Journal of Integer Sequences, Vol. 3 (2000),
Article 00.2.2

Some Easily Derivable Integer Sequences

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Abstract

We propose and discuss several simple ways of obtaining new enumerative sequences from existing ones. For instance, the number of graphs considered up to the action of an involutory transformation is expressible as the semi-sum of the total number of such graphs and the number of graphs invariant under the involution. Another, less familiar idea concerns even- and odd-edged graphs: the difference between their numbers often proves to be a very simple quantity (such as $n!$). More than 30 new sequences will be constructed by these methods.

Acknowledgement. This research was supported by INTAS (Grant INTAS-BELARUS 97-0093).
Mathematics Subject Classification (1991): 05C30, 05A19

Contents

1 Introduction

New realities set up new tasks. The *On-line Encyclopedia of Integer Sequences* [18] (in the sequel referred to as the *OEIS*) is a rapidly growing facility, which has been playing a more and more important role in mathematical research. To be a comprehensive reference source, the OEIS needs to include as many naturally defined sequences as possible. The efforts of numerous enthusiasts have been directed towards promoting this aim. The present work has been motivated by the same goals.

A fruitful idea is to generate new sequences from known ones. To implement it, various useful transformations of sequences have been proposed — see [4, 5, 19, 20]. In most cases discussed hitherto, these operations transform one sequence to another.

Here we consider some other operations of a similar type but which are less general, producing new enumerative sequences for graphs from *two* other sequences (in most cases, as their semi-sum). The corresponding relations between the objects being counted are very simple, and, as a rule, already known. However, they have never been analyzed *systematically* (this can be partially explained just by their simplicity: serious researchers rarely considered them as deserving an independent formulation). As we will see, our operations do result in new and interesting sequences. In a sense, they might be considered as already implicitly present in the OEIS. However, they cannot be extracted by a formal rule and thus need to be presented in the OEIS *explicitly*. At the same time, we should avoid trivial sequences — not all new sequences deserve to be added to the OEIS. We will return to this question in Section 5.

1.1 Definitions, classes of graphs

In what follows, n denotes the *order* of a graph, i.e. the number of nodes (or vertices). For uniformity, we always start with the case $n = 1$, and usually n takes all natural values. In other words, we deal with sequences (or lists) of the form $[a(1), a(2), a(3), \ldots]$. N denotes the number of edges (in digraphs they are usually called *arcs*) and if there are n nodes and N edges we will sometimes speak of an (n, N) graph.

Φ stands for an arbitrary class of graphs, undirected or directed. Graphs may have loops but not multiple edges (except for planar maps). The most important specific classes to be considered will be denoted by the following capital Greek letters, sometimes equipped with a symbolic subscript:

- Γ (simple) undirected graphs
- Γ_{l} (undirected) graphs with loops, i.e. symmetric reflexive relations
- Γ_{e} even (i.e. eulerian) graphs
- Γ_{m} median graphs, i.e. (n, N)-graphs with $N = \lceil n(n-1)/4 \rceil$ edges
- Γ_{r} regular graphs with unspecified degrees
- Γ_{t} (vertex-) transitive graphs
- Γ_{c} circulant graphs (i.e. Cayley graphs of cyclic groups)

- Δ digraphs
- Δ_{l} (binary) relations, i.e. digraphs with loops
- Δ_{e} balanced digraphs (i.e. eulerian digraphs: in-degree = out-degree for any vertex)
- Δ_{c} circulant digraphs
- Ω oriented graphs, i.e. antisymmetric relations
- Θ tournaments, i.e. complete oriented graphs
- Λ planar maps (order = #(edges)).

1.2 Enumerative functions

Lower case letters will be used for the cardinalities (denoted by #) of subsets of labeled graphs, and the corresponding capital letters will be used for unlabeled graphs of the same kind. The most important specific quantities to be mentioned are the following:

- a, A = #(all graphs) in a class Φ
- c, C = #(connected graphs)
- d, D = #(disconnected graphs)
- b, B = #(doubly connected graphs) (both the graph and its complement are connected)
- s, S = #(strongly connected digraphs, or strong digraphs)
- G = #(unlabeled self-complementary undirected graphs)
- K = #(unlabeled graphs up to complementarity)
- $f_{\mathrm{E}}, F_{\mathrm{E}}$ = #(graphs with even number of edges (or arcs)) and
- $f_{\mathrm{O}}, F_{\mathrm{O}}$ = #(graphs with odd number of edges (or arcs)), where $f = a, c, \ldots, F = A, C, \ldots$

We denote the corresponding functions for n-graphs and (n, N)-graphs by $f(\Phi, n)$, $F(\Phi, n)$ and $f(\Phi, n, N)$, $F(\Phi, n, N)$ (or merely $f(n)$, $f(n, N)$, etc. if the class is understood), where f and F refer to labeled and unlabeled graphs respectively The corresponding exponential generating functions (e.g.f.) for labeled graphs and ordinary generating functions (o.g.f.) for unlabeled graphs are denoted by $\mathbf{f}(z)$, $\mathbf{f}(n, x)$, $\mathbf{f}(z, x)$ and $\mathbf{F}(z)$, $\mathbf{F}(n, x)$, $\mathbf{F}(z, x)$), where the formal variable z corresponds to n and x corresponds to N. In particular, in the labeled case,

$$\mathbf{f}(z, x) = \sum_{n \geq 1} \mathbf{f}(n, x) \frac{z^n}{n!} = \sum_{n} \sum_{N} f(n, N) x^N \frac{z^n}{n!}$$

(so as not to confuse $\mathbf{f}(n, x)$ with $\mathbf{f}(z, x)|_{z=n}$, the latter expression will not be used here).

We identify any function $f(n)$ with the sequence of its values $[f(1), f(2), f(3), \ldots]$.

Sequences in [18] will be referred to by their A-numbers. (Many of these sequences were added as a result of the present paper.)

2 Subtraction

We begin with the most trivial case: the subtraction method for calculating objects that do not belong to a given subset of a set. In principle, this is an inexhaustible source of new sequences, but we restrict ourselves to several interesting classes, some of which will be used in what follows.

