Hankel or Turánian determinants, Fourier series, Lambert series, inclusion/exclusion, Laplace expansion formula for determinants, and Schur functions. (For this background material, see refs. 1 and 3–16.)

The problem of representing an integer as a sum of squares of integers has had a long and interesting history, which is surveyed in ref. 17 and chapters 6–9 of ref. 18. The review article (19) presents many questions connected with representations of integers as sums of squares. Direct applications of sums of squares to lattice point problems and crystallography can be found in ref. 20. One such example is the computation of the constant Z_N, which occurs in the evaluation of a certain Epstein zeta function, needed in the study of the stability of rare gas crystals and in that of the so-called Madelung constants of ionic salts.

The s squares problem is to count the number $r_s(n)$ of integer solutions $(x_1, \ldots, x_s)$ of the Diophantine equation

$$x_1^2 + \cdots + x_s^2 = n, \quad (1)$$

in which changing the sign or order of the x_i's gives distinct solutions.

Diophantus (325–409 A.D.) knew that no integer of the form $4n - 1$ is a sum of two squares. Girard conjectured in 1632 that n is a sum of two squares if and only if all prime divisors q of n with $q \equiv 3 \pmod 4$ occur in n to an even power. Fermat in 1641 gave an "irrefutable proof" of this conjecture. Euler gave the first known proof in 1749. Early explicit formulas for $r_2(n)$ were given by Legendre in 1798 and Gauss in 1801. It appears that Diophantus was aware that all positive integers are sums of four integral squares. Bachet conjectured this result in 1621, and Lagrange gave the first proof in 1770.

Jacobi, in his famous *Fundamenta Nova* (1) of 1829, introduced elliptic and theta functions, and used them as tools in the study of Eq. **1**. Motivated by Euler's work on 4 squares, Jacobi knew that the number $r_s(n)$ of integer solutions of Eq. **1** was also determined by

$$\vartheta_3(0, -q)^s := 1 + \sum_{n=1}^{\infty} (-1)^n r_s(n) q^n, \tag{2}$$

where $\vartheta_3(0, q)$ is the $z = 0$ case of the theta function $\vartheta_3(z, q)$ in ref. 21 given by

$$\vartheta_3(0, q) := \sum_{j=-\infty}^{\infty} q^{j^2}. \tag{3}$$

Jacobi then used his theory of elliptic and theta functions to derive remarkable identities for the $s = 2, 4, 6, 8$ cases of $\vartheta_3(0, -q)^s$. He immediately obtained elegant explicit formulas for $r_s(n)$, where $s = 2, 4, 6, 8$. We recall Jacobi's identities for $s = 4$ and 8 in the following theorem.

Theorem 1.1 (Jacobi).

$$\vartheta_3(0, -q)^4 = 1 - 8 \sum_{r=1}^{\infty} (-1)^{r-1} \frac{rq^r}{1+q^r} = 1 + 8 \sum_{n=1}^{\infty} (-1)^n \left[\sum_{\substack{d|n,\, d>0 \\ 4\nmid d}} d \right] q^n, \tag{4}$$

and

$$\vartheta_3(0, -q)^8 = 1 + 16 \sum_{r=1}^{\infty} (-1)^r \frac{r^3 q^r}{1-q^r}$$

$$= 1 + 16 \sum_{n=1}^{\infty} \left[\sum_{d|n, d>0} (-1)^{d} d^{3} \right] q^{n}. \tag{5}$$

Consequently, we have

$$r_4(n) = 8 \sum_{\substack{d|n, d>0 \\ 4\nmid d}} d \; and \; r_8(n) = 16 \sum_{d|n, d>0} (-1)^{n+d} d^{3}, \tag{6}$$

respectively.

In general it is true that

$$r_{2s}(n) = \delta_{2s}(n) + e_{2s}(n), \tag{7}$$

where $\delta_{2s}(n)$ is a divisor function and $e_{2s}(n)$ is a function of order substantially lower than that of $\delta_{2s}(n)$. If $2s = 2, 4, 6, 8$, then $e_{2s}(n) = 0$, and Eq. **7** becomes Jacobi's formulas for $r_{2s}(n)$, including Eq. **6**. On the other hand, if $2s > 8$ then $e_{2s}(n)$ is never 0. The function $e_{2s}(n)$ is the coefficient of q^n in a suitable "cusp form." The difficulties of computing Eq. **7**, especially the nondominate term $e_{2s}(n)$, increase rapidly with $2s$. The modular function approach to Eq. **7** and the cusp form $e_{2s}(n)$ is discussed in ref. 13. For $2s > 8$, modular function methods such as those in refs. 22–27, or the more classical elliptic function approach of refs. 28–30, are used to determine general formulas for $\delta_{2s}(n)$ and $e_{2s}(n)$ in Eq. **7**. Explicit, exact examples of Eq. **7** have been worked out for $2 \leq 2s \leq 32$. Similarly, explicit formulas for $r_s(n)$ have been found for (odd) $s < 32$. Alternate, elementary approaches to sums of squares formulas can be found in refs. 31–36.

We next consider classical analogs of Eqs. **4** and **5** corresponding to the $s = 8$ and 12 cases of Eq. **7**.

Glaisher (37, 62–64) used elliptic function methods rather than modular functions to prove the following theorem.

Theorem 1.2 (Glaisher).

$$\vartheta_3(0,\,-q)^{16} = 1 + \frac{32}{17}\sum_{y_1,m_1\geq 1}(-1)^{m_1}m_1^7q^{m_1y_1} \quad \mathbf{8a}$$

$$-\frac{512}{17}q(q;\,q)_\infty^8(q^2;\,q^2)_\infty^8, \quad \mathbf{8b}$$

where we have

$$(q;\,q)_\infty := \prod_{r\geq 1}(1-q^r). \quad \mathbf{9}$$

Glaisher took the coefficient of q^n to obtain $r_{16}(n)$. The same formula appears in ref. 13 (equation 7.4.32).

To find $r_{24}(n)$, Ramanujan (ref. 2, entry 7, table VI; see also ref. 13, equation 7.4.37) first proved *Theorem 1.3*.

Theorem 1.3 (Ramanujan). *Let* $(q;\,q)_\infty$ *be defined by Eq.* **9**. *Then*

$$\vartheta_3(0,\,-q)^{24} = 1 + \frac{16}{691}\sum_{y_1,m_1\geq 1}(-1)^{m_1}m_1^{11}q^{m_1y_1} \quad \mathbf{10a}$$

$$-\frac{33152}{691}q(q;\,q)_\infty^{24} - \frac{65536}{691}q^2(q^2;\,q^2)_\infty^{24}. \quad \mathbf{10b}$$

One of the main motivations for this paper was to generalize *Theorem 1.1* to $4n^2$ or $4n(n+1)$ squares, respectively, without using cusp forms such as Eqs. **8b** and **10b** but still using just sums of products of at most n Lambert series similar to either Eq. **4** or Eq. **5**, respectively. This is done in *Theorems 2.1* and *2.2* below. Here, we state the $n = 2$ cases, which determine different formulas for 16 and 24 squares.

Theorem 1.4.

$$\vartheta_3(0,\,-q)^{16} = 1 - \frac{32}{3}(U_1 + U_3 + U_5)$$

$$+\frac{256}{3}(U_1U_5 - U_3^2), \quad \mathbf{11}$$

where

$$U_s \equiv U_s(q) := \sum_{r=1}^{\infty} (-1)^{r-1} \frac{r^s q^r}{1+q^r}$$

$$= \sum_{n=1}^{\infty} \left[\sum_{d|n,d>0} (-1)^{d+n/d} d^s q^n \right.$$

$$= \sum_{y_1, m_1 \geq 1} (-1)^{y_1+m_1} m_1^s q^{m_1 y_1}. \qquad \mathbf{12}$$

Analogous to *Theorem 1.3*, we have *Theorem 1.5*.

Theorem 1.5.

$$\vartheta_3(0, -q)^{24} = 1 + \frac{16}{9}(17G_3 + 8G_5 + 2G_7)$$

$$+\frac{512}{9}(G_3G_7 - G_5^2), \qquad \mathbf{13}$$

where

$$G_s \equiv G_s(q) := \sum_{r=1}^{\infty} (-1)^r \frac{r^s q^r}{1-q^r}$$

$$= \sum_{n=1}^{\infty} \left[\sum_{d|n,d>0} (-1)^d d^s q^n \right.$$

$$= \sum_{y_1, m_1 \geq 1} (-1)^{m_1} m_1^s q^{m_1 y_1}. \qquad \mathbf{14}$$

An analysis of Eq. **10b** depends upon Ramanujan's (2) tau function τ(*n*), defined by

$$q(q; q)_{\infty}^{24} := \sum_{n=1}^{\infty} \tau(n) q^n. \qquad \mathbf{15}$$

For example, τ(1) = 1, τ(2) = –24, τ(3) = 252, τ(4) = –1472, τ(5) = 4830, τ(6) = –6048, and τ(7) = –16744. Ramanujan (ref. 2, equation 103) conjectured, and Mordell (38) proved, that τ(*n*) is multiplicative.

In the case where *n* is an odd integer (in particular an odd prime), equating Eqs. **10a**, **10b**, and **13** yields two formulas for τ(*n*) that are different from Dyson's (39) formula. We first obtain *Theorem 1.6*.

Theorem 1.6. *Let* τ(n) *be defined by Eq.* **15** *and let* n *be odd. Then*

$$259\tau(n) = \frac{1}{2^3 \cdot 3^2}[17 \cdot 691\sigma_3(n) + 8 \cdot 691\sigma_5(n) + 2 \cdot 691\sigma_7(n) - 9\sigma_{11}(n)] - \frac{691 \cdot 2^2}{3^2}\sum_{m=1}^{n-1}[\sigma_3^{\dagger}(m)\sigma_7^{\dagger}(n-m) - \sigma_5^{\dagger}(m)\sigma_5^{\dagger}(n-m)], \quad \mathbf{16}$$

where

$$\sigma_r(n) := \sum_{d|n, d>0} d^r \; and \; \sigma_r^{\dagger}(n) := \sum_{d|n, d>0} (-1)^d d^r \quad \mathbf{17}$$

Remark: We can use Eq. **16** to compute τ(*n*) in ≤6*n* ln *n* steps when *n* is an odd integer. This may also be done in $n^{2+\epsilon}$ steps by appealing to Euler's infinite-product-representation algorithm (40) applied to $(q; q)_{\infty}^{24}$ in Eq. **15**.

A different simplification involving a power series formulation of Eq. **13** leads to the following theorem.

Theorem 1.7. *Let* τ(n) *be defined by Eq.* **15** *and let* n ≥ 3 *be odd. Then*

$$259\tau(n) = \frac{1}{2^3}\sum_{d|n, d>0}(-1)^d d^{11} - \frac{691}{2^3 \cdot 3^2}\sum_{d|n, d>0}(-1)^d d^3(17 + 8d^2 + 2d^4) \quad \mathbf{18a}$$

$$- \frac{691 \cdot 2^2}{3^2}\sum_{\substack{m_1 > m_2 \geq 1 \\ m_1 + m_2 \leq n \\ \gcd(m_1, m_2)|n}} (-1)^{m_1+m_2}(m_1 m_2)^3 \times (m_1^2 - m_2^2)^2 \sum_{\substack{y_1, y_2 \geq 1 \\ m_1 y_1 + m_2 y_2 = n}} 1. \quad \mathbf{18b}$$

Remark: The inner sum in Eq. **18b** counts the number of solutions (y_1, y_2) of the classical linear

Diophantine equation $m_1y_1 + m_2y_2 = n$. This relates Eqs. **18a** and **18b** to the combinatorics in sections 4.6 and 4.7 of ref. 15.

In the next section, we present the infinite families of explicit exact formulas that generalize *Theorems 1.1*, *1.4*, and *1.5*.

Our methods yield (elsewhere) many additional infinite families of identities analogous to the η-function identities in appendix I of Macdonald's work (41). A special case of our analysis gives a proof (presented elsewhere) of the two identities conjectured by Kac and Wakimoto (42); these identities involve representing a positive integer by sums of $4n^2$ or $4n(n + 1)$ triangular numbers, respectively. The $n = 1$ case gives the classical identities of Legendre (ref. 43; see also ref. 3, equations ii and iii).

Theorems 1.4 and *1.5* were originally obtained via multiple basic hypergeometric series (44–51) and Gustafson's* C^{ℓ} nonterminating ${}_6\varphi_5$ summation theorem combined with Andrews' (52) basic hypergeometric series proof of Jacobi's 4 and 8 squares identities. We have (elsewhere) applied symmetry and Schur function techniques to this original approach to prove the existence of similar infinite families of sums of squares identities for n^2 or $n(n + 1)$ squares, respectively.

