# On the solutions to '$px+1$ is square'

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In this short note, we construct the closed form of positive solutions to '$px+1$ is square'. A short expression in terms of squares and triangular numbers is found.

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**Lemma.** *Let $p$ be any odd prime, and $x_p(n) > 0$ is the $n$-th solution to $px+1 = m^2$, where $m, n > 0$ is integer. Then*

$$x_p(n) = -\frac{1}{2} + \frac{1}{8}\left(p(2n^2+2n+1) - (p-4)(2n+1)(-1)^n\right). \tag{1}$$

*Proof.* Because the sequence $px_p(n)+1$ lists all solutions to $y^2 \equiv 1 \bmod p$, there are only two of them (see e.g. [1]) for each interval $[kp+1, (k+1)p], k \geq 0$, namely $kp+1$ and $kp+p-1$, and their sequence $y_p(n)$ satisfies

$$\begin{aligned} y_p(n) &= \begin{cases} p \cdot \frac{n}{2} + 1 & n \text{ even,} \\ p \cdot \frac{n-1}{2} + p - 1 = p \cdot \frac{n}{2} + \frac{p}{2} - 1 & n \text{ odd,} \end{cases} \\ &= \frac{np}{2} + \frac{p}{4} - \left(\frac{p}{4} - 1\right)(-1)^n. \end{aligned}$$

Squaring the expression yields

$$px_p(n) + 1 = \left(\frac{(2np+p)^2}{16}\right) - \left(\frac{(2np+p)(p-4)}{8}\right)(-1)^n + \frac{(p-4)^2}{16},$$

from which the assertion follows. ∎

**Theorem.**

$$x_p(n) = (p-2)\lceil n/2 \rceil^2 + 4T_{\lfloor n/2 \rfloor}, \qquad \textit{where } T_n = n(n+1)/2. \tag{2}$$

*Proof.* With [GKP, Wilf] or the exercise collection [2], it is not difficult to get the power series generating function of (1):

$$X_p(z) = \frac{(p-2)(z^3+z)+4z^2}{(1-z)^3(1+z)^2}.$$

∎

## REFERENCES


[1] A060594 in OEIS.

[GKP] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, 2nd ed., Addison-Wesley, 1994

[OEIS] N. J. A. Sloane, editor (2003), *The On-Line Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences/`

[2] R. Stephan, *Exercises with Linear Recurrences*, `http://www.ark.in-berlin.de/rgf.html`

[Wilf] H. S. Wilf, Generatingfunctionology, Academic Press, NY, 1990. `http://www.math.upenn.edu/~wilf/DownldGF.html`


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