The relations between the Fibonacci and the Lucas numbers

Leonardo Fibonacci of Pisa was mathematican in the 13 th century, Italy. By charting the population of rabbits, he discovered a number series from which one can derive the Golden Mean. Here' s the beginning of the sequence :

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ….

Each number is the sum of the two proceeding numbers

$$F(n+2) = F(n+1) + F(n)$$

where

$$F(1) = 1 \qquad F(6) = 8$$
$$F(2) = 1 \qquad F(7) = 13$$
$$F(3) = 2 \qquad F(8) = 21$$
$$F(4) = 3 \qquad F(9) = 34$$
$$F(5) = 5 \qquad F(10) = 55$$

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etc.

We shall now introduce operator of the finite differences that associates the function

$$Hy(x) = y(x+1) - y(x)$$

with the function

$$y = f(x)$$

It is easy to verify that

$$H = e^{D} - 1$$

or

$$H = \sum_{k=0}^{\infty} \frac{D^{k}}{k!}$$

where D is the operator of differentation

$$Dy(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h} = \ln(1+H)\, y(x)$$

and where

$$H^{n} = (-1+e^{D})^{n} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} e^{kD}$$

If we prefer operator of the finite differences in the formula for the Fibonacci numbers, than here is another form

$$(1+H)^{2} F(n) = (1+H)F(n) + F(n)$$

or

$$(1+H)^{2} = (1+H) + 1$$

It may be noted that it is possible to derive operator's formulas for the Fibonacci numbers.
As will be seen,we usually deal with this operator's equations for the Fibonacci and Lucas series

$$H = \frac{1}{1+H}$$

and

$$1+H = \frac{1}{H}$$

Here,we have the operator's equation

$$H^2 + H = 1$$

which is analog with the well known equation

$$x^2 + x = 1$$

The solution of this equations is the value of the Golden Mean

$$phi = \frac{(\sqrt{5}-1)}{2}$$

Here we have the indentity

$$phi \equiv H$$

As we know, the general formula for the n-th degree of the Golden Mean, is

$$(-1)^{n+1}(phi)^{2n} + L(n)\,phi^{n} = 1$$

where L(n) are the Lucas numbers.
The Lucas numbers are formed in the same way as the Fibonacci numbers – by adding the lates two to get the next but instead of starting at 1 and 1 (the Fibonacci numbers), then start with 1 and 3 (the Lucas numbers).
Each Lucas number is the sum of the two proceeding numbers :

$$L(n+2) = L(n+1) + L(n)$$

where

$L(1) = 1$	$L(6) = 18$
$L(2) = 3$	$L(7) = 29$
$L(3) = 4$	$L(8) = 47$
$L(4) = 7$	$L(9) = 76$
$L(5) = 11$	$L(10) = 123$

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etc.

Now, setting

$$phi \equiv H$$

In the general formula for the n-th degree of the Golden Mean, we get formula

$$(-1)^{n+1}H^{2n} + L(n)H^{n} = 1$$

As

$$H = \frac{1}{1+H}$$

We get, for the Fibonacci series, this relation

$$(-1)^{n+1} F(x-2n) + L(n) F(x-n) = F(x)$$

This is the relation between the Lucas and Fibonacci numbers

$$L(n) = \frac{F(x) + (-1)^n F(x-2n)}{F(x-n)}$$

In case x=2n,we get relation

$$L(n) = \frac{F(2n)}{F(n)}$$

For the Lucas series , we get this relation

$$L(n) = \frac{L(x) + (-1)^n L(x-2n)}{L(x-n)}$$

In case x=n, we have relation

$$L(-n) = (-1)^n L(n)$$

In case x=2n, we have relation

$$L(2n) = L^2(n) - 2(-1)^n$$

We can find a formula for F(n) which involves only n and does not need any other Fibonacci values. Binet's formula involves the Golden Mean number phi and its reciprocal Phi.

$$\sqrt{5}F(n) = Phi^n - (-1)^n\, phi^n$$

If we start with the Binet's formula in form

$$\left[2\, phi^{n+1} + phi^n\right]F(n) = 1 - (-1)^n\, phi^{2n}$$

and if we put in it the identity

$$phi \equiv \frac{1}{1+H}$$

we get the operator's equation

$$F(n)\left[2\frac{1}{(1+H)^{n+1}} + \frac{1}{(1+H)^n}\right] = 1 - (-1)^n \frac{1}{(1+H)^{2n}}$$

For the Lucas numbers we have the relation

$$F(n)\left[2L(x-n-1) + L(x-n)\right] = L(x) - (-1)^n L(x-2n)$$

Now , we get the relation between the Fibonacci and Lucas numbers

$$F(n) = \frac{L(x) - (-1)^n L(x-2n)}{2L(x-n-1) + L(x-n)}$$

For the Fibonacci series we get this relation

$$F(n) = \frac{F(x) - (-1)^n F(x - 2n)}{2F(x - n - 1) + F(x - n)}$$

In case x=n we have relation

$$F(-n) = (-1)^{n+1} F(n)$$

In case x=2n we have relation

$$F(2n) = F^2(n) + 2F(n)F(n-1)$$