Our sums of more than 8 squares identities are not the same as the formulas of Mathews (31), Glaisher (37, 62–64), Sierpinski (32), Uspensky (33–35), Bulygin (28, 53), Ramanujan (2), Mordell (26, 54), Hardy (23, 24), Bell (55), Estermann (56), Rankin (27, 57), Lomadze (25), Walton (58), Walfisz (59), Ananda-Rau (60), van der Pol (61), Krätzel (29, 30), Gundlach (22), and Kac and Wakimoto (42).

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2. The $4n^2$ and $4n(n + 1)$ Squares Identities

To state our identities, we first need the Bernoulli numbers B_n defined by

$$\frac{t}{e^t - 1} := \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad \text{for } |t| < 2\pi. \tag{19}$$

We also use the notation $I_n := \{1, 2, \ldots, n\}$; $\|S\|$ is the cardinality of the set S, and $\det(M)$ is the determinant of the $n \times n$ matrix M.

The determinant form of the $4n^2$ squares identity is *Theorem 2.1*.

Theorem 2.1. *Let* n = 1, 2, 3, … . *Then*

$$\vartheta_3(0, -q)^{4n^2} = 1 + \sum_{p=1}^{n} (-1)^p 2^{2n^2+n} \prod_{r=1}^{2n-1} (r!)^{-1} \sum_{\substack{\phi \subset S \subseteq I_n \\ \|S\| = p}} \det(M_{n,S}), \tag{20}$$

where $\vartheta_3(0, -q)$ *is determined by Eq.* **3**, *and* $M_{n,S}$ *is the* $n \times n$ *matrix whose* i*th row is*

$$U_{2i-1}, U_{2(i+1)-1}, \ldots, U_{2(i+n-1)-1}, \mathit{if}\ i \in S \tag{21}$$

$$\mathit{and}\ c_i, c_{i+1}, \ldots, c_{i+n-1}, \mathit{if}\ i \notin S,$$

where U_{2i-1} *is determined by Eq.* **12**, *and* c_i *is defined by*

$$c_i := (-1)^{i-1} \frac{(2^{2i} - 1)}{4i} \cdot |B_{2i}|, \quad \mathit{for}\ i = 1, 2, 3, \ldots, \tag{22}$$

with B_{2i} *the Bernoulli numbers defined by Eq.* **19**.

We next have *Theorem 2.2*.

Theorem 2.2. *Let* n = 1, 2, 3, … . *Then*

$$\vartheta_3(0,\,-q)^{4n(n+1)} = 1 + \sum_{p=1}^{n} (-1)^{n-p} 2^{2n^2+3n} \prod_{r=1}^{2n} (r!)^{-1} \sum_{\substack{\phi \subset S \subseteq I_n \\ \|S\| = p}} \det(M_{n,S}), \qquad 23$$

where $\vartheta_3(0, -q)$ *is determined by Eq.* **3**, *and* $M_{n,S}$ *is the* $n \times n$ *matrix whose* i*th row is*

$$G_{2i+1},\, G_{2(i+1)+1},\, \ldots,\, G_{2(i+n-1)+1},\, if\ i \in S \qquad 24$$

$$and\ a_i,\, a_{i+1},\, \ldots,\, a_{i+n-1},\, if\ i \notin S,$$

where G_{2i+1} *and* $a_i := c_{i+1}$ *are determined by Eqs.* **14** *and* **22**, *respectively*.

We next use Schur functions $s_\lambda(x_1, \ldots, x_p)$ to rewrite *Theorems 2.1* and *2.2*. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots)$ be a partition of nonnegative integers in decreasing order, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \ldots$, such that only finitely many of the λ_i are nonzero. The length $\ell(\lambda)$ is the number of nonzero parts of λ.

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_p)$ of length $\leq p$,

$$s_\lambda(x) \equiv s_\lambda(x_1, \ldots, x_p) := \frac{\det(x_i^{\lambda_j + p - j})}{\det(x_i^{p-j})} \qquad 25$$

is the Schur function (12) corresponding to the partition λ. [Here, $\det(a_{ij})$ denotes the determinant of a $p \times p$ matrix with (i, j)th entry a_{ij}]. The Schur function $s_\lambda(x)$ is a symmetric polynomial in $x_1, \ldots, x_p$ with nonnegative integer coefficients. We typically have $1 \leq p \leq n$.

We use Schur functions in Eq. 25 corresponding to the partitions λ and ν, with

$$\lambda_r := \ell_{p-r+1} - \ell_1 + r - p$$

$$\text{and}\ \nu_r := j_{p-r+1} - j_1 + r - p,$$

$$\text{for}\ r = 1, 2, \ldots, p, \qquad 26$$

where the ℓ_r and j_r are elements of the sets *S* and *T*, with

$$S := \{\ell_1 < \ell_2 < \cdots < \ell_p\}$$

$$\text{and} \quad S^c := \{\ell_{p+1} < \cdots < \ell_n\}, \qquad 27$$

$$T := \{j_i < j_2 < \cdots < j_p\}$$

$$\text{and} \quad T^c := \{j_{p+1} < \cdots < j_n\}, \qquad 28$$

where $S^c := I_n - S$ is the compliment of the set *S*. We also have

$$\sum(S) := \ell_1 + \ell_2 + \cdots + \ell_p$$

$$\text{and} \quad \sum(T) := j_1 + j_2 + \cdots + j_p. \qquad 29$$

Keeping in mind Eqs. **25–29**, symmetry and skew-symmetry arguments, row and column operations, and the Laplace expansion formula (9) for a determinant, we now rewrite *Theorem 2.1* as *Theorem 2.3*.

Theorem 2.3. *Let* n = 1, 2, 3, … . *Then*

$$\vartheta_3(0, -q)^{4n^2} = 1 + \sum_{p=1}^{n} (-1)^p 2^{2n^2+n} \prod_{r=1}^{2n-1} (r!)^{-1}$$

$$\times \sum_{\substack{y_1, \ldots, y_p \geq 1 \\ m_1 > m_2 > \cdots > m_p \geq 1}} (-1)^{y_1+\cdots+y_p}$$

$$\cdot (-1)^{m_1+\cdots+m_p} q^{m_1 y_1+\cdots+m_p y_p} \prod_{1 \leq r < s \leq p} (m_r^2 - m_s^2)^2$$

$$\cdot (m_1 m_2 \cdots m_p) \sum_{\substack{\phi \subset S, T \subseteq I_n \\ \|S\| = \|T\| = p}} (-1)^{\Sigma(S)+\Sigma(T)} \Bigg) \cdot \det(D_{n-p, S^c, T^c})$$

$$\cdot (m_1 m_2 \cdots m_p)^{2\ell_1+2j_1-4} s_\lambda(m_1^2, \ldots, m_p^2) s_v(m_1^2, \ldots, m_p^2), \qquad 30$$

where $\vartheta_3(0, -q)$ *is determined by Eq.* **3**; *the sets* S, S^c, T, *and* T^c *are given by Eqs.* **27** *and* **28**; Σ

Σ(S) *and* Σ(T) *are given by Eq.* **29**; *the* (n – p) × (n – p) *matrix* $\mathbf{D}_{n-p,S^c,T^c} := [c_{(\ell_{p+r}+j_{p+s}-1)}]_{1\le r,s\le n-p}$, *where the* c_i *are determined by Eq.* **22**, *with the* B_{2i} *in Eq.* **19**; *and* s_λ *and* s_ν *are the Schur functions in Eq.* **25**, *with the partitions* λ *and* ν *given by Eq.* **26**.

We next rewrite *Theorem 2.2* as *Theorem 2.4*.

Theorem 2.4. *Let* n = 1, 2, 3, … . *Then*

$$\vartheta_3(0, -q)^{4n(n+1)} = 1 + \sum_{p=1}^{n} (-1)^{n-p} 2^{2n^2+3n} \prod_{r=1}^{2n} (r!)^{-1} \times \sum_{\substack{y_1,\dots,y_p \ge 1 \\ m_1 > m_2 > \cdots > m_p \ge 1}} (-1)^{m_1+\cdots+m_p} \cdot q^{m_1y_1+\cdots+m_py_p} (m_1m_2\cdots m_p)^3 \prod_{1\le r<s\le p} (m_r^2 - m_s^2)^2 \cdot \sum_{\substack{\phi\subset S,T\subseteq I_n \\ \|S\| = \|T\| = p}} (-1)^{\Sigma(S)+\Sigma(T)} \cdot \det(\mathbf{D}_{n-p,S^c,T^c}) \cdot (m_1m_2\cdots m_p)^{2\ell_1+2j_1-4} s_\lambda(m_1^2,\dots,m_p^2) s_\nu(m_1^2,\dots,m_p^2), \qquad \mathbf{31}$$

where the same assumptions hold as in Theorem 2.3, *except that the* (n – p) × (n – p) *matrix* $\mathbf{D}_{n-p,S^c,T^c} := [a_{(\ell_{p+r}+j_{p+s}-1)}]_{1\le r,s\le n-p}$, *where the* $a_i := c_{i+1}$ *are determined by Eq.* **22**.

We close this section with some comments about the above theorems. To prove *Theorem 2.1*, we first compare the Fourier and Taylor series expansions of the Jacobi elliptic function $f_1(u, k) := sc(u, k)dn(u, k)$, where k is the modulus. An analysis similar to that in refs. 3, 4, and 16 leads to the relation $U_{2m-1}(-q) = c_m + d_m$, for $m = 1, 2, 3, \dots$, where $U_{2m-1}(-q)$ and c_m are defined by Eqs. **12** and **22**, respectively, and d_m is given by $d_m = [(-1)^m z^{2m}/2^{2m+1}]\cdot(sd/c)_m(k^2)$, where $z := {}_2F_1(1/2, 1/2; 1; k^2) = 2\boldsymbol{K}(k)/\pi \equiv 2\boldsymbol{K}/\pi$, with $\boldsymbol{K}(k) \equiv \boldsymbol{K}$ the complete elliptic integral of the first kind in ref. 21, and $(sd/c)_m(k^2)$ is the coefficient of $u^{2m-1}/(2m-1)!$ in the Taylor series expansion of $f_1(u, k)$ about $u = 0$.

An inclusion/exclusion argument then reduces the $q \to -q$ case of Eq. **20** to finding suitable product formulas for the $n \times n$ Hankel determinants $\det(d_{i+j-1})$ and $\det(c_{i+j-1})$. Row and column operations immediately imply that

$$\det(d_{i+j-1}) = (z^{2n^2}(-1)^n/2^{2n^2+n})\det[(sd/c)_{i+j-1}(k^2)]. \tag{32}$$

From theorem 7.9 of ref. 4, we have $z = \vartheta_3(0, q)^2$, where $q = \exp[-\pi \boldsymbol{K}(\sqrt{1-k^2})/\boldsymbol{K}(k)]$. Setting $z = \vartheta_3(0, q)^2$ in Eq. **32** and then taking $q \to -q$ produces the $\vartheta_3(0, -q)^{4n^2}$ in Eq. **20**. The proof of *Theorem 2.1* is complete once we show that

$$\det[(sd/c)_{i+j-1}(k^2)] = \prod_{r=1}^{2n-1}(r!)$$

and

$$\det(c_{i+j-1}) = 2^{-(2n^2+n)} \cdot \prod_{r=1}^{2n-1}(r!). \tag{33}$$

By a classical result of Heilermann (7, 8), more recently presented in ref. 10 (theorem 7.14), Hankel determinants whose entries are the coefficients in a formal power series L can be expressed as a certain product of the "numerator" coefficients of the associated Jacobi continued fraction J corresponding to L, provided that J exists. Modular transformations, followed by row and column operations, reduce the evaluation of $\det[(sd/c)_{i+j-1}(k^2)]$ in Eq. **33** to applying Heilermann's formula to Rogers' (14) J-fraction expansion of the Laplace transform of $sd(u, k)cn(u, k)$. The evaluation of $\det(c_{i+j-1})$ can be done similarly, starting with $sc(u, k)$ and the relation $sc(u, 0) = \tan(u)$.

The proof of *Theorem 2.2* is similar to *Theorem 2.1*, except that we start with $sc^2(u, k)dn^2(u, k)$.

Our proofs of the Kac and Wakimoto conjectures do not require inclusion/exclusion, and the analysis involving Schur functions is simpler than in those in Eqs. **30** and **31**.

We have (elsewhere) written down the $n = 3$ cases of *Theorems 2.3* and *2.4* which yield explicit formulas for 36 and 48 squares, respectively.

Acknowledgments

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Footnotes

*Gustafson, R. A., Ramanujan International Symposium on Analysis, Dec. 26–28, 1987, Pune, India, pp. 187–224.

