WHEN IS A SCHUBERT VARIETY GORENSTEIN?

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1. INTRODUCTION

The main goal of this paper is to give an explicit combinatorial characterization of which Schubert varieties in the complete flag variety are Gorenstein.

Let $\mathrm{Flags}(\mathbb{C}^n)$ denote the variety of complete flags $F_\bullet : \langle 0 \rangle \subseteq F_1 \subseteq \ldots \subseteq F_n = \mathbb{C}^n$. Fix a basis $e_1, e_2, \ldots, e_n$ of $\mathbb{C}^n$ and let $E_\bullet$ be the *anti*-canonical reference flag $E_\bullet$, that is, the flag where $E_i = \langle e_{n-i+1}, e_{n-i+2}, \ldots, e_n \rangle$. For every permutation w in the symmetric group S_n, there is the **Schubert variety**

$$X_w = \big\{ F_\bullet \mid \dim(E_i \cap F_j) \geq \#\{k \geq n-i+1, w(k) \leq j\} \big\}.$$

These conventions have been arranged so that the codimension of X_w is $\ell(w)$, that is, the length of any expression for w as a product of simple reflections $s_i = (i \leftrightarrow i+1)$.

The Gorenstein property gives a well-known measurement of how far an algebraic variety is from being smooth; all smooth varieties are Gorenstein, while all Gorenstein varieties are Cohen-Macaulay. In general, a variety is **Gorenstein** if it is Cohen-Macaulay and its canonical sheaf is a line bundle. (Throughout this paper we freely identify vector bundles and their sheaves of sections for convenience.) Recall that on a *smooth* variety X, the **canonical sheaf**, denoted ω_X is $\bigwedge^{\dim(X)} \Omega_X$, where Ω_X is the cotangent bundle of X. For a possibly singular but normal variety, the canonical sheaf is the pushforward of the canonical sheaf $\omega_{X_{\mathrm{smooth}}}$ on the smooth part X_{smooth} of X under the inclusion map. Since every Schubert variety is normal [13, 29] and Cohen-Macaulay [30], the remarks above suffice to define Gorensteinness in the context of this paper. In Section 2.1, we will give the more commonly seen local definition of Gorensteinness. However, combining the above definition together with the results of Ramanathan [30, 31] is what provides our starting point for determining which Schubert varieties are Gorenstein.

Smoothness and Cohen-Macaulayness of Schubert varieties have been extensively studied in the literature; see, for example, [4, 30] and the references therein. While all Schubert varieties are Cohen-Macaulay, very few Schubert varieties are smooth. (See the table at the end of this Introduction.) Explicitly, X_w is smooth if and only if w is "1324-pattern avoiding" and "2143-pattern avoiding" [23]; we give more details on pattern avoidance below.

Our main result (Theorem 1) gives an explicit combinatorial characterization of which Schubert varieties are Gorenstein similar to the above smoothness criteria. This answers a question raised by M. Brion and S. Kumar and passed along to us by A. Knutson; see also [31, p. 88]. Our answer uses a generalized notion of pattern avoidance that we introduce.

To describe our ideas in a simpler case, we first compare the classical smoothness criterion with a characterization of which Schubert varieties in the Grassmannian $\mathrm{Gr}(\ell, n)$ of ℓ-planes in $\mathbb{C}^n$ are Gorenstein. (This is a special case of our main result, as we will explain in Section 3.1.) Schubert varieties X_λ of $\mathrm{Gr}(\ell, n)$ are indexed by partitions λ sitting inside an $\ell \times (n - \ell)$ rectangle.[1] The smooth Schubert varieties are those indexed by partitions λ whose complement in $\ell \times (n - \ell)$ is a rectangle, as explained in, for example, [4] and the references therein. For example, $\lambda = (7, 7, 2, 2, 2)$ indexes a smooth Schubert variety in $\mathrm{Gr}(5, 12)$.
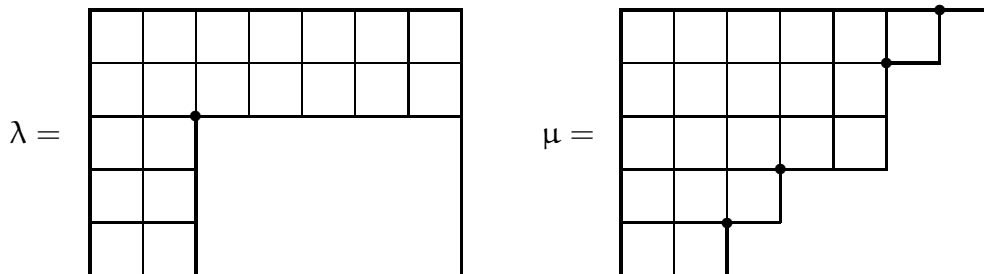
Alternatively, smooth Schubert varieties are those with at most one inner corner. View the lower border of partition as a lattice path from the lower left-hand corner to the upper right-hand corner of $\ell \times (n - \ell)$; an **inner corner** is then a lattice point on this path with lattice points of the path both directly below and directly to the right of it. The inner corners for the partitions λ and μ above are marked by "dots".