2.1 Disconnection

Consider an arbitrary class of graphs Φ. Using the above notation, we have for disconnected labeled graphs,

$$d(\Phi, n) = a(\Phi, n) - c(\Phi, n) \tag{1}$$

and for disconnected unlabeled graphs,

$$D(\Phi, n) = A(\Phi, n) - C(\Phi, n) \tag{1*}$$

Usually $c(n)$ is expressible in terms of $a(n)$ and $C(n)$ in terms of $A(n)$, and vice versa, in one of several ways depending on the labeling type and the repetition restrictions. See for example the transformations EULERi/EULER/WEIGH for unlabeled graphs and LOG/EXP for labeled ones [4, 20]. Therefore $d(n)$ (and $D(n)$) can usually be expressed solely in terms of $a(n)$ *or* $c(n)$ (resp., in terms of $A(n)$ or $C(n)$). In any case, (1) and (1∗) are much easier for calculations if both $a(n)$ and $c(n)$ (resp., $A(n)$ and $C(n)$) have already been calculated.

2.2 Weak and strong digraphs

In the directed case (including the case of relations), connected digraphs are called *weakly* connected in order to distinguish them from *strongly* connected ones. As in Section 2.1 we may consider two further quantities: digraphs that are not strongly connected and (weakly) connected digraphs that are not strongly connected. Only the latter quantity makes sense for tournaments, because all tournaments are weakly connected. Neither notion makes sense for balanced digraphs, in which case weakly connected digraphs are all strongly connected.

This idea is quite fruitful not only for most of the classes of digraphs defined above but also for example for *semi-regular* digraphs: ones with the same out-degree at all vertices[1].

One further notion, which we will use below (4.1.6), is that of a semi-strong digraph. A digraph is called *semi-strong* if all its weakly connected components are strongly connected (in particular, strong digraphs are semi-strong). In the unlabeled case, moreover, one should make a distinction between (at least) two kinds of semi-strong digraphs: with or without repetitions (i.e. isomorphic components). Again, using the ordinary enumerative relationship "connected – disconnected", one can easily count semi-strong digraphs in any class for which the number of strongly connected ones is known.

In practice, these transformations are less productive since strongly connected digraphs (especially unlabeled ones) have been counted only for few types of digraphs (see, in particular, [26, 11, 12]); two of them will be discussed in 4.1.5.

3 Involutory equivalence

Diverse involutory operations on graphs serve as a source of new sequences.

3.1 Complementarity

Several interesting enumerative sequences are related to the notion of complementary graph.

Many classes of graphs contain a uniquely defined *complete graph* (for every order). In particular, complete graphs exist in the families of ordinary undirected graphs Γ, undirected graphs with loops Γ_1, directed graphs Δ and relations Δ_1. This notion allows us to introduce the *complement* of a graph. This is the graph on the same vertices in which the edges are those not in the complete graph.

[1] And for abstract *automata* [7] (Sect. 6.5). Fully defined automata without outputs and initial states are semi-regular digraphs which may be identified with tuples of mappings of the set of states to itself [12].

3.1.1 Double connection

It is clear that the complement of a disconnected graph is connected. This simple assertion allows us to easily count connected graphs (of given type Φ) whose complement is also connected *and* belongs to the same class. We call them *doubly connected*. In the labeled case their number $b\,(\Phi, n)$ is given by

$$c\,(\Phi, n) \;=\; b\,(\Phi, n) \;+\; d\,(\Phi, n),$$

whence by (1),

$$b\,(\Phi, n) \;=\; 2c\,(\Phi, n) \;-\; a\,(\Phi, n). \qquad (2)$$

Likewise for unlabeled graphs,

$$B\,(\Phi, n) \;=\; 2C\,(\Phi, n) \;-\; A\,(\Phi, n). \qquad (2*)$$

Now, for labeled simple undirected graphs,
$a\,(\Gamma, n) \;=\; [1, 2, 8, 64, 1024, 32768, 2097152, \ldots] \;=\;$ A006125 and
$c\,(\Gamma, n) \;=\; [1, 1, 4, 38, 728, 26704, 1866256, \ldots] \;=\;$ A001187, resulting in
$b\,(\Gamma, n) \;=\; [1, 0, 0, 12, 432, 20640, 1635360, \ldots] \;=\;$ A054913.
For labeled digraphs,
$a\,(\Delta, n) \;=\; [1, 4, 64, 4096, 1048576, \ldots] \;=\;$ A053763 and
$c\,(\Delta, n) \;=\; [1, 3, 54, 3834, 1027080, \ldots] \;=\;$ A003027, resulting in
$b\,(\Delta, n) \;=\; [1, 2, 44, 3572, 1005584, \ldots] \;=\;$ A054914.
For unlabeled undirected graphs,
$A\,(\Gamma, n) \;=\; [1, 2, 4, 11, 34, 156, 1044, 12346, 274668, \ldots] \;=\;$ A000088,
$C\,(\Gamma, n) \;=\; [1, 1, 2, 6, 21, 112, 853, 11117, 261080, \ldots] \;=\;$ A001349, and we obtain
$B\,(\Gamma, n) \;=\; [1, 0, 0, 1, 8, 68, 662, 9888, 247492, \ldots] \;=\;$ A054915.
For unlabeled undirected regular graphs,
$A\,(\Gamma_{\rm r}, n) \;=\; [1, 2, 2, 4, 3, 8, 6, 22, 26, 176, \ldots] \;=\;$ A005176,
$C\,(\Gamma_{\rm r}, n) \;=\; [1, 1, 1, 2, 2, 5, 4, 17, 22, 167, \ldots] \;=\;$ A005177 and
$B\,(\Gamma_{\rm r}, n) \;=\; [1, 0, 0, 0, 1, 2, 2, 12, 18, 158, \ldots] \;=\;$ A054916.
For vertex-transitive graphs,
$A\,(\Gamma_{\rm t}, n) \;=\; [2, 2, 4, 3, 8, 4, 14, 9, 22, \ldots] \;=\;$ A006799,
$C\,(\Gamma_{\rm t}, n) \;=\; [1, 1, 2, 2, 5, 3, 10, 7, 18, \ldots] \;=\;$ A006800 and
$B\,(\Gamma_{\rm t}, n) \;=\; [0, 0, 0, 1, 2, 2, 6, 5, 14, \ldots] \;=\;$ A054917.
For unlabeled digraphs,
$A\,(\Delta, n) \;=\; [1, 3, 16, 218, 9608, 1540944, \ldots] \;=\;$ A000273,
$C\,(\Delta, n) \;=\; [1, 2, 13, 199, 9364, 1530843, \ldots] \;=\;$ A003085 and
$B\,(\Delta, n) \;=\; [1, 1, 10, 180, 9120, 1520742, \ldots] \;=\;$ A054918.
For unlabeled (reflexive) relations,
$A\,(\Delta_{\rm l}, n) \;=\; [2, 10, 104, 3044, 291968, \ldots] \;=\;$ A000595, therefore, by the EULERi transformation [20],
$C\,(\Delta_{\rm l}, n) \;=\; [2, 7, 86, 2818, 285382, \ldots] \;=\;$ A054919 and
$B\,(\Delta_{\rm l}, n) \;=\; [2, 4, 68, 2592, 278796, \ldots] \;=\;$ A054920.
For unlabeled symmetric relations (undirected graphs with loops),
$A\,(\Gamma_{\rm l}, n) \;=\; [2, 6, 20, 90, 544, 5096, 79264, \ldots] \;=\;$ A000666, therefore, by the EULERi transformation,
$C\,(\Gamma_{\rm l}, n) \;=\; [2, 3, 10, 50, 354, 3883, 67994, \ldots] \;=\;$ A054921 and
$B\,(\Gamma_{\rm l}, n) \;=\; [2, 0, 0, 10, 164, 2670, 56724, \ldots] \;=\;$ A054922.