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(2,1,{11/4,13/4},1)

Degree 2, Level 1, Weight 12

Ramanujan tau L-function

- 1000 tau coefficients
- Special values
- Theta_p
- Comparison between calculated Theta_p and predicted Sato-Tate
- A plot of Z[t] on [0,25],[0,50]

Level 1 , Degree 2 , Start

The simplest L-function of degree 2 corresponds to the first cuspform one encounters on the full modular group; it has weight 12. It is called Δ and is the generating function for the famous Ramanujan tau function:

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n.$$

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

$$\begin{aligned}\Delta(z) &= \frac{E_4(z)^3 - E_6(z)^2}{1728} \\ &= q - 24\,q^2 + 252\,q^3 - 1472\,q^4 + 4830\,q^5 - \\ &\quad 6048\,q^6 - 16744\,q^7 + 84480\,q^8 - 113643\,q^9 - 115920\,q^{10} + \\ &\quad 534612\,q^{11} - 370944\,q^{12} - 577738\,q^{13} + \\ &\quad 401856\,q^{14} + 1217160\,q^{15} + 987136\,q^{16} - \\ &\quad 6905934\,q^{17} + 2727432\,q^{18} + 10661420\,q^{19} - \\ &\quad 7109760\,q^{20} - 4219488\,q^{21} - 12830688\,q^{22} + \\ &\quad 18643272\,q^{23} + 21288960\,q^{24} - 25499225\,q^{25} + \\ &\quad 13865712\,q^{26} - 73279080\,q^{27} + 24647168\,q^{28} + \\ &\quad 128406630\,q^{29} - 29211840\,q^{30} + \ldots\end{aligned}$$

Lehmer's unproved conjecture is that $\tau(n)$ is never 0.

Level 1, Degree 2, Start

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My Travel Schedule
Sage -- Free Open Source Mathematics Software

Math 581F: Algebraic Number Theory
Seminars: Number Theory, Sage, and Crypto
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My Papers and Books
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The Modular Forms Database
SAGE, MECCAH and Neron (computers)

Personal Info
My Photos and Video

Recordings SGA Mathematics Gadgets
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Much of my work is based upon work supported by the National Science Foundation under Grant No. 0555776. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

wing mountain

The Modular Forms Database

The version of these tables at Harvard with dynamic content work much better.

Related data about elliptic curves, abelian varieties, etc.

William A. Stein

Collection of Tables Related to Modular Forms, Elliptic Curves, and Abelian Varieties
The Modular Forms Explorer (PostgreSQL)
The Interactive Modular Forms Explorer (uses ZOPE)
The Stein-Watkins tables of elliptic curves (see also the paper).

Elliptic Curves Information Calculator
PARI & MAGMA Calculator

Bibtex Reference for the Database
Talks About the Database
Lecture Notes
Software

If you have any questions or comments, please send me email was@math.harvard.edu.

In order to make these tables, I made extensive use of MAGMA, C++, LiDIA, and PARI, and greatly appreciate the work of the designers of these packages. If you find these tables useful when writing a paper, please feel free to acknowledge them; however, I don't require this. If you are seriously going to use some of these computations in a paper, it would be very wise to email me so I can double check their accuracy using my newest software.

Some computational resources used to make these tables:

1. **Modular -- A dual opteron Sun Fire V20Z Server with 8GB RAM acquired using my NSF grant.**
2. **Neron: A Sun Fire V480 Server with 22GB RAM acquired using a Sun Education Grant.**
3. **Meccah: a cluster of 6 dual Athlons funded by W.R. Hearst III and Harvard.**
4. **Chaucer: Mark Watkins's dual athlon at Penn State**
5. **Modular: A dual pentium 933 that Barry Mazur purchased for me. This is the machine that the databases**

and web pages currently run on. Joe Harris also purchased two 120GB hard drives for modular.

6. Kevin Buzzard's computer crackpipe, at Imperial College, UK.
7. The Berkeley cluster of 3 dual celerons that Wayne Whitney built.
8. The MAGMA group's Sun Enterprise E450 at University of Sydney.

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If you did not find what you are looking for in this index,
you can always **search the database for a particular word,**
or even look at the **full database**.

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AT&T Labs Research

Note on the Tau Function
N. A. Carella, September 2005.

Abstract: This note proposes an improved upper bound of the coefficients $\tau(n)$ of the modular function $\Delta(z)$ using elementary method. It improves a well known estimate of Deligne on the tau function $\tau(n)$.

The tau function $\tau(n)$ is defined as nth coefficient of the Fourier series

$$f(z)=\sum_{n=1}^{\infty}\tau(n)q^n = q-24q^2+252q^3-1472q^4+4830q^5-6048q^6+\cdots,$$

where $q = e^{i2\pi z}$, $z \in \mathbb{C}$. The function $f(z)$ is identical (up to a constant) to the discriminant modular form

$$\Delta(z)=(2\pi)^{12}f(z)=g_2(z)^3-27g_3(z)^2=(2\pi)^{12}q\prod_{n=1}^{\infty}(1-q^n)^{24},$$

which is a cusp form of weight $2k = 12$. The corresponding Dirichlet series and Euler product expansions are

$$L(s)=\sum_{n=1}^{\infty}\frac{\tau(n)}{n^s}=\prod_{\text{prime } p}(1-\tau(p)p^{-s}+p^{11-2s})^{-1}.$$

The tau function satisfies the identities

$\tau(p^n) = \tau(p)\tau(p^{n-1}) - p^{11}\tau(p^{n-2})$, $\quad$ p prime,

$\tau(mn) = \tau(m)\tau(n)$, $\quad$ $\gcd(m, n) = 1$.

Let $f(z)=\sum_{n=1}^{\infty}c(n)q^n$ be the Fourier series of a modular function of weight $2k \geq 2$. An earlier work on the magnitudes of the coefficients $c(n)$ states the following.

***Theorem* 1.** (Hecke 1939) The coefficients $c(n)$ of the Fourier series of a cusp form of weight $2k$ satisfy $|c(n)| \leq cn^k$, where $c > 0$ is a constant, and $n \geq 1$.

The proof based on elementary method is well known, see [5], [6, p. 239]. Subsequent authors have obtained better estimates of order of magnitudes close to $O(n^{k-1/4+\varepsilon})$, see [1,

p. 136]. Further, the work of Deligne on the Riemann hypothesis for varieties over finite fields improves this estimate to $| c(n) | \leq \sigma_0(n)n^{k-1/2}$, see [2], [6, p. 239]. In particular, there is the following well known result, which proved a conjecture of Ramanujan.

***Theorem* 2.** (Deligne 1969) For all $n \geq 1$, $| \tau(n) | \leq \sigma_0(n)n^{11/2}$, where $\sigma_0(n)$ is the number of divisors of n.

This estimate has not been improved since its introduction, see [9, p. 31], [3, p. 297] and similar sources. The new estimate is a straight forward application of the formula below, and a sharp estimate of the divisor function.

***Theorem* 3.** ([8]) For all $n \geq 1$ the tau function is given by

$$\tau(n) = n^4\sigma(n) - 24\sum_{k=1}^{n-1}(35k^4 - 52k^3n + 18k^2n^2)\sigma(k)\sigma(n-k).$$

The divisor function $\sigma(n)$ has an average value of $\pi^2 n/6$, this follows from its power series expansion, and the asymptotic value $e^\gamma n\log\log(n)$, see [2].

***Theorem* 4.** ([7], [10]) Let $n \geq 3$, then $\sigma(n) < e^\gamma n\log\log n + \dfrac{.6482n}{\log\log n}$. In particular, $\sigma(n) < cn\log\log(n)$, where $c > 0$ an absolute constant.

The proposed improvement is the following.

***Theorem* 5.** For all $n \geq 1$, $| \tau(n) | \leq (2c^2 + c)n^5\log\log(n)^2$.

Proof: Applying $\sigma(n) < cn\log\log(n)$ in the tau formula yields

$$\begin{aligned}
|\tau(n)| &\leq n^4\sigma(n) + 24\left|\sum_{k=1}^{n-1}(35k^4 - 52k^3n + 18k^2n^2)\sigma(k)\sigma(n-k)\right| \\
&\leq cn^5\log\log(n) + 24\left|c^2\sum_{k=1}^{n-1}(35k^4 - 52k^3n + 18k^2n^2)k(n-k)\log\log(k)\log\log(n-k)\right| \\
&\leq cn^5\log\log(n) + 24c^2\log\log(n)^2\left|\sum_{k=1}^{n-1}(18k^3n^3 - 70k^4n^2 + 87k^5n - 35k^6)\right| \\
&\leq (2c^2 + c)n^5\log\log(n)^2.
\end{aligned}$$

The last sum is a linear combination of elementary power sums

$$S_t(n) = \sum_{k=1}^{n-1}k^t = c_t n^{t+1} + c_{t-1}n^t + \cdots + c_1 n + c_0,$$

where the c_i are rational constants depending on $t > 0$.

The partial sum of the coefficients of a modular function of weight $2k$ is known to satisfy the asymptotic expression

$$\sum_{n \le N} |c(n)|^2 \sim cN^{2k}$$

as $N \to \infty$, with $c > 0$ constant, see [5, p. 71].

The new estimate appears to give either a counterexample or an improvement of the partial sum of coefficients of the modular function $\Delta(z)$ of weight 12. Specifically, the partial sum of the tau function satisfies the relations

$$\sum_{n \le N} |\tau(n)|^2 \le c_0 \sum_{n \le N} | \, n^5 (\log\log n)^2 \, |^2 \le c_0 N^{11} (\log\log N)^4 \; \nsim cN^{12}.$$

.

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My Favorite Integer Sequences

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Abstract. This paper gives a brief description of the author's database of integer sequences, now over 35 years old, together with a selection of a few of the most interesting sequences in the table. Many unsolved problems are mentioned.

This paper was published (in a somewhat different form) in *Sequences and their Applications (Proceedings of SETA '98)*, C. Ding, T. Helleseth and H. Niederreiter (editors), Springer-Verlag, London, 1999, pp. 103-130. Enhanced pdf version prepared Apr 28, 2000.

The paragraph on "Sorting by prefix reversal" on page 7 was revised Jan. 17, 2001.

1 How it all began

I started collecting integer sequences in December 1963, when I was a graduate student at Cornell University, working on perceptrons (or what are now called neural networks). Many graph-theoretic questions had arisen, one of the simplest of which was the following.

Choose one of the n^{n-1} rooted labeled trees with n nodes at random, and pick a random node: what is its expected height above the root? To get an integer sequence, let a_n be the sum of the heights of all nodes in all trees, and let $W_n = a_n/n$. The first few values $W_1, W_2, \ldots$ are

$$0, 1, 8, 78, 944, 13800, 237432, \ldots,$$

a sequence engraved on my memory. I was able to calculate about ten terms, but I needed to know how W_n grew in comparison with n^n, and it was impossible to guess this from so few terms. So instead I tried to guess a formula for the nth term. Again I was unsuccessful. Nor could I find this sequence in Riordan's book [86], although there were many sequences that somewhat resembled it.

So I started collecting all the sequences I could find, entering them on punched cards, thinking that if any of these sequences came up in another problem, at least I would know what *they* were.

I never did find that sequence in the literature, but I learned Pólya's theory of counting and (with John Riordan's help) obtained the answer, which appears in [87]. There *is* a simple formula, although maybe not simple enough to be guessed:

$$W_n = (n-1)! \sum_{k=0}^{n-2} \frac{n^k}{k!} \ .$$

An old formula of Ramanujan [99] then implies that

$$\frac{W_n}{n^n} \sim \sqrt{\frac{2\pi}{n}}, \quad \text{as} \quad n \to \infty \ ,$$

which was what I needed. That sequence became number A435 in the collection.

The idea of a "dictionary" of integer sequences was received with enthusiasm by many people, and in 1973 I published [91], containing about 2400 sequences, arranged lexicographically. One correspondent commented on the book by saying "There's the Old Testament, the New Testament and the Handbook of Integer Sequences".

Over the next twenty years an enormous amount of new material arrived, preprints, reprints, postcards, typewritten letters, handwritten letters, etc., and it was not until 1995 that — with Simon Plouffe's help — a sequel [93] appeared. This contained 5500 sequences.

Around the same time I set up two services that can be used to consult the database via electronic mail. The first has the address sequences@research.att.com, and simply looks up a sequence in the table. The second email address, which is superseeker@research.att.com, tries hard to find an explanation even for a sequence not in the table.

A large number of new sequences started arriving as soon as [93] appeared, and when in 1996 the total number reached 16,000, three times the number in the book, I decided to set up a web site for the database [92]. New sequences still continue to pour in, at about 10000 per year. At present, in April 1999, the database contains about 50000 sequences, and the web site receives over 2500 hits per day.