Therefore, the partition $\mu = (6, 5, 5, 3, 2)$ above does not index a smooth Schubert variety. However, it does index a Gorenstein Schubert variety; in general, a Grassmannian Schubert variety X_μ is Gorenstein if and only if all of the inner corners of μ lie on the same antidiagonal. We mention that this condition can also be derived from [33, (5.5.5)].

In order to state our main result for $\mathrm{Flags}(\mathbb{C}^n)$, we will need some preliminary definitions. First we associate a Grassmannian permutation to each descent of a permutation w. Let d be a **descent** of w, which is an index such that $w(d) > w(d+1)$. Now write w in one-line notation as $w(1)w(2) \cdots w(n)$, and construct a subword $v_d(w)$ of w by concatenating the right-to-left minima of the segment strictly to the left of $d + 1$ with the left-to-right maxima of the segment strictly to the right of d. In particular, $v_d(w)$ will necessarily include $w(d)$ and $w(d + 1)$. Let $\widetilde{v}_d(w)$ denote the **flattening** of $v_d(w)$, which is defined to be the unique permutation whose entries are in the same relative position as those of $v_d(w)$.

Example 1. Let $w = 314972658 \in S_9$. This permutation has descents at positions 1, 4, 5 and 7. We see that $v_1(w) = 3149$, $v_4(w) = 14978$, $v_5(w) = 147268$, and $v_7(w) = 12658$, so therefore $\widetilde{v}_1(w) = 2134$, $\widetilde{v}_4(w) = 12534$, $\widetilde{v}_5(w) = 135246$, and $\widetilde{v}_7(w) = 12435$.

By construction, $\widetilde{v}_d(w) \in S_m$ is a **Grassmannian permutation**, meaning that it has a unique descent at some position we denote e. For any Grassmannian permutation $w \in S_m$ with its unique descent at e, let $\lambda(w) \subseteq e \times (m - e)$ denote the associated partition. The partition $\lambda(w)$ is the one whose lower border is obtained by drawing a lattice path which starts at the lower left corner of $e \times (m - e)$ and continues by a unit horizontal line segment at step i (for some $i \in \{1, \ldots, m\}$) if i appears strictly after position e (or, in other words, if $w^{-1}(i) > e$), and a unit vertical line segment otherwise. For example,

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[1]Consistent with our convention on Schubert varieties in $\mathrm{Flags}(\mathbb{C}^n)$, we index these Schubert varieties so that $|\lambda|$ is the codimension of X_λ.

the Grassmannian permutation $w = 3589\,11 \mid 12467\,10\,12$ corresponds to the partition $\lambda(w) = \mu = (6, 5, 5, 3, 2)$ depicted above. Now, given an inner corner of a partition $\lambda(w)$, let its **inner corner distance** be the sum of the distances from the top and left edges of the rectangle $e \times (m - e)$ to the inner corner. For example, in μ above, all the inner corner distances equal 6. Furthermore, suppose that $\lambda(w)$ has all its inner corners on the same antidiagonal; this is equivalent to requiring that the inner corner distance be the same for all inner corners. In this case we call this common inner corner distance $\mathfrak{I}(w)$; if there are no inner corners, we set $\mathfrak{I}(w) = 0$ by convention. For our example permutation w, $\mathfrak{I}(w) = 6$.