Undirected graphs with the median number of edges $\Gamma_{\rm m}$ need a slight modification of the present approach. Nothing unusual arises for orders $n = 4k$ or $4k + 1$. However for $n \equiv 2, 3 \pmod 4$, the graph and its complement have *different* numbers of edges, namely $\lceil n(n-1)/4\rceil$ and $\lceil n(n-1)/4\rceil - 1$. We will use a prime $'$ in the symbols for the latter case. Now, in order to count doubly connected median graphs, one should, instead of doubling $C\,(\Gamma_{\rm m}, n)$ as in (2∗), take the sum $C\,(\Gamma_{\rm m}, n) + C'(\Gamma_{\rm m}, n)$. In other words we have

$$B\,(\Gamma_{\rm m}, n) \;=\; C\,(\Gamma_{\rm m}, n) \;+ C'(\Gamma_{\rm m}, n) \;-\; A\,(\Gamma_{\rm m}, n). \qquad (2')$$

Indeed, we have $C = B + D'$ and $A' = C' + D'$. By definition, A' counts graphs that are complementary to ones counted by A, i.e. $A = A'$. These equalities give $(2')$.

Numerically, for unlabeled undirected graphs with n nodes and $N = \lceil n(n-1)/4 \rceil$ edges,
$A(\Gamma_m, n) =$ [1, 1, 1, 3, 6, 24, 148, 1646, 34040, ...] = A000717,
$C(\Gamma_m, n) =$ [1, 1, 1, 2, 5, 22, 138, 1579, 33366, ...] = A001437
and by the two-parameter table A054924,
$C'(\Gamma_m, n) =$ [1, 0, 0, 2, 5, 19, 132, 1579, 33366, ...] = A054926, whence
$B(\Gamma_m, n) =$ [1, 0, 0, 1, 4, 17, 122, 1512, 32692, ...] = A054927.

Of course, such a generalization can be applied to other similar classes of graphs (for example, regular of prescribed degree).

3.1.2 Self-complementarity

Next we consider various classes of graphs that are *invariant* with respect to complementarity. Apart from the classes mentioned in 3.1.1, complementarity is applicable, e.g., to the class of regular graphs of unspecified degrees Γ_r, regular undirected graphs of degree $(n-1)/2$ (n odd), median n-graphs for $n(n-1)$ divisible by 4, undirected eulerian graphs Γ_e of *odd* order, balanced digraphs Δ_e, arbitrary tournaments Θ and regular tournaments Θ_r. On the other hand, e.g., the following classes are not invariant with respect to complementarity: undirected eulerian graphs of even order, graphs with one cycle, graphs without 1-valent nodes, regular undirected graphs of a given degree (not equal to $(n-1)/2$), oriented graphs (except for tournaments), functional digraphs, acyclic digraphs and so on.

For a class of unlabeled graphs Φ counted by $A(\Phi, n)$, let $G(\Phi, n)$ count *self-complementary* graphs (i.e. graphs isomorphic to their complements). We may ask: what is the number $K(\Phi, n)$ of graphs in Φ considered *up to complementarity*?

The complement of a graph looks even more natural if one deals with the pair consisting of a graph and its complement: this may be interpreted as a complete graph with edges of two colors. In these terms, $K(\Phi, n)$ means the number of edge-2-colored unlabeled complete graphs whose colors are *interchangeable* and both one-colored edge subgraphs belong to Φ. The answer to the last question is now very simple:

$$K(\Phi, n) = \frac{A(\Phi, n) + G(\Phi, n)}{2}. \tag{3}$$

Indeed, every graph appears twice in different pairs (graph, complement) as the first or second component, except for the self-complementary graphs, which appear in only one pair. Each pair presents one graph up to complementarity, so $2K(n) = A(n) + G(n)$ (cf. [6]).

This composition can be applied:
to undirected graphs, where $A(\Gamma, n) =$ A000088 is given above and
$G(\Gamma, n) =$ [1, 0, 0, 1, 2, 0, 0, 10, 36, ...] = A000171, resulting in the sequence
$K(\Gamma, n) =$ [1, 1, 2, 6, 18, 78, 522, 6178, 137352, ...] = A007869;
to digraphs, where $A(\Delta, n) =$ A000273 and
$G(\Delta, n) =$ [1, 1, 4, 10, 136, 720, 44224, ...] = A003086, resulting in
$K(\Delta, n) =$ [1, 2, 10, 114, 4872, 770832, ...] = A054928;
to tournaments, where
$A(\Theta, n) =$ [1, 1, 2, 4, 12, 56, 456, 6880, 191536, ...] = A000568 and
$G(\Theta, n) = G(\Omega, n) =$[1, 1, 2, 2, 8, 12, 88, 176, 2752, ...]) = A002785, resulting in
$K(\Theta, n) =$ [1, 1, 2, 3, 10, 34, 272, 3528, 97144, ...] = A059735;
to median n-graphs for $n = 4k$ or $4k+1$ (that is, $n = 1, 4, 5, 8, 9 \ldots$), where
$A(\Gamma_m, n) =$ [1, 3, 6, 1646, 34040, ...] = the corresponding subsequence of A000717 (see 3.1.1) and
$G(\Gamma_m, n) = G(\Gamma, n) =$ [1, 1, 2, 10, 36, ...] = A000171 without zeros (see above), resulting in
$K(\Gamma_m, n) =$ [1, 2, 4, 828, 17038, ...], $n \equiv 0, 1 \pmod 4$;
to circulant graphs, where
$A(\Gamma_c, n) =$ [1, 2, 2, 4, 3, 8, 4, 12, 8, 20, 8, 48, 14, 48, 44, 84, 36, 192, ...]) = A049287 and
$G(\Gamma_c, n) =$ [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 4, 0, ...] = A049289, resulting in
$K(\Gamma_c, n) =$ [1, 1, 1, 2, 2, 4, 2, 6, 4, 10, 4, 24, 8, 24, 22, 42, 20, 96, ...] = A054929;
and to circulant digraphs, where
$A(\Delta_c, n) =$ [1, 2, 3, 6, 6, 20, 14, 46, 51, 140, 108, ...] = A049297 and