The main reason for this rapid expansion is that in the two books I only included sequences that had been or were about to be published. For the on-line version, where storage space is no longer a limitation, any well-defined and sufficiently interesting sequence is eligible for inclusion.

There is now also an electronic *Journal of Integer Sequences* [40] and a mailing list for sequence fans [89].

The following sections describe how the database is used (Section 2) and the kinds of sequences it contains (Section 3). Section 4 discusses a few "hard" sequences and Section 5 some recursive examples. Then Sections 6–8 describe sequences associated with meandering rivers and stamp-folding, extremal codes and lattices, and Levine's sequence.

Besides integer sequences, the table also contains many examples of arrays of numbers, Pascal's triangle, for example — see Section 9. The final three sections discuss sequences associated with the Wythoff array, the boustrophedon transformation of sequences, and Tchoukaillon solitaire.

2 How the database is used

The main applications of the database are in identifying sequences or in finding out the current status of a known sequence.

The database has been called a mathematical analogue of a "fingerprint file" [10]. You encounter a number sequence, and wish to know if anyone has ever come across it before. If your sequence is in the database, the reply will provide a description, the first 50 or so terms (usually enough to fill three lines on the screen), and, when available, formulae, recurrences, generating functions, references, computer code for producing the sequence, links to relevant web sites, etc.

Let me illustrate how the database is used with a typical story. Last summer the following question arose at AT&T Labs in connection with a quantization problem [90]. Given an n-dimensional lattice Λ, for which integers N does Λ have a sublattice of index N that is geometrically similar to N?

For the two-dimensional root lattice A_2, for example, it is easy to see that a necessary and sufficient condition is that N be of the form $a^2 + ab + b^2$.

However, for the four-dimensional lattice A_4 the situation is more complicated. The first thing we did was to run a computer search, which showed that A_4 has a similar sublattice of index $N = c^2$ if and only if c is one of the numbers

$$1, 4, 5, 9, 11, 16, 19, 20, 25, 29, 31, 36, \ldots$$

This turned out to be sequence A31363 in the database, with a reference to Baake [3], where it appears in an apparently different context as the indices of coincidence site sublattices in a certain three-dimensional quasicrystal. [3] identifies these numbers as those positive integers in which all primes congruent to 2 or 3 (modulo 5) appear to an even power. This was a very useful hint in getting started on our problem. We were able to show that this is the correct condition for the A_4 lattice, and to find analogous results for a number of other lattices [15]. (However, we have not yet found a direct connection between A_4 and the quasicrystal problem. Nevertheless, the occurrence of the same numbers in the two problems cannot be entirely coincidental.)

My files contain many similar stories. Some other examples can be found in Chapter 3 of [93].

Most of the applications however are less dramatic. One encounters a sequence in the middle of a calculation, perhaps

$$1,1,2,3,6,10,20,35,70,126,\ 252,\ \ldots,$$

and one wants to know quickly what it is — preferably a formula (in this case it is $\binom{n}{[n/2]}$) or generating function.

It is worth emphasizing a special case of this: the simplification of binomial coefficient sums. Powerful methods are available for simplifying such sums by computer [75], [77]. But if one is in a hurry, one can first try evaluating the initial terms and looking up the sequence in the table. E.g. the sum

$$a(n) = \sum_{k=0}^{n} \binom{2n-2k}{n-k}^2 \binom{2k}{k}$$

produces

$$1, 8, 88, 1088, 14296, 195008, \ldots$$

(A36917), and the entry in the table supplies a recurrence

$$n^3 a(n) = 16\left(n - \frac{1}{2}\right)(2n^2 - 2n + 1)a(n-1) - 256(n-1)^3 a(n-2)$$

and a reference ([77], p. ix, although the sum is misstated in the first printing). I have begun entering into the database all the sequences corresponding to the singly-indexed identities in Gould's table [32] of binomial coefficient identities.

A related application is in identifying arithmetic inequalities. Suppose you suspect that

$$\sigma(n) \geq d(n) + \phi(n), \quad \text{for} \quad n \geq 2\ , \tag{1}$$

where σ, d and ϕ are respectively the sum of divisors, number of divisors, and Euler totient functions. You could evaluate the sequence formed by the difference of the two sides, which for $n \geq 1$ is

$$-1, 0, 0, 2, 0, 6, 0, 7, 4, 10, 0, 18, 0, 14, 12, \ldots$$

(A46520). The table then points you to a reference ([73], §I.3.1) where this is stated as a theorem. (Again I would like to get more examples of such sequences.)

Another important use for the database is in finding out the present state of knowledge about some problem. A second story will illustrate this. The number of Latin squares of order n (Q_n) is given by sequence A315 (see Fig. 1). Computing Q_n is one of the famous hard problems in combinatorics.

n	Q_n
1	1
2	1
3	1
4	4
5	56
6	9408
7	16942080
8	535281401856
9	377597570964258816
10	7580721483160132811489280

Fig. 1. Sequence A315, the number of Latin squares of order n (McKay and Rogoyski [71]).

In 1991 Brendan McKay, at the Australian National University in Camberra, computed Q_{10} (which as can be seen is a rather large number). When he checked the database he found that the same value had recently been obtained by Eric Rogoyski of Cadence Design Systems in San Jose, California. As it turned out the two methods were similar but not identical, and he and Rogoyski ended up writing a joint paper [71], acknowledging the database for bringing them together.

3 Types of Sequences

The database contains sequences from all branches of science, including

- enumeration problems (combinatorics, graph theory, lattices, etc.)
- number theory (number of solutions to $x^2 + y^2 + z^2 = n$, etc.)
- game theory (winning positions, etc.)
- physics (paths on lattices, etc.)
- chemistry (sizes of clusters of atoms, etc.)
- computer science (number of steps to sort n things, etc.)
- communications (m-sequences, weight distributions of codes, etc.)
- puzzles
- etc.

To be accepted into the database, a sequence must be well-defined and interesting. However, when in doubt, my tendency is to accept rather than to reject. The amazing coincidences of the Monstrous Moonshine investigations [14] make it difficult to say that a particular sequence, no matter how obscure, will never be of interest.

Readers are urged to send me any sequences they come across that are not at present in the database. There is a convenient electronic form for this purpose in [92]. Some of the reasons for sending in your sequence are as follows:

- this stakes your claim to it
- your name is immortalized
- the next person who comes across it will be grateful and, not least,
- you may benefit from this yourself, when you come across the same sequence some weeks from now.

Often one finds that a particular project may involve dozens of sequences, all variants of a few basic ones. Ideally you should send them all to the database!

4 Hard sequences

One of the keywords used in the database is “hard”, which indicates that the term following those given is not known. Besides the Latin squares problem mentioned above, some other classic hard sequences are the following:

Projective planes. The number of projective planes of orders $n = 2, 3, \ldots, 10$ (A1231):

$$1, 1, 1, 1, 0, 1, 1, 4, 0,$$

where the last term refers to the result of Clement Lam et al. [58] (completing work begun in [66]) that there is no projective plane of order 10.

The Poincaré conjecture. The number of differential structures on the n-sphere, for $n = 1, 2, \ldots, 16$ (A1676):

$$1, 1, 1?, 1, 1, 1, 28, 2, 8, 6, 992, 1, 3, 2, 16256, 2,$$

as given by Kervaire and Milnor [41]. The *Poincaré conjecture* is that the third term is 1.

Dedekind's problem. The number of monotone Boolean functions of n variables, for $n = 1, 2, 3, \ldots, 8$ (A7153):

$$1, 4, 18, 166, 7579, 7828352, 2414682040996,$$
$$56130437228687557907786$$

where the last term was computed by Wiedemann [100]. (As is the case for many of these examples, there are several other versions of this sequence in the database.)

The Hadamard maximal determinant problem. What is the maximal determinant of an $n \times n$ $\{0, 1\}$-matrix? The values for $n = 1, 2, \ldots, 13$ are (A3432):

$$1, 1, 2, 3, 5, 9, 32, 56, 144, 320, 1458, 3645, 9477,$$

where the last two terms are due to Ehlich, and Ehlich and Zeller [25], [26]. For $n \equiv -1 \pmod 4$, Hadamard showed that the nth term is equal to

$$(n + 1)^{(n+1)/2}/2^n \ ,$$

provided that what is now called a Hadamard matrix of order n exists. In some cases conference matrices give the answer when $n \equiv 1 \ (\bmod\ 4)$, but the problem of finding the other terms in the sequence seems to have been untouched for over 35 years. It would be nice to have confirmation of the above values as well as some more terms!

Enumerating Hadamard matrices. The number of Hadamard matrices of orders $n = 4, 8, 12, \ldots, 28$ (A7299) is

$$1, 1, 1, 5, 3, 60, 487,$$

where the last entry is the work of Kimura [50], [51], [52], [53]. The *Hadamard conjecture* is that such a matrix always exists if n is a multiple of 4. Judging by how rapidly these numbers are growing, this should not be hard to prove, yet it has remained an open question for a century. Of course, as the example in Section 7 shows, such numerical evidence can be misleading.

The kissing number problem. How many spheres can touch another sphere of the same size? For arrangements that occur as part of a lattice packing, the answers are known for $n = 1, 2, \ldots, 9$ (A1116):

$$2, 6, 12, 24, 40, 72, 126, 240, 272,$$

the last term being due to Watson (see [16]). For nonlattice packings, all we know is

$$2, 6, 12, ?, ?, ?, ?, 240, \geq 306 \ .$$

The best bounds known in dimensions 4, 5, 6 and 7 are respectively

24–25, 40–46, 72–82   and   126–140

— see [16] for further information.

Sorting by prefix reversal. If you can only reverse segments that include the initial term of the current permutation, how many reversals are needed to transform an arbitrary permutation of n letters to the identity permutation? To state this another way [24]:

> The chef in our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes. Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest at the bottom) by grabbing several from the top and flipping them over, repeating this (varying the number I flip) as many times as necessary. If there are n pancakes, what is the maximum number of flips (as a function $f(n)$ of n) that I will ever have to use to rearrange them?

The only exact values known are $f(1), \ldots, f(9)$:

$$0, 1, 3, 4, 5, 7, 8, 9, 10,$$

due to Garey, Johnson and Lin, and Robbins (see [24], [29]). It is also known that $f(n) \geq n+1$ for $n \geq 6$, $f(n) \geq 17n/16$ if n is a multiple of 16 (so $f(32) \geq 34$), and $f(n) \leq (5n+5)/3$, the last two bounds[1] being due to Gates and Papadimitriou [29]. Again it would be nice to have more terms.

[1] (Added May 5, 2000.) The bound $f(n) \leq (5n+5)/3$ was independently obtained by E. Györi and G. Turán, Stack of pancakes, *Studia Sci. Math. Hungar.*, **13** (1978), 133–137. I thank László Lovász for pointing out this reference.

Note added Jan. 17, 2001. John J. Chew III (Department of Mathematics, University of Toronto) has found that $f(10)$ through $f(13)$ are $11, 13, 14, 15$, respectively, so the sequence begins:

$$0, 1, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15$$

This has now been added to the database as sequence A058986.

5 Recursive sequences

Whereas the sequences in the previous section enumerated some class of objects, the following are self-generated.

Differences = complement (A5228):

$$1, 3, 7, 12, 18, 26, 35, 45, 56, 69, 83, 98, 114, \ldots$$

The differences (A30124)

$$2, 4, 5, 6, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, \ldots$$

are the terms not in the sequence! This is one of many fine self-generating sequences from Hofstadter [38].

Golomb's sequence. The nth term is the number of times n appears (A1462):

$$1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, 7, 8, \ldots$$

The nth term is the nearest integer to (and converges to)

$$\tau^{2-\tau} n^{\tau-1} ,$$

where $\tau = (1 + \sqrt{5})/2$ [31], [33, Section E25].

Wilson's primeth recurrence. a_{n+1} is the a_n-th prime (A7097), shown in Fig. 2. The sequence was sent in by R. G. Wilson V [102], and the last few terms were computed by P. Zimmermann [103] and M. Deléglise [18]. Their algorithm is a slightly speeded up version of an algorithm for computing $\pi(x)$, the number of primes not exceeding x, due to J. C. Lagarias, V. S. Miller and A. M. Odlyzko (see [57]). It is quite remarkable that it is possible to compute so many terms of this sequence.