Next we proceed to define **Bruhat-restricted pattern avoidance**. Recall that, classically, for $v \in S_\ell$ and $w \in S_n$, with $\ell \leq n$, an **embedding of v into w** is a sequence of indices $i_1 < i_2 < \cdots < i_\ell$ such that, for all $1 \leq a < b \leq \ell$, $w(i_a) > w(i_b)$ if and only if $v(a) > v(b)$. Then the classical definition of pattern avoidance is that w **pattern avoids** v if there are no embeddings of v into w.

Now recall the **Bruhat order** $\succ$ on S_n. First we say that $w(i \leftrightarrow j)$ **covers** w if $i < j$, $w(i) < w(j)$, and, for each k with $i < k < j$, either $w(k) < w(i)$ or $w(k) > w(j)$; then the Bruhat order is the transitive closure of this covering relation. The Bruhat order is graded by the length of a permutation, and one can check that v can cover w only if $\ell(v) = \ell(w) + 1$.

Given a permutation $v \in S_\ell$, let $\mathcal{T}_v = \{(m_1 \leftrightarrow n_1), \ldots, (m_k \leftrightarrow n_k)\}$ be a set of **Bruhat transpositions** in v, by which we mean a subset of transpositions such that $v \cdot (m_j \leftrightarrow n_j)$ covers v in the Bruhat order. We define a $\mathcal{T}_v$-**restricted embedding** of v into w to be an embedding of v into w such that $w \cdot (i_{m_j} \leftrightarrow i_{n_j})$ covers w for all $(m_j \leftrightarrow n_j) \in \mathcal{T}_v$. Then we say that w **pattern avoids** v **with Bruhat restrictions** $\mathcal{T}_v$ if there are no $\mathcal{T}_v$-restricted embeddings of v into w.

Now we are ready to state our combinatorial characterization of which Schubert varieties in $\mathrm{Flags}(\mathbb{C}^n)$ are Gorenstein:

Theorem 1. *Let $w \in S_n$. The Schubert variety X_w is Gorenstein if and only if*

- *for each descent d of w, $\lambda(\widetilde{v}_d(w))$ has all of its inner corners on the same antidiagonal, and*
- *the permutation w pattern avoids both 31524 and 24153 with Bruhat restrictions $\{(1 \leftrightarrow 5), (2 \leftrightarrow 3)\}$ and $\{(1 \leftrightarrow 5), (3 \leftrightarrow 4)\}$ respectively.*

In comparing the smoothness characterization of [23] with Theorem 1, considering our description of the Grassmannian case allows one to check that the 1324-pattern avoidance condition of the former implies the "inner corner condition" of the latter. It is also easy to see that the 2143-pattern avoidance condition of the former implies both of the Bruhat-restricted pattern avoidance conditions of the latter. We mention that Fulton [15] has characterized 2143-pattern avoidance in terms of the essential set of a permutation. A similar characterization can be given for the Bruhat-restricted pattern avoidance conditions of Theorem 1.

Example 2. The permutation $w = \underline{51}3\underline{28}4\underline{67} \in S_8$ has descents at positions $1, 3$ and 5 and we have

$$\widetilde{v}_1(w) = 3124, \widetilde{v}_3(w) = 1324, \text{ and } \widetilde{v}_5(w) = 126345.$$

Hence one checks that w satisfies the inner corner condition with

$$\mathfrak{I}(\widetilde{v}_1(w)) = 2, \ \mathfrak{I}(\widetilde{v}_3(w)) = 1, \text{ and } \mathfrak{I}(\widetilde{v}_5(w)) = 1.$$

The Schubert variety X_w is Gorenstein, since there are no forbidden 31524 and 24153 patterns with Bruhat restrictions $\{(1 \leftrightarrow 5), (2 \leftrightarrow 3)\}$ or $\{(1 \leftrightarrow 5), (3 \leftrightarrow 4)\}$ respectively. Note that the underlined subword of w is a 31524-pattern, but since $w(1 \leftrightarrow 8)$ does not cover w, it does not prevent X_w from being Gorenstein.

By combining Theorem 1 with the descriptions of the singularities along the "maximal singular locus" of a Schubert variety X_w given in [12, 25], we obtain the following geometric corollary.

