Boring proof of a nonlinearity

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We prove classically, and step-by-step, that certain integer sequences satisfy the recurrence $a_{n-1}a_{n+1} = (a_n - 1)^2$.

Proposition. *The coefficients $s_k(n)$ in the power series expansion of the generating function*

$$S_{k\geq 1}(z) = \sum_{n\geq 0} s_k(n)z^n = \frac{z+z^2}{1-(k-1)z+(k-1)z^2-z^3}$$

satisfy $s_k(0) = 0$, $s_k(1) = 1$, $s_k(2) = k$, and

$$s_k(n-1)s_k(n+1) = (s_k(n)-1)^2. \qquad (1)$$

Proof. We will prove the proposition formally by deriving the elementary closed form for $s_k(n)$, from which the final step is easily done.

First, we want to write the denominator of $S_k(z)$ as a product of its factors,

$$S_k(z) = \frac{z+z^2}{(1-z)(1-\theta_1 z)(1-\theta_2 z)},$$

where the $\theta_{1,2}$ are the solutions to the quadratic equation

$$1-(k-2)z+z^2 = 0 \qquad \Longrightarrow \quad \theta_{1,2} = \frac{k-2\pm\sqrt{k(k-4)}}{2}.$$

By expressing the denominator of $S_k(z)$ as product of its smallest factors (even if they were complex), we will be able to write the fraction as sum of its partial fractions, like this:

$$S_k(z) = \frac{A}{1-z} + \frac{B}{1-\theta_1 z} + \frac{C}{1-\theta_2 z}.$$

Graham, Knuth, and Patashnik(GKP) give a formula for the case when none of the roots of the denominator are multiple. This happens here only with $k = 4$. For the other cases we get

$$A = \frac{2}{(\theta_1-1)(\theta_2-1)}, \qquad B, C = \frac{\theta_{1,2}+1}{(\theta_{1,2}-1)(\theta_{1,2}-\theta_{2,1})},$$

and therefore, with $\theta_1\theta_2 = 1$,

$$A = \frac{-2}{k-4}, \qquad B = C = \frac{1}{k-4}.$$

So the closed form for $s_k(n)$ is

$$s_k(n) = \tfrac{-2+\theta_1^n+\theta_2^n}{k-4}$$
$$= \tfrac{1}{k-4}\left[-2+\left(\tfrac{k-2+\sqrt{k(k-4)}}{2}\right)^n + \left(\tfrac{k-2-\sqrt{k(k-4)}}{2}\right)^n\right].$$

It is not difficult now to see that the values for $n = 0, 1, 2$ are as proposed, so it remains to prove the identity in (1). Multiplying with the denominator, we have for the left hand side of (1)

$$(-2+\theta_1^{n-1}+\theta_2^{n-1})(-2+\theta_1^{n+1}+\theta_2^{n+1}) = 4+\theta_1^2+\theta_2^2$$
$$-2\theta_1^{n-1}(1+\theta_1^2) - 2\theta_2^{n-1}(1+\theta_2^2) + \theta_1^{2n} + \theta_2^{2n}$$

and for the right hand side

$$(2-k+\theta_1^n+\theta_2^n)^2 = (2-k)^2 + 2(2-k)(\theta_1^n+\theta_2^n)$$
$$+1+1+\theta_1^{2n}+\theta_2^{2n}.$$

Together with the fact that $(k-2)\theta_{1,2} = 1+\theta_{1,2}^2$, equality holds. The missing special case $k = 4$ turns out to be the square numbers, for which the proposition is trivially true. □

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