Recamán's sequences. (i) $a_n = a_{n-1} - n$ if $a_{n-1} - n > 0$ and $a_{n-1} - n$ has not already occurred in the sequence, otherwise $a_n = a_{n-1} + n$ (A5132):

$$1, 3, 6, 2, 7, 13, 20, 12, 21, 11, 22, 10, 23, 9, \ldots$$

(ii) $a_{n+1} = a_n/n$ if n divides a_n, otherwise $a_{n+1} = na_n$ (A8336):

$$1, 1, 2, 6, 24, 120, 20, 140, 1120, 10080, \ldots$$

1
2
3
5
11
31
127
709
5381
52711
648391
9737333
174440041
3657500101
88362852307
2428095424619
75063692618249
2586559730396077

Fig. 2. a_{n+1} is the a_n-th prime.

These were sent in by B. Recamán [84]. How fast do they grow?

The $10,000 sequence. In a colloquium talk at AT&T Bell Labs [12], John Conway discussed the sequence (A4001)

$$1, 1, 2, 2, 3, 4, 4, 4, 5, 6, 7, 7, 8, 8, 8, 8, 9, \ldots$$

defined by (for $n \geq 3$)

$$a(n+1) = a(a(n)) + a(n+1-a(n)) \ .$$

(In words, $a(n+1)$ is the $a(n)$th term in from the left plus the $a(n)$th term in from the right.) This sequence seems to have been introduced by either David Newman or Douglas Hofstadter around 1986. In his talk, Conway said that he could prove that $\frac{a(n)}{n} \to \frac{1}{2}$, and offered $10,000 for finding the exact n at which $\left| \frac{a(n)}{n} - \frac{1}{2} \right|$ last exceeds $\frac{1}{20}$. My colleague Colin Mallows did not take long to analyze the sequence, and came up with an answer of 6083008742 [67].

Colin tells me that in fact the problem is actually much easier than either he or John Conway had believed, and the true answer is 1489. A recent paper by Kubo and Vakil [56] also studies this sequence and its generalizations. See also [33, Section E31].

Many variants of this sequence have not yet been analyzed. Even the rate of growth of this one is not known: $a(1) = a(2) = 1$, $a(n) = a(a(n-2)) + a(n-a(n-2))$, A5229 [67], which begins

$$1, 1, 2, 3, 3, 4, 5, 6, 6, 7, 7, 8, 9, 10, 10, 10, 11, 12, 12, \ldots$$

The Prague clock sequence. This is not really recursive in the same sense as the preceding sequences, but it seems to fit in here. It is, in any case, delightful. This is A28354: 1, 2, 3, 4, 32, 123, 43, 2123, 432, 1234, 32123, 43212, 34321, 23432, 123432, 1234321, 2343212, 3432123, 4321234, 32123432, 123432123, 43212343, 2123432123, 432123432, 1, 2, 3, 4, 32, ... According to [39], the sequence indicates how the astronomical clock in Prague strikes the hours. There is a single bell which (i) makes from 1 to 4 strokes at a time, (ii) the number of strokes follows the sequence

$$\ldots 3212343212343 \ldots ,$$

(iii) at the nth hour, for $n = 1, 2, \ldots, 24$, the strokes add to n, and (iv) at the 25th hour there is a single stroke (so the sequence has period 24). As the reader will see by studying the sequence, its existence depends on two coincidences!

Cayley's mistake. Since the sequences in the database are numbered A1, A2, A3, ..., several people humorously proposed the "diagonal" sequence (now A31135) in which the nth term is equal to the nth term of An:

$$1, 2, 1, 0, 2, 3, 0, 6, 8, 4, 63, 1, 316, 42, 16, \ldots,$$

and the even less well-defined sequence (now A37181) with nth term equal to A31135$(n) + 1$. I resisted adding these sequences for a long time, partly out of a desire to maintain the dignity of the database, and partly because A22 was only known to 11 terms!

Sequence A22 gives the number of "centered hydrocarbons with n atoms", and is based on an 1875 paper of Cayley [9]. The paper is extremely hard to follow, and gives incorrect values for $n = 12$ and 13. The errors it contains were reproduced in [7], [91] and [93], even though Herrmann [37] had pointed out these errors in 1880. As far as we can tell, a correct verion of this sequence was never published until Eric Rains and I did so in 1999 [83]. Although Henze and Blair, Pólya and many others have written articles enumerating related families of chemical compounds (see [83] for a brief survey), this sequence seems to have been forgotten for over 100 years.

Once we determined what it was that Cayley was trying to enumerate, Pólya's counting theory quickly gave the answer, and the correct version of A22 is now in the database (as are the two diagonal sequences mentioned above – mostly because users of the database kept proposing them).

6 Meanders and stamp-folding

The meandric and stamp-folding numbers are similar to the better-known Catalan numbers (sequence A108) in that they are fundamental, easily described combinatorial quantities that arise in many different parts of mathematics, but differ from them in that there is no known formula, and in fact seem to be quite hard to compute.

The stamp-folding problem has a history going back to at least Lucas [63] in the nineteenth century, while the meandric problem seems to have been first mentioned by Poincaré [80]. The meandric problem asks: in how many different ways can a river (starting in the South-West and flowing East) cross a road n times? For $n = 5$ crossings there are $M_5 = 8$ possibilities, shown in Fig. 3. The sequence M_1, M_2, M_3, ... (A5316) begins

$$1, 1, 2, 3, 8, 14, 42, 81, 262, 538, 1828, \ldots$$

These are called *meandric* numbers, since the river *meanders* across the road. The even-numbered terms M_2, M_4, $M_6, \ldots$ give the number of different ways an

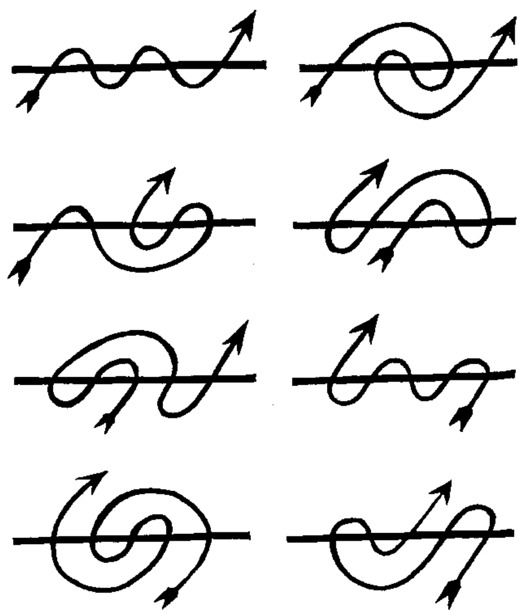

Fig. 3. The eight ways a river (going from South-West to North-East) can cross a road five times.

oriented line can cross a Jordan curve (A5315). There are several other interpretations, one of which is the number of "simple alternating transit mazes" [78], [79].

The stamp-folding problem asks the same question, but now the line is only semi-infinite. Equivalently, how many ways are there to fold a strip of n stamps? Pictures illustrating the first few terms of the stamp-folding sequence (A1011) can be found on the front cover of [91] and in Figure M4587 of [93]. The sequence begins

$$1, 1, 2, 5, 14, 38, 120, 353, 1148, 3527, \ldots$$

No polynomial-time algorithm is known for computing either sequence (on the other hand it is not known that such algorithms do not exist). The best algorithms known require on the order of nC_n steps, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number (A108): these algorithms are due to Koehler [55] for the stamp-folding problem and to Knuth and Pratt [54] and Reeds [85] for the meandric problem. They are too complicated to describe here.

Using these algorithms, Stephane Legendre [61] has extended the stamp-folding sequence to 26 terms and the meandric sequence to 25 terms. Knuth and Pratt [54] have computed 17 terms of the M_2, M_4, $M_6, \ldots$ subsequence. Lando and Zvonkin [59], [60] and Di Francesco, Golinelli and Guitter [21], [20], [22] have also studied these sequences.

The exact rate of growth of these sequences is not known. The best bounds for M_{2n} presently known appear to be due to Reeds, Shepp and McIlroy [85]. It is easy to see that M_{2n} is submultiplicative, and that $C_n \le M_{2n} \le C_n^2$, which implies that

$$\mu = \lim_{n\to\infty} M_{2n}^{1/n}$$

exists and satisfies $4 \le \mu \le 16$. In [85] it is shown that

$$8.8 \le \mu \le 13.01 \ .$$

Besides the papers already mentioned, the meandric and stamp-folding problems have recently been discussed by Arnold [1], Di Francesco [19], Harris [36], Lando and Zvonkin [59], [60], Lunnon [64], Sade [88], Smith [95] and Touchard [98].

7 Extremal codes and lattices

Let C be a binary linear self-dual code of length n in which the weight of every codeword is a multiple of 4, and let

$$W_C(x,y) = \sum_{c\in C} x^{n-wt(c)} y^{wt(c)}$$

be its weight enumerator. Examples are the Hamming code of length 8, with weight enumerator

$$f = x^8 + 14x^4y^4 + y^8 \ ,$$

and the Golay code of length 24, with weight enumerator

$$g = x^{24} + 759x^{16}y^8 + 2576x^{12}y^{12} + 759x^8y^{16} + y^{24} \ .$$

A remarkable theorem of Gleason says that the weight enumerator of any such code C is a polynomial in f and g. (References for this section are [16], [65], [82].) E.g. if C has length $n = 72$, its weight enumerator can be written as

$$W_C = a_0 f^9 + a_1 f^6 g + a_2 f^3 g^2 + a_3 g^3 \ ,$$

for rational numbers $a_0, \ldots, a_3$. If we choose these numbers so that the minimal distance of this (hypothetical) code C is as large as possible, we find that

$$W_C = 1 + 0x^4 + 0x^8 + 0x^{12} + 249849x^{16} + 18106704x^{20} + 462962955x^{24} + \cdots ,$$

so that C would have minimal distance 16 (the coefficients form sequence A18236). The coefficients in this "extremal" weight enumerator are all nonnegative integers, but whether a code exists with this weight enumerator is an important unsolved question.

One can perform this calculation for any length that is a multiple of 8, and in [70] it was shown that when as many initial terms as possible are set to zero, the next term in the extremal weight enumerator is always positive (so the minimal distance of the hypothetical extremal code is precisely $4[n/24] + 4$).

Lengths that are multiples of 24 are especially interesting. Figure 4 shows the leading term in the extremal weight enumerator for a binary self-dual code of length $n = 24m$ (this is sequence A34414). It is "obvious" that these numbers

n	Coefficient
0	1
24	759
48	17296
72	249849
96	3217056
120	39703755
144	481008528
168	5776211364
192	69065734464

Fig. 4. Number of codewords of minimal weight $n/6 + 4$ in extremal weight enumerator of length n.

are growing rapidly, and in fact it is shown in [70] that if $n = 24m$ then the leading coefficient is

$$\binom{24m}{5}\binom{5m-2}{m-1}/\binom{4m+4}{5} .$$

The next term in the extremal weight enumerator (the number of codewords of weight $4[n/24] + 8$) also grows rapidly — see Fig 5 (sequence A34415).

Again it is "obvious" that these numbers also grow rapidly, especially if one examines a more extensive table, where the number of digits continues to increase with each term. Yet — this shows the danger of drawing conclusions just from numerical data — it is proved in [69] that this sequence goes negative at $n = 3696$ and stays negative forever. (See Theorem 29 of [82] for more precise information.)

n	Coefficient
0	1
24	2576
48	535095
72	18106704
96	369844880
120	6101289120
144	90184804281
168	1251098739072
192	16681003659936

Fig. 5. Number of codewords of next-to-minimal weight $n/6 + 8$ in extremal weight enumerator of length n.

So codes corresponding to these extremal weight enumerators certainly do not exist for $n \geq 3696$ (since the weight enumerator of a genuine code must have nonnegative coefficients). They are known to exist for $n = 24$ (the Golay code) and 48 (a quadratic residue code), but every case from 72 to 3672 is open. For further details, and in particular for a description of the analogous situation when n is a multiple of 8 but not of 24, see [82].

The situation for lattices is similar. Consider an even unimodular lattice Λ of dimension n, with theta series

$$\Theta_\Lambda(q) = \sum_{u \in \Lambda} q^{u \cdot u} \ .$$

Examples are the eight-dimensional root lattice E_8, with theta series

$$\begin{aligned} f &= 1 + 240q^2 + 2160q^4 + 6720q^6 + \cdots \\ &= 1 + 240 \sum_{m=1}^{\infty} \sigma_3(m) q^{2m} \ , \end{aligned}$$

where $\sigma_3(m)$ is the sum of the cubes of the divisors of m, and the 24-dimensional Leech lattice, with theta series

$$\begin{aligned} g &= 1 + 196560q^4 + 16773120q^6 + 398034000q^8 + \cdots \\ &= f^3 - 720q^2 \prod_{m=1}^{\infty} (1 - q^{2m})^{24} \ . \end{aligned}$$

The Taylor series expansion of the last product defines the famous Ramanujan numbers. (The coefficients of f and g give sequences A4009 and A8408, respectively; the Ramanujan numbers form sequence A594.)