Corollary 1. *A Schubert variety X_w is Gorenstein if and only if it is Gorenstein along its maximal singular locus.*

In other words, Corollary 1 states that a Schubert variety is Gorenstein if and only if its "smoothest" singularities (those at the generic points of the irreducible components of the singular locus) are Gorenstein.

We now describe the canonical sheaf of a Gorenstein Schubert variety in terms of the Borel-Weil construction of line bundles. Let $T \cong (\mathbb{C}^*)^{n-1}$ be the subgroup of invertible diagonal matrices of determinant 1 in $\mathrm{SL}_n(\mathbb{C})$; the Borel-Weil construction associates to each integral weight $\alpha \in \mathrm{Hom}(T, \mathbb{C}^*)$ a line bundle $\mathcal{L}_\alpha$. Let $\mathcal{L}_\alpha\big|_{X_w}$ denote the restriction of this line bundle to X_w. We will write weights additively in terms of the $\mathbb{Z}$-basis of fundamental weights Λ_r, defined by $\Lambda_r \left(\begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{bmatrix} \right) = t_1 \cdots t_r.$

Theorem 2. *If X_w is Gorenstein, then $\omega_{X_w} \cong \mathcal{L}_\alpha\big|_{X_w}$ where $\alpha = \sum_{r=1}^{n-1} \widetilde{\alpha}_r \Lambda_{n-r}$ and*

$$(1) \qquad \widetilde{\alpha}_r = \begin{cases} -2 + \mathfrak{I}(\widetilde{v}_r(w)) & \text{if } r \text{ is a descent} \\ -2 & \text{otherwise.} \end{cases}$$

The proofs of Theorems 1 and 2 as well as Corollary 1 will be given in Section 2. In Section 3, we end with a number of remarks and applications.

Further study of the relationship between the geometry of Gorensteinness of Schubert varieties and related combinatorics should have potential. We conclude this introduction with some open problems and suggestions for further work. The most natural is:

Problem 1. *Give analogues of Theorems 1 and 2 for generalized flag varieties corresponding to Lie groups other than $\mathrm{GL}_n(\mathbb{C})$.*

We expect that the methods given in this paper will extend to solve Problem 1. It is not difficult to use Theorem 1 to derive an analogue of Theorem 1 for the case of the odd orthogonal groups $\mathrm{SO}(2n + 1, \mathbb{C})$. However, we have found the combinatorial analysis required to be more intricate in general. Consequently, in the interest of brevity, we plan to discuss our investigations for the other Lie types in a subsequent paper.

It should also be interesting to determine the "maximal non-Gorenstein locus" of a non-Gorenstein Schubert variety: Let X be a variety that is Cohen-Macaulay but not Gorenstein; since the rank of any coherent sheaf on X is upper semicontinuous (see, for example, [19, III.12.7.2]), the canonical sheaf has rank strictly greater than 1 at some non-trivial closed subvariety. This subvariety then consists of all points of X at which X is not

Gorenstein by the local definition. Since the canonical sheaf of a Schubert variety is B_--equivariant for the subgroup $B_- \subseteq GL_n(\mathbb{C})$ of lower triangular matrices, this subvariety is a union of Schubert varieties contained in X_w. Therefore we ask:

Problem 2. *Give a combinatorial characterization for the minimal v in the Bruhat order for which X_w is not Gorenstein at X_v.*

In view of Corollary 1, it is natural to propose the following answer:

Conjecture 1. *The maximal non-Gorenstein locus of X_w is the union of those Schubert varieties X_v in the maximal singular locus of X_w for which the generic point is not Gorenstein in X_w.*

One can give a combinatorial rule characterizing the set of X_v appearing in Conjecture 1 using the explicit description of the singular locus of Schubert varieties [5, 12, 17, 22, 23, 24, 25] and facts mentioned in the proof of Corollary 1.

A geometric explanation was recently given in [3] for the appearance of pattern avoidance in characterizations of smooth Schubert varieties. However, this explanation does not have an obvious modification to take into account Bruhat-restrictions. This leads to the following:

Problem 3. *Give a geometric explanation of Bruhat-restricted pattern avoidance which explains its appearance in Theorem 1.*

Lastly, for those interested in combinatorial enumeration:

Problem 4. *Give a combinatorial formula (for example, a generating series) computing the number of Gorenstein Schubert varieties in* $\mathrm{Flags}(\mathbb{C}^n)$.