$G(\Delta_c, n) = [1, 0, 1, 0, 2, 0, 2, 0, 3, 0, 4, \ldots] =$ A049309, resulting in
$K(\Delta_c, n) = [1, 1, 2, 3, 4, 10, 8, 23, 27, 70, 56, \ldots] =$ A054930.

In the last two cases, $G(n)$ differ from the corresponding sequences in OEIS by additional zeros interspersed appropriately in order to cover all orders.

One further class of graphs worth mentioning in this respect is that of bipartite graphs; we refer to [16] for enumerative results concerning the function G for such graphs.

In general, this idea can be productively applied to a class of graphs whenever we know any two out of the three corresponding sequences.

3.1.3 A combination

Somewhat more artificially we can apply the same approach to connected graphs, i.e. we consider the number $L(n)$ of unlabeled connected graphs up to complementarity. Complementarity clearly preserves the subclass of connected graphs whose complement is also connected. Thus formula (3) is applicable, giving rise to $L(n) = (B(n) + G(n))/2$, where $B(n)$ is determined by formula (2∗). Thus

$$L(\Phi, n) = C(\Phi, n) - \frac{A(\Phi, n) - G(\Phi, n)}{2}. \tag{4}$$

So, for unlabeled undirected connected graphs, we obtain
$L(\Gamma, n) = [1, 0, 0, 1, 5, 34, 331, 4949, 123764, \ldots] =$ A054931,
and for digraphs,
$L(\Delta, n) = [1, 1, 7, 95, 4628, 760731, \ldots] =$ A054932.

3.2 Arc reversal

We can apply the same idea to other involutory transformations.

Consider first the reversal of arcs in digraphs. Now

$$K_R(\Phi, n) = \frac{A(\Phi, n) + G_R(\Phi, n)}{2}, \tag{3R}$$

where G_R stands for the number of self-converse digraphs and K_R for the number of (unlabeled) digraphs considered up to reversing the arcs.

For digraphs, $A(\Delta, n) =$ A000273 (see 3.1.1),
$G_R(\Delta, n) = [1, 3, 10, 70, 708, 15224, \ldots] =$ A002499 and we obtain
$K_R(\Delta, n) = [1, 3, 13, 144, 5158, 778084, \ldots] =$ A054933.
For relations, $A(\Delta_l, n) =$ A000595,
$G_R(\Delta_l, n) = [2, 8, 44, 436, 7176, 222368, \ldots] =$ A002500 and
$K_R(\Delta_l, n) = [2, 9, 74, 1740, 149572, 48575680, \ldots] =$ A029849.
For oriented graphs,
$A(\Omega, n) = [1, 2, 7, 42, 582, 21480, 2142288, \ldots] =$ A001174,
$G_R(\Omega, n) = [1, 2, 5, 18, 102, 848, 12452, \ldots] =$ A005639 and we obtain
$K_R(\Omega, n) = [1, 2, 6, 30, 342, 11164, 1077370, \ldots] =$ A054934.

3.3 Planar maps

Equation (3) has a form which is intrinsic for unlabeled objects possessing an additional involutory transformation. Such transformations occur in particular for geometric and topological objects like planar maps.[2]

[2] We notice incidentally that formula (3) is a particular case (for the group of order 2) of the result known as Burnside's Lemma. Formulae (2) and (2∗) are also particular cases of (3).

3.3.1 Duality and reflection

The idea can be applied to planar maps (or maps on other surfaces) with respect to topological duality. For the number $A\,(\Phi, n)$ of unrooted (= unlabeled) planar maps with n edges in a class of maps Φ and the corresponding number $G_{\mathrm{D}}(\Phi, n)$ of *self-dual* maps, we have, similarly to (3),

$$K_{\mathrm{D}}(\Phi, n) \;=\; \frac{A\,(\Phi, n) \,+\, G_{\mathrm{D}}(\Phi, n)}{2}, \tag{3D}$$

where $K_{\mathrm{D}}(\Phi, n)$ denotes the number of unrooted maps considered *up to duality*.

At present, a formula for $G_{\mathrm{D}}(\Phi, n)$ seems to be known in only one case, namely, for the class $\Phi \,=\, \Lambda$ of all planar maps considered on the sphere with a distinguished orientation [13]. In this case,

$$K^{+}_{\mathrm{D}}(\Lambda, n) \;=\; \frac{A^{+}(\Lambda, n) \,+\, G^{+}_{\mathrm{D}}(\Lambda, n)}{2}, \tag{3D$^+$}$$

where the superscript $^+$ means enumeration up to *orientation-preserving* transformations. Now,
$A^{+}(\Lambda, n) \,=\, [2, 4, 14, 57, 312, 2071, 15030, 117735, 967850, 8268816, \ldots] \,=\,$ A006384 and
$G^{+}_{\mathrm{D}}(\Lambda, n) \,=\, [0, 2, 0, 9, 0, 69, 0, 567, 0, 5112, \ldots] \,=\,$ A006849 interspersed with 0s. Hence
$K^{+}_{\mathrm{D}}(\Lambda, n) \,=\, [1, 3, 7, 33, 156, 1070, 7515, 59151, 483925, 4136964, \ldots] \,=\,$ A054935.

Instead of duality, let us consider reflections. We obtain the formula

$$A\,(\Lambda, n) \;=\; \frac{A^{+}(\Lambda, n) \,+\, G_{\mathrm{ach}}(\Lambda, n)}{2}, \tag{3a}$$

where $G_{\mathrm{ach}}(\Lambda, n)$ denotes the number of *achiral* maps (i.e. maps isomorphic to their mirror images) considered up to orientation-preserving isomorphisms.