A theorem due essentially to Hecke says that the theta series of any such lattice Λ is a polynomial in f and g. We may then define extremal theta series just as we defined extremal weight enumerators. It is known [69] that the leading term in the extremal theta series is positive, but that again the next coefficient eventually becomes negative (even though the sequence begins with about 1700 exponentially growing terms). The dimension where the first negative coefficient appears is around 41000. The leading terms are shown in Fig. 6 (sequence A34597).

n	Coefficient
0	1
24	196560
48	52416000
72	6218175600
96	565866362880
120	45792819072000

Fig. 6. Number of shortest vectors u (with $u \cdot u = n/12 + 2$) in extremal theta series at dimension n.

Lattices corresponding to the extremal theta series exist for $n = 24$ (the Leech lattice) and 48 (at least three inequivalent lattices exist), but every case from 72 to about 41000 is open. The 24- and 48-dimensional extremal lattices provide record kissing numbers in those dimensions; it would be nice to have a 72-dimensional example.

8 Levine's sequence

In the summer of 1997 Lionel Levine [62] submitted a new sequence to the table, a sequence of such beauty that it immediately caught the attention of several of my colleagues. It is constructed via the array in Fig. 7. If a row of the array

1 1
1 2
1 1 2
1 1 2 3
1 1 1 2 2 3 4
1 1 1 1 2 2 2 3 3 4 4 5 6 7
1 1 1 1 1 1 1 2 2 2 2 2 2 3 3 3 3 3 4 4 4 4 5 5 5 5 6 6 6 7 7 7 8 8 9 9 10 11 12 13 14

Fig. 7. The array that produces Levine's sequence $1, 2, 2, 3, 4, 7, 14, \ldots$.

contains the numbers

$$a_1 \quad a_2 \quad a_3 \quad \ldots \quad a_{k-1} \quad a_k$$

then the next row contains

$$a_k \text{ 1's}, \quad a_{k-1} \text{ 2's}, \quad a_{k-2} \text{ 3's}, \ldots$$

Levine's sequence (A11784) is obtained by taking the last term in each row:

$$\begin{aligned}&1, 2, 2, 3, 4, 7, 14, 42, 213, 2837, 175450, 139759600, 6837625106787,\\ &266437144916648607844, 508009471379488821444261986503540, \ldots\end{aligned} \tag{2}$$

The terms grow unexpectedly rapidly! The nth term L_n is

(i) the sum of the elements in row $n-2$
(ii) the number of elements in row $n-1$
(iii) the last element in row n
(iv) the number of 1's in row $n+1$
...

Furthermore, if $s(n,i)$ denotes the sum of the first i elements in row n, then we have

(v) $L_{n+2} = s(n, L_{n+1})$

(vi) $L_{n+3} = \sum_{i=1}^{L_{n+1}} s(n,i)$

(vii) $L_{n+4} = \sum_{i=1}^{L_{n+1}} \binom{s(n,i)+1}{2}$.

The latter identity was found by Allan Wilks [101], who also found a more complicated formula for L_{n+5}, and used it to compute the last two terms shown in (2). No other terms are known!

As to the rate of growth, we have only a crude estimate. Bjorn Poonen and Eric Rains [81] showed that

$$\log L_n \sim c\tau^n \ , \tag{3}$$

where $\tau = (1+\sqrt{5})/2$. *Sketch of Proof.* (a) $L_{n+2} \le L_{n+1}L_n$, and so $\log L_n$ is bounded above by a Fibonacci-like sequence. (b) The sum of the $(n+1)$st row is at most

$$\binom{\left[\frac{L_{n+2}}{L_n}\right]+1}{2} L_n \ ,$$

which implies

$$\frac{L_{n+3}}{2L_{n+2}} \ge \frac{L_{n+2}}{2L_{n+1}} \frac{L_{n+1}}{2L_n}$$

and so $\log(L_{n+1}/2L_n)$ is bounded below by a Fibonacci-like sequence.

Colin Mallows [68] has determined numerically that a reasonably good approximation to L_n is given by

$$\frac{1}{c_1} e^{c_2 \tau^n}$$

where $c_1 \approx 0.277$, $c_2 \approx 0.05427$. It would be nice to have better estimates for L_n, and one or more additional terms.

9 Arrays of Numbers

Besides number *sequences*, the database also contains *arrays* of numbers that have been converted to sequences. Triangular arrays are read by rows, in the obvious way. E.g. Pascal's triangle of binomial coefficients

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

becomes sequence A7318:

$$1, 1, 1, 1, 2, 1, 1, 3, 3, 1, 1, 4, 6, 4, 1, \dots$$

Square arrays are read by antidiagonals, usually in this order:

$$
\begin{array}{l}
a_0\ a_2\ a_5\ a_9\ \dots \\
a_1\ a_4\ a_8\ \dots \\
a_3\ a_7\ \dots \\
a_6\ \dots \\
\dots
\end{array}
$$

E.g. the Nim-addition table [11]

$$
\begin{array}{l}
0\,1\,2\,3\,4\dots \\
1\,0\,3\,2\,5\dots \\
2\,3\,0\,1\,6\dots \\
3\,2\,1\,0\,7\dots \\
\cdot\ \cdot\ \cdot
\end{array}
$$

becomes sequence A3987:

$$0, 1, 1, 2, 0, 2, 3, 3, 3, 3, 4, 2, 0, 2, 4, \dots$$

Other classical arrays are the Stirling numbers of both kinds, Eulerian numbers, etc.

A less well-known array arises from **Gilbreath's conjecture**. This conjecture states that if one writes down the primes in a row, and underneath the absolute values of the differences, as in Fig. 8, then the leading terms (shown underlined) of all rows except the first are equal to 1 ([33], §A10). The corresponding sequence (A36262) is

$$2, 1, 3, 1, 2, 5, 1, 0, 2, 7, 1, 2, 2, 4, 11, 1, \dots$$

Odlyzko [76] has verified the conjecture out to 3×10^{11}.

2	3	5	7	11	13	17	19	23	...					
	<u>1</u>	2	2	4	2	4	2	4	...					
		<u>1</u>	0	2	2	2	2	2	...					
			<u>1</u>	2	0	0	0	0	...					
				<u>1</u>	2	0	0	0	...					
					<u>1</u>	2	0	0	...					
						...								

Fig. 8. Gilbraith's conjecture is that the leading terms of all rows in this array except the first are always 1 (the top row contains the primes, subsequent rows are the absolute values of the differences of the previous row).

10 The Wythoff array

This array shown in Fig. 9 has many wonderful properties, some of which are mentioned here. I learned about most of these properties from John Conway [13], but this array has a long history — see Fraenkel and Kimberling [28], Kimberling [43], [44], [45], [46], [47], [48], [49], Morrison [74] and Stolarsky [96], [97]. It is related to a large number of sequences in the database (the main entry is A35513).

0	1	1	2	3	5	8	13	21	34	55
1	3	4	7	11	18	29	47	76	..	
2	4	6	10	16	26	42	68	..		
3	6	9	15	24	39	63	..			
4	8	12	20	32	52	..				
5	9	14	23	37	60	..				
6	11	17	28	45	73	..				
7	12	19	31	50	81	..				
.	..	..	..	..	..	..				

Fig. 9. The Wythoff array.

Construction (1). The two columns to the left of the vertical line consist respectively of the nonnegative integers n, and the *lower Wythoff sequence* (A201), whose nth term is $[(n+1)\tau]$. The rows are then filled in by the Fibonacci rule that each term is the sum of the two previous terms. The entry n in the first column is the *index* of that row.

Definition. The *Zeckendorf expansion* of a number n is obtained by repeatedly subtracting the largest possible Fibonacci number until nothing remains. E.g. $100 = 89 + 8 + 3 = F_{11} + F_6 + F_4$. The *Fibonacci successor* to n, Sn, say, is found by replacing each F_i in the Zeckendorf expansion by F_{i+1}. E.g. the Fibonacci successor to 100 is $S100 = F_{12} + F_7 + F_5 = 144 + 13 + 5 = 162$.

Construction (2). The two columns to the left of the vertical line in Fig. 9 read n, $1 + Sn$; then after the vertical line the row continues

$$m \quad Sm \quad SSm \quad SSSm \quad SSSSm \quad \ldots ,$$

where $m = n + 1 + Sn$.

Construction (3). Let $\{S1, S2, S3, \ldots\} = \{2, 3, 5, 7, 8, 10, 11, \ldots\}$ be the sequence of Fibonacci successors (A22342). The first column to the right of the line consists of the numbers not in that sequence: 1, 4, 6, 9, 12, ... (A7067). The rest of each row is filled in by repeatedly applying S.

Construction (4). The entry in row n and column k is

$$[(n + 1)\tau F_{k+2}] + F_{k+1}n$$

(where $k = 0$ indicates the first column to the right of the vertical line).

Some properties of the array to the right of the line are the following:

(i) The first row consists of the Fibonacci sequence $1, 2, 3, 5, 8, \ldots$
(ii) Every row satisfies the Fibonacci recurrence.
(iii) The leading term in each row is the smallest number not found in any earlier row.
(iv) Every positive integer appears exactly once.
(v) The terms in any row or column are monotonically increasing.
(vi) Every positive Fibonacci-type sequence (i.e. satisfying $a(n) = a(n-1) + a(n-2)$ and eventually positive) appears as some row of the array.
(vii) The terms in any two adjacent rows alternate.

There are infinitely many arrays with Properties 1–7, see [47].

The nth term of the *vertical para-Fibonacci sequence*

$$0, 0, 0, 1, 0, 2, 1, 0, 3, 2, 1, 4, 0, 5, 3, 2, 6, 1, 7, 4, 0, 8, 5, \ldots$$

(A19586) gives the index (or parameter) of the row of the Wythoff array that contains n. This sequence also has some nice fractal-like properties:

(a) If you delete the first occurrence of each number, the sequence is unchanged. Thus if we delete the underlined numbers from

$$\underline{0}, 0, 0, \underline{1}, 0, \underline{2}, 1, 0, \underline{3}, 2, 1, \underline{4}, 0, \underline{5}, 3, 2, \underline{6}, 1, \underline{7}, 4, 0, \underline{8}, 5, \ldots$$

we get

$$0, 0, 0, 1, 0, 2, 1, 0, 3, 2, 1, 4, 0, 5, 3, 2, 6, 1, 7, 4, 0, 8, 5, \ldots$$

again!

(b) Between any two consecutive 0's we see a permutation of the first few positive integers, and these nest, so the sequence can be rewritten as (read across the rows):

0												
0												
0								1				
0					2			1				
0			3		2			1			4	
0		5	3		2		6	1		7	4	
0	8	5	3	9	2	10	6	1	11	7	4	12

The nth term of the *horizontal para-Fibonacci sequence*

$$1, 2, 3, 1, 4, 1, 2, 5, 1, 2, 3, 1, 6, 1, 2, 3, 1, 4, 1, 2, 7, 1, 2, \ldots$$

(A35612) gives the index (or parameter) of the column of the Wythoff array that contains n. This sequence also has some nice properties.

I hope I have said enough to convince you that the Wythoff array is well worth studying and full of surprises.

11 The Boustrophedon transform

The Taylor series for $\sin x$ and $\cos x$ are easily remembered, but most people have trouble with

$$\tan x = 1x + 2\frac{x^3}{3!} + 16\frac{x^5}{5!} + 272\frac{x^7}{7!} + \cdots ,$$
$$\sec x = 1 + 1\frac{x^2}{2!} + 5\frac{x^4}{4!} + 61\frac{x^6}{6!} + \cdots .$$

However, their coefficients can be calculated from the array in Fig. 10. The

							1							
						0		1						
					1		1		0					
				0		1		2		2				
			5		5		4		2		0			
		0		5		10		14		16		16		
	61		61		56		46		32		16		0	
0		61		122		178		224		256		272		272
							...							

Fig. 10. The secant-tangent triangle.

nonzero entries on the left are the *secant numbers* (A364):

$$1, 1, 5, 61, 1385, 50521, 2702765, \ldots,$$

and those on the right are the *tangent numbers* (A182)

$$1, 2, 16, 272, 7936, 353792, 22368256, \ldots$$

while the combination of the two sequences (A111):

$$1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, \ldots \tag{4}$$

are usually called the *Entringer numbers*. The latter count permutations of $\{1, 2, \ldots, n\}$ that alternately fall and rise.

This array is filled by a rule somewhat similar to that for Pascal's triangle: the rows are scanned alternately from right to left and left to right, the leading entry in each row is 0, and every subsequent entry is the sum of the previous entry in the same row and the entry above it in the previous row. (This is the *boustrophedon* or "ox-plowing" rule.) The earliest reference I have seen to this triangle is Arnold [2], who calls it the Euler-Bernoulli triangle. However, it may well be much older origin. [72] gives many other references.