Using the methods of this paper, we computed the number of Gorenstein Schubert varieties in $\mathrm{Flags}(\mathbb{C}^n)$ for some small values of n (see below). We compare this to the number of smooth Schubert varieties computed using the result of [23] (by the recursive formulas found in [6, 32]).

n	$n! = $ # Cohen-Macaulay X_w	# Gorenstein X_w	# Smooth X_w
1	1	1	1
2	2	2	2
3	6	6	6
4	24	24	22
5	120	116	88
6	720	636	366
7	5040	3807	1552
8	40320	24314	6652
9	362880	163311	28696

2. Proof of Theorems 1 and 2

2.1. **Geometry to combinatorics.** First we explain the algebraic definition of Gorenstein-ness and reduce the algebro-geometric problem of determining when a Schubert variety is Gorenstein to a problem in linear algebra; we will then solve this linear algebra problem combinatorially. This reduction to linear algebra appears to be folklore (and was told to us by M. Brion, A. Knutson and S. Kumar); we could not locate an explicit reference for it in the literature. Therefore, we include an argument for the sake of completeness. While we treat only $\mathrm{Flags}(\mathbb{C}^n)$ explicitly, the arguments of this section generalizes easily to all semi-simple Lie groups with the substitution of the appropriate Monk-Chevalley formula [10]. We found that an excellent resource for facts about the geometry of Schubert varieties to be [8].

A local ring $(R, \mathfrak{m}, \mathbb{k})$ is said to be **Cohen-Macaulay** if $\mathrm{Ext}^i_R(\mathbb{k}, R) = 0$ for $i \leq \dim R$; it is **Gorenstein** if, in addition, $\dim_{\mathbb{k}} \mathrm{Ext}^{\dim R}_R(\mathbb{k}, R) = 1$. A variety is Cohen-Macaulay (respectively Gorenstein) if the local ring at every point is Cohen-Macaulay (respectively Gorenstein). Using the Kozsul complex on a regular sequence, one can show that every regular local ring is Gorenstein; hence smooth varieties are Gorenstein. See [9] for details.

One might naively expect that, in order to check if a Schubert variety is Gorenstein, one would need to check if it is Gorenstein at all, or at least some, of its points. However, the alternative equivalent definition of the Gorenstein property alluded to in the introduction, which is based on Grothendieck duality theory (see [20] or [1]), allows for a different approach using the global geometry of Schubert varieties. Each projective variety has a dualizing complex (of sheaves) which play a role analogous to that of the canonical bundle ω_X, that is, the top exterior power of the cotangent bundle $\bigwedge^{\dim(X)} \Omega_X$, of a smooth variety in Serre duality. A connected projective variety is Cohen-Macaulay if and only if the dualizing complex is a sheaf, and Gorenstein if and only if the dualizing sheaf is locally free of rank one. For a normal, Cohen-Macaulay variety, one can realize the dualizing sheaf as the pushforward of the canonical sheaf $\omega_{X_{\mathrm{smooth}}}$ of the smooth part X_{smooth} under the inclusion map. As mentioned in the introduction, all Schubert varieties are known to be normal [13, 29] and Cohen-Macaulay [30], so we can then use the calculation of the canonical sheaf of Schubert varieties by Ramanathan [30, 31] to determine which Schubert varieties are Gorenstein.

We now need some standard definitions which can be found in [19, II.6]. Let $\mathrm{Cl}(X_w)$ denote the Weil divisor class group of X_w; its elements are linear equivalence classes $[Z]$ of formal sums of codimension 1 subvarieties Z of X_w. There is a natural group homomorphism $\mathrm{div} : \mathrm{Pic}(X_w) \to \mathrm{Cl}(X_w)$, where $\mathrm{Pic}(X_w)$ is the group of isomorphism classes of line bundles under tensor product. On a Schubert variety X_w (or, in general, any normal irreducible variety over a field), div is injective and its image in $\mathrm{Cl}(X_w)$ is the **Cartier class group** $\mathrm{CaCl}(X_w)$. (This is an unorthodox definition of the Cartier class group, but for convenience we have identified it with its isomorphic image in the Weil class group.) For smooth varieties, div is an isomorphism, so $\mathrm{CaCl} = \mathrm{Cl}$.