Thus from
$A\,(\Lambda, n) \,=\, [2, 4, 14, 52, 248, 1416, 9172, 66366, 518868, 4301350, \ldots] \,=\,$ A006385 we have
$G_{\mathrm{ach}}(\Lambda, n) \,=\, [2, 4, 14, 47, 184, 761, 3314, 14997, 69886, 333884, \ldots] \,=\,$ A054936. Here it is perhaps more natural to consider maps of the complementary class, i.e. *chiral* maps, i.e.

$$G_{\mathrm{ch}}(\Lambda, n) \;=\; A^{+}(\Lambda, n) \,-\, A\,(\Lambda, n) \;=\; A\,(\Lambda, n) \,-\, G_{\mathrm{ach}}(\Lambda, n).$$

Hence
$G_{\mathrm{ch}}(\Lambda, n) \,=\, [0, 0, 0, 5, 64, 655, 5858, 51369, 448982, 3967466, \ldots] \,=\,$ A054937.

It would also be interesting to investigate planar maps with respect to the *central symmetry*.

3.3.2 Circular objects

By circular objects we refer to various classes of geometric figures defined inside a disk, or, more concretely, inside a convex (regular) polygon. Examples are *necklaces* (i.e. strings considered up to rotations), triangulations of a polygon and other types of dissections (that is, non-separable outerplanar maps).

Enumerative results for necklaces are well known and widely represented in the OEIS. In particular, there are many sequences enumerating necklaces that can be turned over; such necklaces are sometimes called *bracelets*. For any type of necklace, the same semi-sum formula connects three corresponding sequences that enumerate, respectively, necklaces, bracelets and strings up to both rotations *and* turning over (i.e. reversal or reflection). So whenever two sequences are known, the third can immediately be obtained. Moreover, just as for maps (see 3.3.1), instead of bracelets it is sometimes useful to switch to their complementary set, i.e. to count necklaces that are not isomorphic to their reversals.

Another natural transformation of necklaces is an interchange between bead colors (or string letters). Again, if this is an involution (such as the transposition of two colors), then three appropriate quantities arise which are connected by the same formula (see [6]). Moreover, one may combine this involution with the reversal and count necklaces up to this combined transformation as well as those invariant with respect to it.

An unusual instance of the semi-sum formula arises for two-color necklaces with $2n$ beads in which opposite beads have different colors. In other words, these are necklaces that are self-dual with respect to a

180° rotation combined with the transposition of the colors. According to [14], the number of such self-dual necklaces is given by the expression

$$Q(n) = \frac{h(n) + 2^{\lfloor (n-1)/2 \rfloor}}{2},$$

where

$$h(n) = \frac{1}{2n} \sum_{k|n,\ k \text{ odd}} \phi(k)\, 2^{n/k}$$

involving the Euler totient function $\phi(n)$. This is the sequence
$Q(n) = Q(\Psi, n) = [1, 1, 2, 2, 4, 5, 9, 12, 23, 34, 63, \ldots] =$ A007147.

At the same time,
$h(n) = h(\Theta, n) = [1, 1, 2, 2, 4, 6, 10, 16, 30, 52, 94, \ldots] =$ A000016 enumerates so-called vortex-free labeled tournaments (see in particular [8], p. 14). It is curious to notice that there is also a sensible shift transformation of $Q(n)$: according to [1],

$$Q(n) - [n^2/12] - 1$$

enumerates a class of polytopal spheres, where square brackets mean the nearest integer. Numerically this is
$[0, 0, 0, 0, 1, 1, 4, 6, 15, 25, 52, \ldots] =$ A059736.

Other specific examples of self-dual necklaces can be found, e.g., in [14, 17]. Instead of discussing them here, we turn to an important but less familiar class Ξ of circular object called chord diagrams. A *chord diagram* is a set of chords between pairwise different nodes lying on an oriented circle. Chords may intersect and their sets are considered up to an isotopy transforming the circle to itself. If no restrictions are imposed, the number of chord diagrams $A^+(\Xi, n)$ with n chords and the number of reversible (achiral) chord diagrams $G_{\text{ach}}(\Xi, n)$ can easily be evaluated (see details in [25, 2]). The corresponding (3a)-type formula has $A(\Xi, n)$ on the left-hand side, where $A(\Xi, n)$ denotes the number of chord diagrams considered up to reflection.

Numerically,
$A^+(\Xi, n) = [1, 2, 5, 18, 105, 902, 9749, 127072, 1915951, \ldots] =$ A007769 and
$G_{\text{ach}}(\Xi, n) = [1, 2, 5, 16, 53, 206, 817, 3620, 16361, \ldots] =$ A018191, therefore
$A(\Xi, n) = [1, 2, 5, 17, 79, 554, 5283, 65346, 966156, \ldots] =$ A054499. So, for the complementary sequence of *chiral* chord diagrams $G_{\text{ch}}(\Xi, n) = A(\Xi, n) - G_{\text{ach}}(\Xi, n)$ we obtain
$G_{\text{ch}}(\Xi, n) = [0, 0, 0, 1, 26, 348, 4466, 61726, 949795, \ldots] =$ A054938.

4 Even- and odd-edged graphs

Consider a specific type of sequence: the numbers $f_{\text{E}}(n)$ and $f_{\text{O}}(n)$ of graphs (of a given class with unspecified numbers of edges) with *even* and *odd* numbers of edges. In some non-trivial cases one can easily express both numbers in terms of the numbers of the corresponding graphs. We use a formal approach based on generating functions. The formulae arising in this way are fairly uniform, but require individual proofs. The general idea (going back to [6]) is to evaluate the difference $f_{\text{E}}(\Phi, n) - f_{\text{O}}(\Phi, n)$ (in other words, this is a weighted enumeration of graphs, where an (n, N)-graph gets the weight $(-1)^N$). It is clearly equal to $\mathbf{f}(\Phi, n, -1)$ and often turns out to be a very simple function.

We also consider analogous sequences $F_{\text{E}}(n)$ and $F_{\text{O}}(n)$ for unlabeled graphs, but here fewer results have been obtained.

