ON SOME 1-ADDITIVE SEQUENCES

MATTHEW AKERAN

ABSTRACT. We give a characterization for numbers in a class of 1-additive sequences and thus solve a conjecture by Stephan and, more generally, a problem posed by Finch.

1-additive sequences have the definition "a_n is smallest number which is uniquely $a_j + a_k, j < k$". Our interest in these sequences was sparked by Stephan[S] who observed that, for start values $2, 7$, the first differences seemed to have period 26 (this is sequence `A003668` from [OEIS]). However, Finch already proved[F] that all sequences with start values $(2, v), v \geq 5$ have periodic differences. In the following, we will give an elementary proof of a more general proposition, namely

Theorem 1. *The 1-additive sequences with start values* $2, 2^k - 1$*, for* $k \geq 3$ *are identical to sets* $\{2, 2^k + 1\} \cup B$*, where* B *is defined to consist of numbers of the form*

$$2x + 2^{k+1}y + 2^k - 3 + (2^{2k+1} - 2)m,$$

where the conditions hold

$$\begin{aligned} 0 \leq x \leq 2^k - 1, \quad 0 \leq y \leq 2^k - 1, \\ x + y > 0, \quad m \geq 0, \quad x \,\&\, y = 0, \end{aligned} \tag{1}$$

and & *denotes the bitwise-and operator.*

As x and y are k-bit binary numbers, and the possible pairs of corresponding bits in the two numbers are $(0, 0), (0, 1)$, and $(1, 0)$ (with the case $x = y = 0$ excluded), then from the theorem would follow two corollaries, stated already as conjectures by Finch, and also the first, in the case $k = 3$, by Stephan.

Corollary 1. *The 1-additive sequences with start values* $2, 2^k - 1$*, for* $k \geq 3$ *have differences with period* $3^k - 1$*.*

Corollary 2. *The span between periods of first differences of 1-additive sequences with start values* $2, 2^k - 1$*, for* $k \geq 3$ *is* $2^{2k+1} - 2$*.*

In the rest of the paper, we will prove the theorem using four lemmata, where the last one coincides with Conjecture 2 in Finch's paper[F].

Definition. Let

$$\mathcal{O}(x, y, m) = \mathcal{O}(x, y, m; k) = 2x + 2^{k+1}y + 2^k - 3 + (2^{2k+1} - 2)m$$

$$\mathcal{E}(x, y, m) = \mathcal{O}(x, y, m) + 2^k - 3.$$

First see that, if the conditions in (1) hold, then every odd number $2^k - 1$ and above has a unique representation $\mathcal{O}(x, y, m)$. Also, every even number $2^{k+1} - 4$ and above has a unique representation in the form $\mathcal{E}(x, y, m)$.

Lemma 1. *Suppose $x \,\&\, y = 0, x > 0, y > 0$. Then exactly one of $(x - 1) \,\&\, y, x \,\&(y - 1)$ is zero.*

Proof. Let c be the position of the lowest bit set in both x or y. If the c-th bit of x is set, then $(x-1) \,\&\, y = 0$ but $x \,\&(y-1) \neq 0$. Exchange x and y. □

Lemma 2. *For any number $b \in B$, exactly one of $b - 2$ and $b - 2^{k+1}$ is in B.*

Proof. Let $b = \mathcal{O}(x, y, m) \in B$,

(i) if $x > 0$ and $y > 0$, then $b - 2 = \mathcal{O}(x - 1, y, m)$ and $b - 2^{k+1} = \mathcal{O}(x, y - 1, m)$. Since $b \in B, x \,\&\, y = 0$, so by Lemma 1, one of $b - 2$ and $b - 2^{k+1}$ is in B;

(ii) if $x = 0, y = 1$, then $b - 2 = \mathcal{O}(2^k - 1, 0, m) \in B$, $b - 2^{k+1} = \mathcal{O}(0, 0, m) = \mathcal{O}(2^k - 1, 2^k - 1, m - 1) \notin B$;

(iii) if $x = 1, y = 0$, then $b - 2 = \mathcal{O}(0, 0, m) \notin B$, $b - 2^{k+1} = \mathcal{O}(0, 2^k - 1, m - 1) \in B$;

(iv) if $x = 0, y > 1$, then $b - 2 = \mathcal{O}(2^k - 1, y - 1, m) \notin B$, $b - 2^{k+1} = \mathcal{O}(0, y - 1, m) \in B$;

(v) if $x > 1, y = 0$, then $b - 2 = \mathcal{O}(x - 1, 0, m) \in B$, $b - 2^{k+1} = \mathcal{O}(x - 1, 2^k - 1, m - 1) \notin B$.

□

Lemma 3. *If an odd number b is not in B, then either both or neither of $b - 2$ and $b - 2^{k+1}$ is in B.*

Proof. Let $b = \mathcal{O}(x, y, m) \notin B$. Then $x \,\&\, y$ is not zero (note the illegal case $x = y = 0$ resolves to $\mathcal{O}(2^k - 1, 2^k - 1, m - 1)$).

(i) If $x \,\&\, y$ has a single nonzero bit, and both x and y are multiples of $x \,\&\, y$, then both $x \,\&(y - 1)$ and $(x - 1) \,\&\, y$ are zero, so $b - 2$ and $b - 2^{k+1}$ are both in B.

(ii) If $x \,\&\, y$ has at least two nonzero bits, then the higher of the two bits is still nonzero in $x - 1$ and $y - 1$, so $(x - 1) \,\&\, y$ and $x \,\&(y - 1)$ are both nonzero, and $b - 2$ and $b - 2^{k+1}$ are neither in B.