We now proceed to describe explicitly $\mathrm{Cl}(X_w)$ and $\mathrm{CaCl}(X_w)$. The Schubert variety X_w is the disjoint union of the open Schubert cell X_w° (which is isomorphic to the affine space $\mathbb{C}^{\binom{n}{2} - \ell(w)}$) together with the codimension 1 subvarieties X_v for v covering w in the Bruhat order. Therefore, by repeatedly applying [19, Prop. II.6.5], we see that $\mathrm{Cl}(X_w)$ is freely generated (as an abelian group) by $[X_v]$ for v covering w.

To describe $\mathrm{CaCl}(X_w)$, we will need the Chow group $A_*(\mathrm{Flags}(\mathbb{C}^n))$ of the flag variety, whose elements are rational equivalence classes $[Z]$ of subvarieties Z of $\mathrm{Flags}(\mathbb{C}^n)$; see for example [16, Ch. 1]. Since $\mathrm{Flags}(\mathbb{C}^n)$ is smooth, the Chow ring $A^*(\mathrm{Flags}(\mathbb{C}^n))$ is by definition equal as abelian groups to $A_*(\mathrm{Flags}(\mathbb{C}^n))$ [16, 8.3]. The graded pieces $A_d(\mathrm{Flags}(\mathbb{C}^n)) = A^{\binom{n}{2}-d}(\mathrm{Flags}(\mathbb{C}^n))$ are freely generated by the classes $[X_v]$ of the Schubert varieties X_v of dimension d, which are precisely those for which $d = \binom{n}{2} - l(v)$. Therefore, the natural map $\iota_* : \mathrm{Cl}(X_w) \to A_{\binom{n}{2}-\ell(w)-1}(\mathrm{Flags}(\mathbb{C}^n))$ induced by the inclusion $\iota : X_w \to \mathrm{Flags}(\mathbb{C}^n)$ is injective. Note that, by definition, $A_{\binom{n}{2}-1}(\mathrm{Flags}(\mathbb{C}^n)) = \mathrm{Cl}(\mathrm{Flags}(\mathbb{C}^n))$.

It is known [26, Prop. XII.6] that every line bundle on a Schubert variety is the restriction of a line bundle on $\mathrm{Flags}(\mathbb{C}^n)$. Furthermore, for a line bundle $\mathcal{L}$ on $\mathrm{Flags}(\mathbb{C}^n)$, general facts of intersection theory [16, Ch. 2] tell us that $\iota_*(\mathrm{div}(\mathcal{L}\big|_{X_w})) = \mathrm{div}(\mathcal{L}) \cdot [X_w]$, where the right hand side is a product in $A^*(\mathrm{Flags}(\mathbb{C}^n))$. Therefore, since $\mathrm{CaCl}(\mathrm{Flags}(\mathbb{C}^n)) = \mathrm{Cl}(\mathrm{Flags}(\mathbb{C}^n))$ is generated by $\left\{\left[X_{(r\leftrightarrow r+1)}\right]\right\}_{r=1}^{n-1}$, $\iota_*(\mathrm{CaCl}(X_w)) \subseteq A^*(\mathrm{Flags}(\mathbb{C}^n))$ is generated by $\left\{\left[X_{(r\leftrightarrow r+1)}\right] \cdot [X_w]\right\}_{r=1}^{n-1}$.

By Monk's formula [27],

$$\left[X_{(r\leftrightarrow r+1)}\right] \cdot [X_w] = \sum_{\substack{a \leq r < b \\ \ell(w(a\leftrightarrow b))=\ell(w)+1}} \left[X_{w(a\leftrightarrow b)}\right],$$

so $\mathrm{CaCl}(X_w)$ is generated by these classes for $1 \leq r \leq n-1$. (We can drop the ι_* since it is an injection.)