Richard Guy [34] observed that if the entries at the beginnings of the rows are changed from $1, 0, 0, \ldots$ to say $1, 1, 1, 1, 1, \ldots$ or to $1, 2, 4, 8, 16, \ldots$ then the numbers that appear at the ends of the rows form interesting-looking sequences that were not to be found in [93], and asked if they had a combinatorial interpretation. Using $1, 1, 1, \ldots$ for example the triangle becomes

$$\begin{array}{ccccccccccc} &&&&& 1 &&&&& \\ &&&& 1 && 2 &&&& \\ &&& 4 && 3 && 1 &&& \\ && 1 && 5 && 8 && 9 && \\ & 24 && 23 && 18 && 10 && 1 & \\ 1 && 25 && 48 && 66 && 76 && 77 \\ &&&&& \cdots &&&&& \end{array}$$

yielding the sequence (A667)

$$1, 2, 4, 9, 24, 77, 294, 1309, \ldots \tag{5}$$

We may regard this process as carrying out a transformation (the *boustrophedon transform*) of sequences: if the numbers at the beginnings of the rows are $a_0,\ a_1,\ a_2, \ldots$ (the input sequence) then the numbers at the ends of the rows, $b_0,\ b_1,\ b_2, \ldots$ (say) are the output sequence. In [72] we showed that there is a simple relationship between the input and output sequences: their exponential generating functions

$$\mathcal{A}(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \quad \mathcal{B}(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}$$

are related by

$$\mathcal{B}(x) = (\sec x + \tan x)\mathcal{A}(x) \ .$$

We also give a combinatorial interpretation of the $\{b_n\}$. E.g. in (5), b_n is the number of up-down subsequences of $\{1, \ldots, n\}$, so that $b_3 = 9$ corresponds to $\emptyset, 1, 2, 3, 12, 13, 23, 132, 231$.

The Entringer sequence (4) then has the property that it shifts one place left under the boustrophedon transform. The lexicographically earliest sequence that shifts *two* places left under this transform (A661) is

$$1, 0, 1, 1, 2, 6, 17, 62, 259, 1230, 6592, \ldots$$

We do not know what this enumerates!

Many examples of similar "eigen-sequences" for other transformations of sequences can be found in Donaghey [23], Cameron [8], and especially [4]. E.g. the sequence giving the number of planted achiral trees [30], [35] (A3238):

$$1, 1, 2, 3, 5, 6, 10, 11, 16, 19, 26, \ldots$$

has the property that it shifts left one place under the "inverse Möbius transformation" given by

$$b_n = \sum_{d|n} a_d \ .$$

12 Tchoukaillon solitaire (or Mancala, or Kalahari)

These are ancient board games, with hundreds of variants and many different names. The version to be described here is called Tchoukaillon solitaire. It has been studied by several authors (see for example Betten [5] and Broline and Loeb [6]). It is played on a board with a row of holes numbered $0, 1, 2, \ldots$ (see Fig. 11).

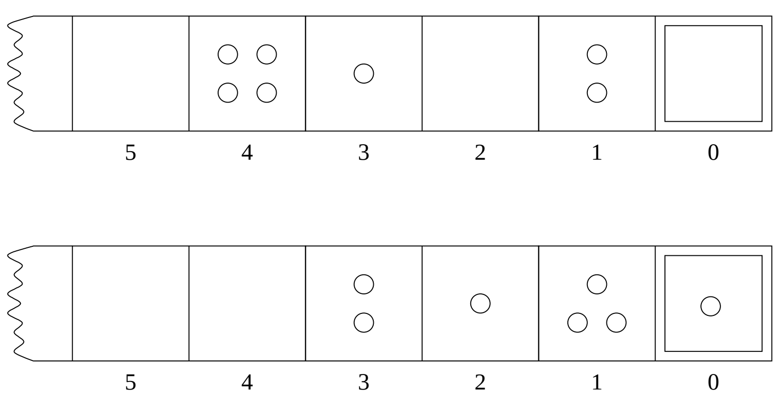

Fig. 11. A move in Tchoukaillon solitaire.

The game begins with n stones placed anywhere except in hole 0. A move consists in picking up the stones in some hole and placing one in each lower-numbered hole. If the last stone falls in hole 0 then play continues, otherwise the game is lost. The objective is to get all the stones into hole 0.

The game is interesting because there is a unique winning position for any number of stones. These winning positions are shown in Fig. 12 (sequence A28932), and can be found by playing the game backwards.

n	Position
0	0
1	1
2	20
3	21
4	310
5	311
6	4200
7	4201
8	4220
9	4221
10	53110
11	53111
12	642000
13	642001
..	...

Fig. 12. The unique winning position for n stones in Tchoukaillon solitaire.

The array can be more explicitly constructed by the rule that if the first 0 in a row (counting from the right) is in position i, then the next row is obtained by writing i in position i and subtracting 1 from all earlier positions. The sequence of successive values of i (A28920) is

$$1, 2, 1, 3, 1, 4, 1, 2, 1, 5, 1, 6, 1, 2, \ldots$$

Let $t(k)$ denote the position where k occurs for the first time in this sequence. The values of $t(1)$, $t(2)$, $t(3), \ldots$ are (sequence A2491):

$$1, 2, 4, 6, 10, 12, 18, 22, 30, 34, 42, \ldots$$

This sequence has some very nice properties. It has been investigated by (in addition to the references mentioned above) David [17], Erdős and Jabotinsky [27] and Smarandache [94].

(i) $t(n)$ can be obtained by starting with n and successively rounding up to the next multiple of $n-1$, $n-2, \ldots, 2, 1$. E.g. if $n = 10$, we obtain

$$10 \to 18 \to 24 \to 28 \to 30 \to 30 \to 32 \to 33 \to 34 \to 34\ ,$$

so $t(10) = 34$.

(ii) The sequence can be obtained by a sieving process: write $1, 2, \ldots$ in a column. To get the second column, cross off 1 and every second number. To get the third column, cross off the first and every third number. Then cross off the first and every fourth number, and so on (see Fig. 13). The top number in column n is $t(n)$. Comparison of Figures 12 and 13 shows that connection with the solitaire game.

1	~~1~~						
2	2	~~2~~					
3	~~3~~						
4	4	4	~~4~~				
5	~~5~~						
6	6	6	6	~~6~~			
7	~~7~~						
8	8	~~8~~					
9	~~9~~						
10	10	10	10	10	~~10~~		
11	~~11~~						
12	12	12	12	12	12	~~12~~	
13	~~13~~						
14	14	~~14~~					
15	~~15~~						
16	16	16	~~16~~				
17	~~17~~						
18	18	18	18	18	18	18	~~18~~
19	~~19~~						
20	20	~~20~~					

Fig. 13. A sieve to generate the sequence $t(1), t(2), \ldots = 1, 2, 4, 6, 10, 12, 18, \ldots$. At stage n, the first number and every nth are crossed off.

(iii) Finally, Broline and Loeb [6] (extending the work of the other authors mentioned) show that, for large n,

$$t(n) = \frac{n^2}{\pi} + O(n) \ .$$

It is a pleasant surprise to see π emerge from such a simple game.

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Arithmetic properties of the Ramanujan function

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Dedicated to T N Shorey on his sixtieth birthday

Abstract. We study some arithmetic properties of the Ramanujan function $\tau(n)$, such as the largest prime divisor $P(\tau(n))$ and the number of distinct prime divisors $\omega(\tau(n))$ of $\tau(n)$ for various sequences of n. In particular, we show that $P(\tau(n)) \geq (\log n)^{33/31+o(1)}$ for infinitely many n, and

$$P(\tau(p)\tau(p^2)\tau(p^3)) > (1+o(1))\frac{\log\log p \log\log\log p}{\log\log\log\log p}$$

for every prime p with $\tau(p) \neq 0$.

Keywords. Ramanujan τ-function; applications of $\mathscr{S}$-unit equations.

1. Introduction

Let $\tau(n)$ denote the *Ramanujan function* defined by the expansion

$$X\prod_{n=1}^{\infty}(1-X^n)^{24} = \sum_{n=1}^{\infty}\tau(n)X^n, \qquad |X| < 1.$$

For any integer n we write $\omega(n)$ for the number of distinct prime factors of n, $P(n)$ for the largest prime factor of n and $Q(n)$ for the largest square-free factor of n with the convention that $\omega(0) = \omega(\pm 1) = 0$ and $P(0) = P(\pm 1) = Q(0) = Q(\pm 1) = 1$.

In this note, we study the numbers $\omega(\tau(n))$, $P(\tau(n))$ and $Q(\tau(n))$ as n ranges over various sets of positive integers.

The following basic properties of $\tau(n)$ underline our approach which is similar to those of [9,13]:

- $\tau(n)$ is an integer-valued multiplicative function; that is, $\tau(m)\tau(n) = \tau(mn)$ if $\gcd(m,n) = 1$.
- For any prime p, and an integer $r \geq 0$, $\tau(p^{r+2}) = \tau(p^{r+1})\tau(p) - p^{11}\tau(p^r)$, where $\tau(1) = 1$.

In particular, the identity

$$\tau(p^2) = \tau(p)^2 - p^{11} \tag{1}$$

plays a crucial role in our arguments.

It is also useful to recall that by the famous result of Deligne

$$|\tau(p)| \le 2p^{11/2} \quad \text{and} \quad |\tau(n)| \le n^{11/2+o(1)} \tag{2}$$

for any prime p and positive integer n (see [7]).

One of the possible approaches to studying arithmetic properties of $\tau(n)$ is to remark that the values $u_r = \tau(2^r)$ form a Lucas sequence satisfying the following binary recurrence relation

$$u_{r+2} = -24u_{r+1} - 2048u_r, \qquad r = 0, 1, \ldots, \tag{3}$$

with the initial values $u_0 = 1$, $u_1 = -24$. By the primitive divisor theorem for Lucas sequences which claims that each sufficiently large term u_r has at least one new prime divisor (see [2] for the most general form of this assertion), we conclude that

$$\omega\left(\prod_{r\le z} \tau(2^r)\right) \ge z + O(1),$$

leading to the inequality

$$\omega\left(\prod_{\substack{n\le x\\ \tau(n)\neq 0}} \tau(n)\right) \ge \left(\frac{1}{\log 2} + o(1)\right)\log x$$

as $x \to \infty$. In particular, we derive that for infinitely many n,

$$P(\tau(n)) \ge \log n \log\log n.$$

A stronger conditional result, under the ABC-conjecture, is given in [10]. We also have

$$Q(\tau(n)) \ge n^{(\log 2 + o(1))/\log\log\log n}$$

for infinitely many n (see eq. (16) in [14]).

Furthermore, since $u_r | u_s$, whenever $r+1 | s+1$, it follows that if for sufficiently large s we set $k = \mathrm{lcm}[2, \ldots, s+1] - 1$, then $\tau(2^k)$ is divisible by $\tau(2^r)$ for all $r \le s$. Thus, setting $n = 2^k$ we get

$$\omega(\tau(n)) \ge s + O(1) = \left(\frac{1}{\log 2} + o(1)\right)\log k \ge \left(\frac{1}{\log 2} + o(1)\right)\log\log n$$

as $n \to \infty$. Here, we use different approaches to improve on these bounds.

Our results are based on some bounds for smooth numbers, that is, integers n with restricted $P(n)$ (see [5,16]). We also use results on $\mathscr{S}$-unit equations (see [3]). We recall that for a given finite set of primes $\mathscr{S}$, a rational $u = s/t \neq 0$ with $\gcd(s,t) = 1$ is called an $\mathscr{S}$-unit if all prime divisors of both s and t are contained in $\mathscr{S}$. Finally, we also use bounds on linear forms in q-adic logarithms (see [17]).

We recall that in [8] it is shown under the extended Riemann hypothesis that $\omega(\tau(p)) \sim \log\log p$ holds for almost all primes p and that $\omega(\tau(N)) \sim 0.5(\log\log N)^2$ holds for almost all positive integers N.

Throughout the paper, the implied constants in the symbols 'O', '$\gg$' and '$\ll$' are absolute (recall that the notations $U \ll V$ and $V \gg U$ are equivalent to the statement that $U = O(V)$ for positive functions U and V). We also use the symbol 'o' with its usual meaning: the statement $U = o(V)$ is equivalent to $U/V \to 0$.

We always use the letters p and q to denote prime numbers.

