An Identity for A103314(n)

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For a given $n > 1$, let U_n be the set containing the n n-th roots of unity. Using the multiplication operation in $\mathbb{C}$, this set is a group. Let $s(n)$ be the squarefree kernel of n; that is, if the prime factorization of n is $p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$, then $s(n) = p_1 p_2 \dots p_r$.

Consider the 2^n sums formed by the subsets of U_n. At least 2 of these sums are zero because both U_n and the empty set sum to zero. In fact, when n is prime, these two are the only subsets that sum to zero. Let K_n denote the number of subsets that sum to zero. Our goal is to show that

$$K_n = K_{s(n)}^{n/s(n)}.$$

If n is squarefree, then $s(n) = n$ and the formula is merely

$$K_n = K_n.$$

Hence, we assume that n is not squarefree; that is, $s(n) < n$. To simplify the notation, let $g = n/s(n)$ and $m = s(n)$. Clearly, because n is not squarefree, $g > 1$.

Let H be the unique subgroup of U_n of size m. It is well-known that there are $n/m = g$ disjoint cosets of H in U_n. There are K_m subsets of H that sum to zero. Hence, for each one of the g cosets, we can find K_m subsets that sum to zero. Because these subsets are independent of each other, we can construct K_m^g subsets of U_n that sum to zero.

Is there another subset of U_n whose members sum to zero, but not one of the K_m^g subsets we constructed above? Suppose x is such a subset of U_n. We may assume that x contains no subset that sums to zero; that is x is primitive (or minimal). However, by Lemma 1 in [1], we find that (after a possible rotation) x must be a subset the subgroup H. This is the same as saying that x is a subset of one of the g cosets of H. However, this contradicts the assumption that x is not one of the K_m^g subsets. Hence, no x exists.

[1] Bjorn Poonen and Michael Rubinstein, The number of intersection points made by the diagonals of a regular polygon, published electronically at www.arXiv.org/math/9508209