Since Schubert varieties are Cohen-Macaulay [28, 13, 30], a Schubert variety X_w is Gorenstein if and only if its canonical sheaf ω_{X_w} is a line bundle. By results of Ramanathan [31, Thm. 4.2], the canonical sheaf of X_w is

$$\omega_{X_w} = \mathcal{L}_{-\rho}\big|_{X_w} \otimes \mathcal{I}(\partial X_w),$$

where $\mathcal{L}_{-\rho}\big|_{X_w}$ is the restriction to X_w of the line bundle associated to the weight $-\rho = -\sum_{r=1}^{n-1} \Lambda_r$ by the Borel-Weil construction, and $\mathcal{I}(\partial X_w)$ is the ideal sheaf of the complement of X_w°, or equivalently, the ideal sheaf of the reduced subscheme $\bigcup_v X_v$ where v ranges over all permutations covering w in the Bruhat order. Since $\mathcal{L}_{-\rho}\big|_{X_w}$ is a line bundle and Pic is a group, ω_{X_w} is a line bundle if and only if $\mathcal{I}(\partial X_w)$ is a line bundle. However, the ideal sheaf of a reduced codimension 1 subscheme Y is a line bundle if and only if $[Y]$ is a Cartier divisor, in which case $\mathrm{div}(\mathcal{I}(Y)) = -[Y]$; see, for example, [19, II.6]. Therefore, X_w is Gorenstein if and only if

$$[\partial X_w] = \sum_{\substack{v \succ w \\ \ell(v)=\ell(w)+1}} [X_v] \in \mathrm{CaCl}(X_w).$$

Hence, by our previous calculation of $\mathrm{CaCl}(X_w)$ as a subgroup of $\mathrm{Cl}(X_w)$, we obtain the following:

Proposition 1. *The Schubert variety X_w is Gorenstein if and only if there exists an integral solution $(\alpha_1, \ldots, \alpha_{n-1})$ to*

$$(2) \qquad \sum_{r=1}^{n-1} \alpha_r \left(\sum_{\substack{a \leq r < b \\ \ell(w(a\leftrightarrow b))=\ell(w)+1}} \left[X_{w(a\leftrightarrow b)}\right] \right) = \sum_{\substack{v=w(a\leftrightarrow b) \\ \ell(v)=\ell(w)+1}} [X_v] \in \mathrm{CaCl}(X_w).$$

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2.2. Interlude: a diagrammatic formulation and two sample problems.

Although it is not used in our proof below, let us give a diagrammatic formulation of the above linear algebra problem (2) that the reader may find useful.

Label n columns by the values $w(1), w(2), \ldots, w(n)$ of a permutation $w \in S_n$. Draw horizontal bars between the midpoints of columns i and j if and only if $w(i \leftrightarrow j)$ covers w in the Bruhat order. Now draw vertical bars between columns i and $i+1$ for $1 \leq i \leq n-1$. Then a solution to (2) is equivalent to an assignment $(\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{Z}^{n-1}$ of integers to the vertical bars (from left to right respectively) such that, for each horizontal bar, the sum of the assignments to the vertical bars that it crosses equals 1.

We encourage the reader to try out the following two sample problems; answers are at the bottom of the page[2]:
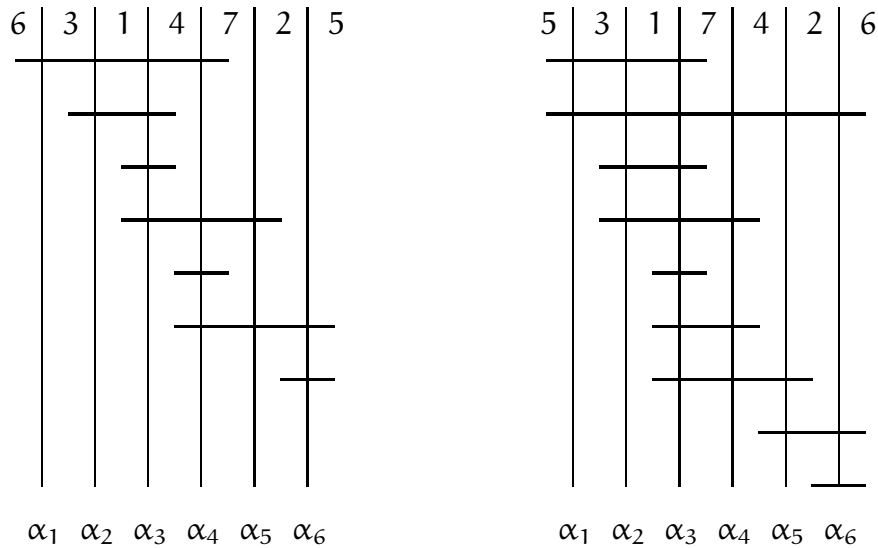
2.3. Necessity of the combinatorial conditions in Theorem 1.