(iii) $x \,\&\, y$ has one nonzero bit, but at least one smaller bit is set in x or y: If a smaller bit is set in x, then $x - 1$ has the $x \,\&\, y$ bit set, so $(x - 1) \,\&\, y > 0$. If no smaller bit is set on in x, then all smaller bits are set in $x - 1$, and at least one of these smaller bits is set in y, so $(x - 1) \,\&\, y > 0$. Therefore $(x - 1) \,\&\, y > 0$, whether x has any smaller bits set or not, so $b - 2 = \mathcal{O}(x - 1, y, m)$ is not in B. Likewise, $b - 2^{k+1}$ is not in B.

□

Lemma 4. *If $a < b \in B$, and $a + b > 2^{k+1} + 2$, then there are $c < d \in B$, with $c \neq a$, and $a + b = c + d$.*

Proof. Let the sums

$$\begin{aligned}
\mathcal{S}_1 &= \mathcal{O}(x,0,0) + \mathcal{O}(0,y,m), \quad \mathcal{S}_2 = \mathcal{O}(x,0,m) + \mathcal{O}(0,y,0),\\
\mathcal{S}_3 &= \mathcal{O}(2^k - y - 1, y, 0) + \mathcal{O}(x + y + 1 - 2^k, 0, m), \text{if } x + y \geq 2^k,\\
\mathcal{S}_4 &= \mathcal{O}(2^k - y, y - 1, 0) + \mathcal{O}(x + y, 0, m), \text{if } x + y < 2^k.
\end{aligned}$$

(i) If none of x, y, m is zero, then $\mathcal{E}(x,y,m) = \mathcal{S}_1 = \mathcal{S}_2$, and the sums have different terms.

(ii) If $m = 0$, and neither x nor y is zero, then $\mathcal{E}(x,y,m) = \mathcal{S}_1 = \{\mathcal{S}_3 \text{ or } \mathcal{S}_4\}$, and the sums have different terms.

(iii) If $m = 0 = y$, then $\mathcal{E}(x,0,0) = \mathcal{O}(x-1,0,0) + \mathcal{O}(1,0,0) = \mathcal{O}(x-2,0,0) + \mathcal{O}(2,0,0)$ are valid and different provided $x > 4$. $\mathcal{E}(4,0,0) = 2^{k+1} + 2 = (2^k - 1) + (2^k + 1) = 2 + (2^{k+1})$.

(iv) If $m = 0 = x$, then $\mathcal{E}(0,y,0) = \mathcal{O}(0,y-1,0) + \mathcal{O}(0,1,0) = \mathcal{O}(0,y-2,0) + \mathcal{O}(0,2,0)$ are valid and different provided $y > 4$. The other cases:

$$\begin{aligned}
\mathcal{E}(0,1,0) &= \mathcal{O}(1,0,0) + \mathcal{O}(2^k - 1,0,0) = \mathcal{O}(2,0,0) + \mathcal{O}(2^k - 2,0,0)\\
\mathcal{E}(0,2,0) &= \mathcal{O}(2,1,0) + \mathcal{O}(2^k - 2,0,0) = \mathcal{O}(4,1,0) + \mathcal{O}(2^k - 4,0,0)\\
\mathcal{E}(0,3,0) &= \mathcal{O}(1,2,0) + \mathcal{O}(2^k - 1,0,0) = \mathcal{O}(0,2,0) + \mathcal{O}(0,1,0)\\
\mathcal{E}(0,4,0) &= \mathcal{O}(4,2,0) + \mathcal{O}(2^k - 4,1,0) = \mathcal{O}(0,1,0) + \mathcal{O}(0,3,0)
\end{aligned}$$

(v) If $m > 0$, $x = 0$, then y is not zero, $\mathcal{E}(0,y,m) = \mathcal{O}(2^k - y, y - 1, m) + \mathcal{O}(y,0,0) = \mathcal{O}(2^k - y, y - 1, 0) + \mathcal{O}(y,0,m)$.

(vi) If $m > 0$, $y = 0$, then x is not zero, $\mathcal{E}(x,0,m) = \mathcal{O}(x-1, 2^k - x, m-1) + \mathcal{O}(0,x,0) = \mathcal{O}(x-1,0,m) + \mathcal{O}(1,0,0)$ provided $x > 1$. $\mathcal{E}(1,0,m) = \mathcal{O}(0,1,0) + \mathcal{O}(0, 2^k - 1, m-1) = \mathcal{O}(0,2,0) + \mathcal{O}(0, 2^k - 2, m-1)$.

□

The conclusion from Lemma 4 is that every even number greater than $2^{k+1}+2$ is the sum of members of B either in no way, or in two or more ways. We also see $2^{k+1}+2 = (2)+(2^{k+1}) = (2^k-1)+(2^k+3)$, while $2^{k+1} = 2^k-1+2^k+1$. No even number less than 2^{k+1} is the sum of two different $\mathcal{O}(x,y,m)$ numbers because the smallest two are $2^k - 1$ and $2^k + 1$. Therefore we need bother only with odd members of B.

By taking this together with lemmata 2 and 3, the theorem is proved.

References

[F] S. R. Finch, *Patterns in 1-additive sequences*, Exp. Math. 1(1992), 57–63.

[OEIS] N.J.A. Sloane, *Online Enyclopedia of Integer Sequences*, `www.research.att.com/~njas/sequences/Seis.html`

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