2. Divisors of the Ramanujan function

Theorem 1. *There exist infinitely many n such that $\tau(n) \neq 0$ and $P(\tau(n)) \geq (\log n)^{33/31+o(1)}$.*

Proof. For a constant $A > 0$ and a real z we define the set

$$\mathscr{S}_A(z) = \{n \leq z \colon P(n) \leq (\log n)^A\}.$$

For every $A > 1$, we have $\#\mathscr{S}_A(z) = z^{1-1/A+o(1)}$, as $z \to \infty$ (see eq. (1.14) in [5] or Theorem 2 in § III.5.1 of [16]).

Let $x > 0$ be sufficiently large. By a result of Serre [11], the estimate $\#\{p \leq y \colon \tau(p) = 0\} \ll y/(\log y)^{3/2}$ holds as y tends to infinity. Applying this estimate with $y = x^{1/2}$, it follows that there are only $o(\pi(y))$ primes $p < y$ such that $\tau(p) = 0$. It is also obvious from (1) that $\tau(p^2) \neq 0$.

Assume that for some A with $1 < A < 33/31$, we have the inequality $P(\tau(p)\tau(p^2)) \leq (\log y)^A$ for all remaining primes $p \leq y$. We see from (1) and (2) that $|\tau(p^2)| = |\tau(p)^2 - p^{11}| \leq 3p^{11} \leq 3y^{11}$. Denoting $z_1 = 3y^{11}$ and $z_2 = 2p^{11/2}$, we deduce that for $(1+o(1))\pi(y) = y^{1+o(1)}$ primes $p < y$ with $\tau(p) \neq 0$, we have a representation $p^{11} = s_1^2 - s_2$, where $s_i \in \mathscr{S}_A(z_i)$, $i = 1, 2$. Thus

$$y^{1+o(1)} \leq \#\mathscr{S}_A(z_1)\#\mathscr{S}_A(z_2) \leq (z_1 z_2)^{1-1/A+o(1)} \leq (6y^{33/2})^{1-1/A+o(1)},$$

which is impossible for $A < 33/31$. This completes the proof. □

We remark in passing that the above proof shows that the inequality $P(\tau(p)\tau(p^2)) > (\log p)^{33/31+o(1)}$ holds for almost all primes p.

Theorem 2. *The estimate*

$$\omega\left(\prod_{\substack{p<x^{1/3}\\ \tau(p)\neq 0}} \tau(p)\tau(p^2)\tau(p^3)\right) \geq \left(\frac{1}{6\log 7} + o(1)\right)\log x$$

holds as x tends to infinity.

Proof. Let x be a large positive integer and put $y = x^{1/3}$. Let $\mathscr{R}$ be the set of odd primes $p \leq y$ such that $\tau(p) \neq 0$. Note that since $\tau(p) \neq 0$, it follows that $\tau(p^2) \neq 0$ and $\tau(p^3) \neq 0$. Let

$$M = \prod_{p \in \mathscr{R}} \tau(p)\tau(p^2)\tau(p^3) \quad \text{and} \quad s = \omega(M).$$

Since $\tau(p^2) = \tau(p)^2 - p^{11}$ and $\tau(p^3) = \tau(p)(\tau(p)^2 - 2p^{11})$, eliminating p^{11}, we get the equation

$$1 = \frac{2\tau(p^2)}{\tau(p)^2} - \frac{\tau(p^3)}{\tau(p)^3}.$$

We claim that the rational numbers $2\tau(p^2)/\tau(p)^2$ are distinct for distinct odd primes. Indeed, if $\tau(p_1^2)/\tau(p_1)^2 = \tau(p_2^2)/\tau(p_2)^2$ for two distinct odd primes p_1, p_2, we get that $p_1^{11}/\tau(p_1)^2 = p_2^{11}/\tau(p_2)^2$, or $p_1^{11}\tau(p_2)^2 = p_2^{11}\tau(p_1)^2$. Therefore, $p_1^{11}|\tau(p_1)^2$. Thus, $p_1^{12}|\tau(p_1)^2$, which is impossible for $p_1 > 3$ because of (2), and can be checked by hand to be impossible for $p_1 = 3$.

Let $\mathscr{S}$ be the set of all prime divisors of M. Thus, $\#\mathscr{S} = s$. We see that the equation $u - v = 1$ has $\#\mathscr{R}$ distinct solutions in the $\mathscr{S}$-units

$$(u, v) = \left(\frac{2\tau(p^2)}{\tau(p)^2}, \frac{\tau(p^3)}{\tau(p)^3} \right). \tag{4}$$

It is known (see [3]), that the number of solutions of such a $\mathscr{S}$-unit equation is $O(7^{2s})$. We thus get that $7^{2s} \gg \#\mathscr{R} = (1 + o(1))\pi(y)$, giving

$$s \geq \frac{1}{6 \log 7}(1 + o(1)) \log x$$

as $x \to \infty$, which finishes the proof. □

Theorem 3. *The estimate*

$$P(\tau(p)\tau(p^2)\tau(p^3)) > (1 + o(1)) \frac{\log\log p \log\log\log p}{\log\log\log\log p}$$

holds as p tends to infinity through primes such that $\tau(p) \neq 0$.

Proof. As in the proof of Theorem 2, we consider the equation $u - v = 1$, having the solution (4) for every prime p with $\tau(p) \neq 0$. Write

$$u = E/D \quad \text{and} \quad v = F/D,$$

where D is the smallest positive common denominator of u and v. Then

$$E = Du = 2D - 2p^{11}D/\tau(p)^2 \quad \text{and} \quad F = Dv = D - 2Dp^{11}/\tau(p)^2$$

are integers with $\gcd(E, F) = 1$, and since $E - F = D$, we also have $\gcd(D, E) = \gcd(D, F) = 1$.

We note the inequalities

$$D \ll p^{11} \quad \text{and} \quad p \ll \max\{|E|, |F|\} \ll p^{22}. \tag{5}$$

Indeed, the upper bounds follow directly from (2). It also follows from (2) that $p^6 \nmid \tau(p)$. This shows that $p^{11}/\tau(p)^2$ is a rational number whose numerator is a multiple of p. In particular,

$$E - 2F = \frac{2Dp^{11}}{\tau(p)^2} \geq p,$$

which implies the lower bound in (5).

We have $P(\tau(p)\tau(p^2)\tau(p^3)) \geq \ell$, where $\ell = P(EDF)$.

Let $t = \omega(\tau(p)\tau(p^2)\tau(p^3))$. By (5), we see that there exists a prime q and a positive integer α such that q^α divides one of E or F and $q^\alpha \gg p^{1/t}$.

First we assume that $q^\alpha | E = D - F$, and write

$$D = \prod_{j=1}^{t} q_j^{\beta_j} \quad \text{and} \quad F = \prod_{j=1}^{t} q_j^{\gamma_j},$$

with some primes q_j and non-negative integers β_j, γ_j such that $\min\{\beta_j, \gamma_j\} = 0$ for all $j = 1, \ldots, t$ (clearly, $\beta_i = \gamma_i = 0$ for $q_i = q$). By (5), we also have

$$B = \max_{j=1,\ldots,t} \{\beta_j, \gamma_j\} \ll \max\{\log D, \log |E|\} \ll \log p.$$

Using the lower bound for linear forms in q-adic logarithms of Yu [17], we derive

$$\alpha \leq qc^t \log B \prod_{j=1}^{t} \log q_j \ll \ell (c \log \ell)^t \log\log p \tag{6}$$

with some absolute constant $c > 0$. Since also

$$\alpha \gg \frac{\log p}{t \log q} \geq \frac{\log p}{t \log \ell},$$

we get

$$\frac{\log p}{\log\log p} \ll \ell t (c \log \ell)^t \ll \ell (2c \log \ell)^t.$$

Hence,

$$\log\log p \leq t(1 + o(1)) \log\log \ell. \tag{7}$$

By the prime number theorem (see [4]), we have

$$t \leq (1 + o(1)) \frac{\ell}{\log \ell},$$

which together with (7) leads us to

$$(1 + o(1)) \frac{\log\log p \log\log\log p}{\log\log\log\log p} \leq t.$$

The case $q^\alpha | F = D - E$ can be considered completely analogously which concludes the proof. □

We recall that the *ABC*-conjecture asserts that for any fixed $\varepsilon > 0$ the inequality

$$Q(abc) \gg (\max |a|, |b|, |c|)^{1-\varepsilon}$$

holds for any relatively prime integers a, b, c with $a + b = c$. Thus, in the notation of the proof of Theorem 3, we immediately conclude from (5) that the *ABC*-conjecture yields

$$Q(\tau(p)\tau(p^2)\tau(p^3)) \geq Q(DEF) \geq p^{1+o(1)}.$$

Thus, by the prime number theorem,

$$P(\tau(p)\tau(p^2)\tau(p^3)) \geq (1+o(1))\log p.$$

The best known unconditional result of Stewart and Yu [15] towards the *ABC*-conjecture implies that

$$Q(\tau(p)\tau(p^2)\tau(p^3)) \geq Q(DEF) \geq (\log p)^{3+o(1)}.$$

3. Factorials and the Ramanujan function

In [6], all the positive integer solutions (m,n) of the equation $f(m!) = n!$ were found, where f is any one of the multiplicative arithmetical functions φ, σ, d, which are the Euler function, the sum of divisors function, and the number of divisors function, respectively. Further results on such problems have been obtained by Baczkowski [1]. Here, we study this problem for the Ramanujan function.

Theorem 4. *There are only finitely many effectively computable pairs of positive integers (m,n) such that $|\tau(m!)| = n!$.*

Proof. Assume that (m,n) are positive integers such that $\tau(m!) = n!$. By (2) and the Stirling formula

$$\begin{aligned} \exp((1+o(1))n\log n) = n! = \tau(m!) &< (m!)^{11/2+o(1)} \\ &< \exp((11/2+o(1))m\log m), \end{aligned}$$

as m tends to infinity. Thus, we conclude that if m is sufficiently large, then $n < 6m$.

Let $\nu(m)$ be the order at which the prime 2 appears in the prime factorization of $m!$. It is clear that $\nu(m) > m/2$ if m is sufficiently large. Since τ is multiplicative, it follows that $u_{\nu(m)} = \tau(2^{\nu(m)})|n!$, where the Lucas sequence u_r is given by (3) with $u_0 = 1$, $u_1 = -24$.

For $r \geq 1$, we put $\zeta_r = \exp(2\pi i/r)$ and consider the sequence $v_r = \Phi_r(\alpha,\beta)$ where

$$\Phi_r(X,Y) = \prod_{\substack{1\leq k\leq r \\ \gcd(k,r)=1}} (X - \zeta_r^k Y).$$

It is known that $v_r|u_r$. It is also known (see [2]), that $v_r = A_r B_r$, where A_r and $B_r > 0$ are integers, $|A_r| \leq 6(r+1)$ and every prime factor of B_r is congruent to $\pm 1 \pmod{r+1}$. Let α and β be the two roots of the characteristic equation $\lambda^2 - 24\lambda - 2048 = 0$. Since both inequalities $|v_k| \leq 2|\alpha|^{k+1}$ and $|v_k| \geq |\alpha|^{k+1-\gamma\log(k+1)}$ hold for all positive integers k with some absolute constant γ (see, for example, Theorem 3.1 on p. 64 in [12]), it follows that

$$\begin{aligned} 6(r+1)B_r &\geq 2^{-\tau(r+1)}\alpha^{\varphi(r+1)-\gamma\tau(r+1)\log(r+1)} \\ &= |\alpha|^{\varphi(r+1)+O(\tau(r+1)\log(r+1))}. \end{aligned}$$

Since $\varphi(r+1) \gg r/\log\log r$, and $\tau(r+1)\log(r+1) = r^{o(1)}$, the above inequality implies that

$$B_r > |\alpha|^{\varphi(r+1)/2}$$

whenever r is sufficiently large.

In particular, we see that $B_{\nu(m)}|\tau(m!)$, has all prime factors $\ell \equiv \pm 1 \pmod{\nu(m)+1}$, and is of the size

$$B_{\nu(m)} > \exp(cm/\log\log m),$$

where c is some positive constant.

However, since $B_{\nu(m)}|n!$ and $n < 6m$, it follows that all prime factors ℓ of $B_{\nu(m)}$ satisfy $\ell < 6m$. Since $\nu(m) > m/2$, there are at most 26 primes $\ell < 6m$ with $\ell \equiv \pm 1 \pmod{\nu(m)+1}$. Furthermore, again since $B_{\nu(m)}|n!$, $n < 6m$, and all prime factors ℓ of $B_{\nu(m)}$ satisfy $\ell \equiv \pm 1 \pmod{\nu(m)+1}$, it follows that $\ell^{14} \nmid B_{\nu(m)}$. Hence,

$$B_{\nu(m)} < (6m)^{26\cdot 13} = m^{O(1)}.$$

Comparing this with the above lower bound on $B_{\nu(m)}$, we conclude that m is bounded.

□

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