Conjecture 36 - from OEIS - a.k.a. A063920

The proof of conjecture 36 from Ralf Stephan's "Prove or Disprove 100 Conjectures from the OEIS" found at http://front.math.ucdavis.edu/math.CO/0409509.

Conjecture 1 *Let* $t(n) = |\phi(n) - n|$. *Then for any* $n > 0$ *we have that*

$$t(n) = t(t(n) - n) \tag{1}$$

iff

$$n = 5 \cdot 2^k \text{ or } n = 7 \cdot 2^k \text{ for some } k > 0.$$

Proof. We first note that (1) is equivalent to

$$\begin{aligned} n - \phi(n) &= t(-\phi(n)) \\ n - \phi(n) &= |\phi(-\phi(n)) + \phi(n)| \\ n - \phi(n) &= \phi(\phi(n)) + \phi(n) \\ n &= \phi(\phi(n)) + 2\phi(n) \end{aligned} \tag{2}$$

Henceforth, we instead work to show that (2) is true only iff $n = 5 \cdot 2^k$ or $n = 7 \cdot 2^k$ for some $k > 0$.

If $n = 2^k 5$, with $k \geq 1$ then $\phi(n) = 4 \cdot 2^{k-1} = 2^{k+`}$ and $\phi(\phi(n)) = 2^k$ which clearly satisfies (2).

If $n = 2^k 7$, with $k \geq 1$ then $\phi(n) = 6 \cdot 2^{k-1} = 3 \cdot 2^k$ and $\phi(\phi(n)) = 2 \cdot 2^{k-1} = 2^k$ which again satisfies (2).

We show that all other values for n fail.

- The cases $1 \leq n \leq 6$ fail by inspection.
- If $n \geq 7$, then $\phi(\phi(n)) + 2\phi(n)$ is even so (2) fails for all odd n.
- If $n = 2^k$ for $k \geq 2$, then $\phi\left(\phi\left(2^k\right)\right) + 2 \cdot \phi\left(2^k\right) = 2^{k-2} + 2 \cdot 2^{k-1} \neq 2^k$. So $n = 2^k$ fails for all $k \geq 0$.
- If $n = 2^k M$ where $M > 1$ is odd and $k \geq 1$,

$$\begin{aligned} n &= \phi(\phi(n)) + 2\phi(n) \\ 2^k M &= \phi\left(\phi\left(2^k M\right)\right) + 2\phi\left(2^k M\right) \\ 2^k M &= \phi\left(2^{k-1} \cdot \phi(M)\right) + 2^k \cdot \phi(M) \\ 2^k M &= 2^{k-1} \cdot \phi(\phi(M)) + 2^k \cdot \phi(M) \text{ since } \phi(M) \text{ is even} \end{aligned}$$

so we get the following further refinement of (2).

$$2M = \phi(\phi(M)) + 2\phi(M) \tag{3}$$

- If there exist two different odd primes p and q such that $pq|M$, then $\phi(p)\phi(q)|\phi(M)$ so it follows that $4|\phi(M)$. In fact, we can write

  $$\phi(p)\phi(q) = (p-1)(q-1) = 2^{\ell} N$$

  for some $\ell \geq 2$ and odd N. Therefore

  $$\phi\left(2^{\ell}\right)\phi(N) = 2^{\ell-1}\phi(N)\,|\phi(\phi(M)).$$

  If $N = 1$, then $p = 2^{\alpha}+1$ and $q = 2^{\beta}+1$ for some $\alpha \neq \beta$ so that $\ell = \alpha+\beta \geq 3$. If $N > 1$ then $\phi(N)$ is even. For both cases $N = 1$ and $N > 1$, we will have $4|\phi(\phi(M))$. So we conclude that (3) fails because 4 divides the RHS of (3) but not the LHS

  Now we only need consider the case where M is an odd prime power..

- If $n = 2^k p$, where $k \geq 1$ and p is an odd prime, then (3) reduces to

  $$\begin{aligned} 2p &= \phi(p-1) + 2(p-1) \\ 2 &= \phi(p-1) \end{aligned}$$

  which fails for all cases other than $p = 5$ or 7.

- If $n = 2^k p^j$, where $k \geq 1$, $j \geq 2$ and p is an odd prime, then (3) reduces to

  $$\begin{aligned} 2^k p^j &= \phi\left(\phi\left(2^k p^j\right)\right) + 2\phi\left(2^k p^j\right) \\ 2^k p^j &= \phi\left(2^{k-1}\cdot(p-1)\,p^{j-1}\right) + 2\cdot 2^{k-1}(p-1)\,p^{j-1} \\ 2^k p^j &= 2^{k-1}\cdot\phi(p-1)\cdot(p-1)\,p^{j-2} + 2^k(p-1)\,p^{j-1} \text{ (since } p-1 \text{ is even and } (p-1,p)=1) \\ 2p^2 &= \phi(p-1)\cdot(p-1) + 2\cdot\left(p^2-p\right) \\ 2p &= \phi(p-1)\cdot(p-1) \end{aligned}$$

  this fails for all odd p since p is clearly not a factor of the RHS.

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