4.1 Labeled graphs

4.1.1 Connected graphs

For the class Γ, as we know, the e.g.f. of the number $c(n, N)$ of labeled connected (n, N)-graphs satisfies the equation

$$\mathbf{c}(z, x) = \log(1 + \mathbf{a}(z, x)),$$

where the corresponding o.g.f. for n-graphs for varying N are $\mathbf{a}(n,x) = (1+x)^{n(n-1)/2}$ and $\mathbf{c}(n,x) = \sum_N c(n,N)x^N$ (Γ is dropped everywhere for simplicity). Thus $\mathbf{a}(n,-1) = 0$ for $n > 1$, $\mathbf{a}(1,-1) = 1$ and $\mathbf{a}(z,-1) = z$. Hence $\mathbf{c}(z,-1) = \log(1+z)$ and

$$c_{\mathrm{E}}(n) - c_{\mathrm{O}}(n) = \mathbf{c}(n,-1) = -(-1)^n(n-1)!.$$

This is *Amer. Math. Monthly* problem #6673, and in [22] one can find another proof and a generalization to k-component graphs. We notice also that $(-1)^{n-1}(n-1)!$ is the Möbius function of the lattice of set partitions.

Finally, $c_{\mathrm{E}}(n) + c_{\mathrm{O}}(n) = c(n)$, hence

$$c_{\mathrm{E}}(\Gamma,n) = \frac{c(\Gamma,n) - (-1)^n(n-1)!}{2}$$

and

$$c_{\mathrm{O}}(\Gamma,n) = \frac{c(\Gamma,n) + (-1)^n(n-1)!}{2}.$$

Numerically (with $c(\Gamma,n) = [1, 1, 4, 38, 728, 26704, 1866256\ldots] =$ A001187,
$c_{\mathrm{E}}(\Gamma,n) = [1, 0, 3, 16, 376, 13292, 933488, \ldots] =$ A054939 and
$c_{\mathrm{O}}(\Gamma,n) = [0, 1, 1, 22, 352, 13412, 932768, \ldots] =$ A054940.

4.1.2 Connected digraphs

The same result is valid for (weakly) connected labeled digraphs Δ (see my comment in [22]); in the proof we need only use the generating function $(1+x)^{n(n-1)}$ instead of $(1+x)^{n(n-1)/2}$.

4.1.3 Symmetric relations

For the class of graphs with loops Γ_1, the same proof with $(1+x)^{n(n+1)/2}$ instead of $(1+x)^{n(n-1)/2}$ results in $\mathbf{a}(z,-1) = 0$ and $\mathbf{c}(n,-1) = 0$. Hence

$$c_{\mathrm{E}}(\Gamma_1,n) = c_{\mathrm{O}}(\Gamma_1,n) = c(\Gamma_1,n)/2$$

(by complementarity, this is evident for $n \equiv 1, 2 \pmod 4$).

4.1.4 Oriented graphs

For oriented graphs Ω, we work with the polynomials $\mathbf{a}(n,x) = (1+2x)^{n(n-1)/2}$, so that $\mathbf{a}(n,-1) = (-1)^{n(n-1)/2}$. Now $\mathbf{a}(z,-1) = \cos(z) + \sin(z) - 1$ and

$$\mathbf{c}(\Omega,z,-1) = \log(\cos(z) + \sin(z)).$$

Therefore
$c_{\mathrm{E}}(\Omega,n) - c_{\mathrm{O}}(\Omega,n) = [1, -2, 4, -16, 80, -512, 3904, -34816, \ldots]$, which is A000831 (the expansion of $(1+\tan x)/(1-\tan x)$) up to alternating signs.
$c_{\mathrm{E}}(\Omega,n) + c_{\mathrm{O}}(\Omega,n) = c(\Omega,n) = [1, 2, 20, 624, 55248, 13982208, \ldots] =$ A054941. Thus
$c_{\mathrm{E}}(\Omega,n) = [1, 0, 12, 304, 27664, 6990848, \ldots] =$ A054942 and
$c_{\mathrm{O}}(\Omega,n) = [0, 2, 8, 320, 27584, 6991360, \ldots] =$ A054943.

4.1.5 Strongly connected digraphs

Proposition. *For labeled strong digraphs,*

$$s_{\mathrm{E}}(\Delta,n) - s_{\mathrm{O}}(\Delta,n) = (n-1)!\,. \tag{5}$$

Remark. This is the *Amer. Math. Monthly* problem [15] mentioned earlier without proof in [22].

Proof. Let $s\,(n,N) = s\,(\Delta,n,N)$. The left-hand difference in (5) is $\mathbf{s}\,(n,-1)$. According to [11] (cf. also [26]),

$$\mathbf{s}\,(z,x) \;=\; -\,\log\,(1\;-\;\mathbf{v}\,(z,x)),$$

where $\mathbf{v}\,(z,x) \;=\; \sum_{n\geq 1}\mathbf{v}\,(n,x)z^n/n!$, $\mathbf{v}\,(n,x) \;=\; \mathbf{a}\,(n,x)\,\mathbf{u}\,(n,x)$, $\mathbf{a}\,(n,x) \;=\; (1+x)^{n(n-1)/2}$, $\mathbf{a}\,(z,x) \;=\; \sum_{n\geq 1}\mathbf{a}\,(n,x)z^n/n!$ (hence $a\,(n,N) \;=\; a\,(\Gamma,n,N)$ is the number of all labeled undirected graphs) and

$$\mathbf{u}\,(z,x) \;=\; \sum_{n\geq 1}\mathbf{u}\,(n,x)\frac{z^n}{n!} \;=\; 1\;-\;\frac{1}{1+\mathbf{a}\,(z,x)}. \tag{6}$$

As we saw in 4.1.1, $\mathbf{a}\,(n,-1) \;=\; 0$ for $n>1$. Moreover, $\mathbf{a}\,(1,-1) \;=\; \mathbf{u}\,(1,-1) \;=\; 1$. Therefore $\mathbf{v}\,(z,-1) \;=\; z$, whence $\mathbf{s}\,(z,-1) \;=\; -\log\,(1-z)$ and $\mathbf{s}\,(n,-1) \;=\; (n-1)!$. ∎

Different proofs can be found in [24].

Corollary.

$$s_{\mathrm{E}}(\Delta,n) \;=\; \frac{s\,(\Delta,n)\;+\;(n-1)!}{2}$$

and

$$s_{\mathrm{O}}(\Delta,n) \;=\; \frac{s\,(\Delta,n)\;-\;(n-1)!}{2}.$$

Thus, from $s\,(\Delta,n) \;=\; [1,\,1,\,18,\,1606,\,565080,\,\ldots] \;=\;$ A003030, we obtain
$s_{\mathrm{E}}(\Delta,n) \;=\; [1,\,1,\,10,\,806,\,282552,\,\ldots] \;=\;$ A054944 and
$s_{\mathrm{O}}(\Delta,n) \;=\; [0,\,0,\,8,\,800,\,282528,\,\ldots] \;=\;$ A054945.

Let

$$v\,(n) \;=\; v\,(\Delta,n) \;=\; 2^{n(n-1)/2}u\,(n),$$

where the e.g.f. $\mathbf{u}\,(z) \;=\; 1\;-\;1/(1+\mathbf{a}\,(z))$ and $\mathbf{a}\,(z) \;=\; \sum_{n\geq 1}2^{n(n-1)/2}z^n/n!$. It is known that $u\,(n)$ enumerates strong labeled tournaments (see, e.g., [7], (5.2.4)). So this is the sequence
$u\,(n) \;=\; s\,(\Theta,n) \;=\; [1,\,0,\,2,\,24,\,544,\,22320,\,1677488,\,\ldots] \;=\;$ A054946. The factors $2^{n(n-1)/2}$ form the sequence
$a\,(\Gamma,n) = a\,(\Theta,n) \;=\; [1,\,2,\,8,\,64,\,1024,\,32768,\,2097152,\,\ldots] \;=\;$ A006125. Thus
$v\,(n) \;=\; [1,\,0,\,16,\,1536,\,557056,\,731381760,\,\ldots] \;=\;$ A054947.

4.1.6 A digression: semi-strong digraphs

As we pointed out in [11], $v\,(n) \;=\; s^{\mathrm{O}}(\Delta,n)\;-\;s^{\mathrm{E}}(\Delta,n)$, where $s^{\mathrm{E}}(\Delta,n)$ and $s^{\mathrm{O}}(\Delta,n)$ are the numbers of semi-strong digraphs (see 2.2) with an even and odd *number of components*. Moreover,
$s^{\mathrm{O}}(\Delta,n)\;+\;s^{\mathrm{E}}(\Delta,n) \;=\; s^{\mathrm{W}}(\Delta,n)$, where $s^{\mathrm{W}}(\Delta,n)$ denotes the number of labeled semi-strong digraphs, which is easily expressed via $s\,(\Delta,n)$ by the EXP transformation [4, 20]. This provides a way to evaluate $s^{\mathrm{E}}(\Delta,n)$ and $s^{\mathrm{O}}(\Delta,n)$. Specifically,
$s^{\mathrm{W}}(\Delta,n) \;=\; [1,\,2,\,22,\,1688,\,573496,\,738218192,\,\ldots] \;=\;$ A054948,
$s^{\mathrm{O}}(\Delta,n) \;=\; [1,\,1,\,19,\,1612,\,565276,\,734799976,\,\ldots] \;=\;$ A054949 and
$s^{\mathrm{E}}(\Delta,n) \;=\; [0,\,1,\,3,\,76,\,8220,\,3418216,\,\ldots] \;=\;$ A054950.

There is a similar formula for the corresponding odd-even difference for unlabeled semi-strong digraphs with *mutually non-isomorphic components*: $V\,(n) \;=\; S^{\mathrm{O}}(\Delta,n)\;-\;S^{\mathrm{E}}(\Delta,n)$. This alternating sum plays a key role in the enumeration of unlabeled strongly connected digraphs [11]:
$1-\sum_n V\,(n)z^n = \prod_n(1-z^n)^{S\,(\Delta,n)}$. From these formulae one can extract $S^{\mathrm{E}}(\Delta,n)$ and $S^{\mathrm{O}}(\Delta,n)$. First we need to evaluate $V\,(n)$. In [11] we gave a direct (though difficult) formula and numerical data for the corresponding two-parametric function $V\,(n,N)$. But now we may proceed in the opposite direction, using the above expression and known values of $S\,(\Delta,n)$. Numerically,
$S\,(\Delta,n) \;=\; [1,\,1,\,5,\,83,\,5048,\,1047008,\,\ldots] \;=\;$ A035512, whence we evaluate

$V(n) = [1, 1, 4, 78, 4960, 1041872, \ldots] =$ A054951. Now $S^{\mathrm{O}}(\Delta, n) + S^{\mathrm{E}}(\Delta, n) = S^{\mathrm{W}}(\Delta, n)$, the number of semi-strong digraphs with pairwise different components. We have $1 + \sum_n S^{\mathrm{W}}(\Delta, n) z^n = \prod_n (1 + z^n)^{S(\Delta, n)}$ (this series corresponds to the WEIGH transformation [4, 5, 20]). Therefore
$S^{\mathrm{W}}(\Delta, n) = [1, 1, 6, 88, 5136, 1052154, \ldots] =$ A054952. Thus
$S^{\mathrm{O}}(\Delta, n) = [1, 1, 5, 83, 5048, 1047013, \ldots] =$ A054953 and
$S^{\mathrm{E}}(\Delta, n) = [0, 0, 1, 5, 88, 5141, \ldots] =$ A054954.

Evidently, other types of disconnected (di)graphs, labeled or unlabeled, specified by the parity of the number of components are also worth considering.

4.1.7 Eulerian digraphs

The next assertion is new.

Proposition. *For labeled balanced digraphs,*

$$a_{\mathrm{E}}(\Delta_{\mathrm{e}}, n) = \frac{a(\Delta_{\mathrm{e}}, n) + n!}{2} \qquad (7_{\mathrm{E}})$$

and

$$a_{\mathrm{O}}(\Delta_{\mathrm{e}}, n) = \frac{a(\Delta_{\mathrm{e}}, n) - n!}{2}. \qquad (7_{\mathrm{O}})$$

For labeled Eulerian (i.e. connected balanced) digraphs,

$$c_{\mathrm{E}}(\Delta_{\mathrm{e}}, n) = \frac{c(\Delta_{\mathrm{e}}, n) + (n-1)!}{2} \qquad (8_{\mathrm{E}})$$

and

$$c_{\mathrm{O}}(\Delta_{\mathrm{e}}, n) = \frac{c(\Delta_{\mathrm{e}}, n) - (n-1)!}{2}. \qquad (8_{\mathrm{O}})$$

Proof. According to Theorem 2 of [10], the o.g.f. $\mathbf{a}(\Delta_{\mathrm{e}}, n, x)$ of balanced digraphs can be expressed by a formula in terms of m-roots of unity, $m \geq n$. Choosing $m = n$, and putting $x := -1$, we have from that formula,

$$\mathbf{a}(\Delta_{\mathrm{e}}, n, -1) = n^{-n} n! \prod_{1 \leq k \neq l \leq n} (1 - w^{k-l}),$$

where w is a primitive n-root of unity. Thus

$$\mathbf{a}(\Delta_{\mathrm{e}}, n, -1) = n^{-n} n! \prod_{r=1}^{n} (1 - w^r)^n.$$

But $\prod_r (1 - w^r) = n$, since this is merely the polynomial $(z^n - 1)/(z - 1)$ evaluated at $z = 1$. Thus,

$$\mathbf{a}(\Delta_{\mathrm{e}}, n, -1) = n!$$

This implies formulae (7_{E}) and (7_{O}).

Now, for connected balanced digraphs, $c_{\mathrm{E}}(\Delta_{\mathrm{e}}, n) - c_{\mathrm{O}}(\Delta_{\mathrm{e}}, n) = \mathbf{c}(\Delta_{\mathrm{e}}, n, -1)$. As usual, $\mathbf{c}(\Delta_{\mathrm{e}}, z, x) = \log(1 + \mathbf{a}(\Delta_{\mathrm{e}}, z, x))$. By the above formulae, $\mathbf{a}(\Delta_{\mathrm{e}}, z, -1) = z/(1 - z)$, thus we have $\log(1 + z/(1 - z)) = \sum_{n \geq 1} z^n / n$ and $\mathbf{c}(\Delta_{\mathrm{e}}, n, -1) = (n - 1)!$. ∎

Numerically we obtain the following sequences:
$a(\Delta_{\mathrm{e}}, n) = [1, 2, 10, 152, 7736, 1375952, \ldots] =$ A007080 whence by (7_{E}),
$a_{\mathrm{E}}(\Delta_{\mathrm{e}}, n) = [1, 2, 8, 88, 3928, 688336, \ldots] =$ A054955, and by (7_{O}),
$a_{\mathrm{O}}(\Delta_{\mathrm{e}}, n) = [0, 0, 2, 64, 3808, 687616, \ldots] =$ A054956. Now (by the LOG transformation),
$c(\Delta_{\mathrm{e}}, n) = [1, 1, 6, 118, 7000, 1329496, \ldots] =$ A054957 so that
$c_{\mathrm{E}}(\Delta_{\mathrm{e}}, n) = [1, 1, 4, 62, 3512, 664808, \ldots] =$ A054958 and
$c_{\mathrm{O}}(\Delta_{\mathrm{e}}, n) = [0, 0, 2, 56, 3488, 664688, \ldots] =$ A054959.

4.2 Unlabeled graphs

Here we restrict ourselves to one class of graphs, Γ (but compare also 4.1.6). Consider the difference $A_{\mathrm{E}}(\Gamma,n)\,-\,A_{\mathrm{O}}(\Gamma,n)$. This is clearly the value at $x = -1$ of the corresponding o.g.f. $\mathbf{A}\,(\Gamma,n,x) \,=\, \sum_N A\,(\Gamma,n,N)x^N$. According to the Pólya enumeration theorem (see for example [7], (4.1.8)),

$$\mathbf{A}\,(\Gamma,n,x) \,=\, \mathbf{Z}\,(\mathrm{S}_n^{(2)},1+x,1+x^2,\ldots),$$

where $\mathbf{Z}\,(\mathrm{S}_n^{(2)},z_1,z_2,\ldots)$ denotes the cycle index of the symmetric group S_n in its induced action on the 2-subsets of vertices. Thus

$$A_{\mathrm{E}}(\Gamma,n)\,-\,A_{\mathrm{O}}(\Gamma,n) \,=\, \mathbf{Z}\,(\mathrm{S}_n^{(2)},0,2,0,2,\ldots). \tag{9}$$

We see that the right-hand side coincides with the formula (6.2.3) in [7] for the number $G\,(\Gamma,n)$ of self-complementary graphs. Thus [23], $A_{\mathrm{E}}(\Gamma,n)\,-\,A_{\mathrm{O}}(\Gamma,n) \,=\, G\,(\Gamma,n)$. But $A_{\mathrm{E}}(\Gamma,n)\,+\,A_{\mathrm{O}}(\Gamma,n) \,=\, A\,(\Gamma,n)$. Therefore

$$A_{\mathrm{E}}(\Gamma,n) \,=\, \frac{A\,(\Gamma,n)\,+\,G\,(\Gamma,n)}{2} \tag{10_{E}}$$

and

$$A_{\mathrm{O}}(\Gamma,n) \,=\, \frac{A\,(\Gamma,n)\,-\,G\,(\Gamma,n)}{2}. \tag{10_{O}}$$

So, comparing formulae (10_{E}) and (3), we obtain the following identity:

$$A_{\mathrm{E}}(\Gamma,n) \,=\, K\,(\Gamma,n).$$

We note also that $A_{\mathrm{E}}(\Gamma,n) \,=\, A_{\mathrm{O}}(\Gamma,n) \,=\, A\,(\Gamma,n)/2$ if $n = 4k+2$ or $4k+3$.

From the numerical data for $A\,(\Gamma,n)$ and $G\,(\Gamma,n)$ (or, instead, $K\,(\Gamma,n)$) presented in 3.1.1, one gets $A_{\mathrm{O}}(\Gamma,n) \,=\,$ [0, 1, 2, 5, 16, 78, 522, 6168, 137316, ...] = A054960.

Similar assertions are valid for arbitrary digraphs and some other classes of graphs.

5 Concluding remark

In principle, there is an easy way to obtain numerous new sequences from known ones. Namely, if $a\,(n)$ and $b\,(n)$ count objects of two types, then of course their product $a\,(n)b\,(n)$ counts ordered pairs of objects, and their sum $a\,(n)\,+\,b\,(n)$ counts objects of their disjoint union. As a rule this can hardly be considered as a really fruitful idea: in general, such pairs and the union are unnatural. But sometimes, the term-by-term product (and, still more often, the sum) of two sequences turns out to have a natural interpretation, though possibly unexpected. In this work we encountered various sequences that can be presented as the semi-sum or sum of two other sequences. Only one sequence (namely, $v\,(n)$ in 4.1.5) was presented as the product of two sequences (one of which is, moreover, primitive). Several more such examples can be found in [9]. As far as I know, no systematic investigations of such meaningful operations has been undertaken so far.

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Received Dec. 30, 1999, revised version received May 24, 2000, published in Journal of Integer Sequences Feb. 9, 2001.

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