It is possible to prove necessity by appealing to the geometric description of the singularities along the maximal singular locus found in [12, 25]; however we will give a simple, purely combinatorial proof.

We will need the following two lemmas, the first of which is immediate:

Lemma 1. *The vector* $(\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{Z}^{n-1}$ *is a solution to (2) if and only if* $\sum_{r=i}^{j-1} \alpha_r = 1$ *for all* $(i \leftrightarrow j)$ *such that* $w(i \leftrightarrow j)$ *covers* w *in the Bruhat order.*

Lemma 2. *If there exists a solution* $(\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{Z}^{n-1}$ *to (2), and* $i < j$ *with* $w(i) < w(j)$, *then* $\sum_{r=i}^{j-1} \alpha_r \geq 1$. *Equality holds if and only if* $w(i \leftrightarrow j)$ *covers* w.

Proof. If $w(i \leftrightarrow j)$ covers w then the claim holds by Lemma 1. Otherwise, it follows from the observation that there are indices

$$i_0 = i < i_1 < i_2 < \ldots < i_{t-1} < j = i_t$$

such that $w(i_s \leftrightarrow i_{s+1})$ covers w for $0 \leq s \leq t-1$. $\qquad\square$

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[2]The problem on the left is solved by $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = (-1, 0, 1, 1, -1, 1)$ while the problem on the right has no solution.

Now suppose that there is an embedding $i_1 < i_2 < i_3 < i_4 < i_5$ of a 31524 pattern with Bruhat restrictions $\{(1 \leftrightarrow 5), (2 \leftrightarrow 3)\}$. Then by Lemma 2, any solution would satisfy

$$\sum_{r=i_1}^{i_3-1} \alpha_r \geq 1, \ \sum_{r=i_2}^{i_4-1} \alpha_r \geq 1, \ \sum_{r=i_4}^{i_5-1} \alpha_r \geq 1, \text{ and } \sum_{r=i_2}^{i_3-1} \alpha_r = 1.$$

Therefore,

(3) $$\sum_{r=i_1}^{i_5-1} \alpha_r = \sum_{r=i_1}^{i_3-1} \alpha_r + \sum_{r=i_2}^{i_4-1} \alpha_r + \sum_{r=i_4}^{i_5-1} \alpha_r - \sum_{r=i_2}^{i_3-1} \alpha_r \geq 2.$$

Thus (3) is a contradiction of Lemma 1 (or Lemma 2) since $w(i_1 \leftrightarrow i_5)$ covers w. Therefore such an embedding cannot exist. A similar argument shows that there cannot exist an embedding into w of a 24153 pattern with Bruhat restrictions $\{(1 \leftrightarrow 5), (3 \leftrightarrow 4)\}$.

It remains to show that for each descent d of w, $\lambda(\tilde{v}_d(w))$ has all of its inner corners on the same antidiagonal. For this purpose, we need:

Lemma 3. *Let v be a Grassmannian permutation with descent at position d. Then the transpositions $(i \leftrightarrow j)$ with $i \leq d < j$ such that $v(i \leftrightarrow j)$ covers v are in bijection with the inner corners of $\lambda(v)$. Moreover, if $(i \leftrightarrow j)$ corresponds to an inner corner of $\lambda(v)$ under this bijection, then the corresponding inner corner distance equals $j - i - 1$.*

Proof. In terms of the lattice path description of $\lambda(v)$ given on page 2, an inner corner of $\lambda(v)$ occurs exactly when there is an "up step" at time a, followed by a "right step" at time $a + 1$. In terms of v, this means a and $a + 1$ appear in positions i and j satisfying the hypotheses. Conversely, if $i \leq d < j$ and $v(i \leftrightarrow j)$ covers v, then $v(j) = v(i) + 1$. The claims then follow. $\square$

The next lemma is clear from the definition of $v_d(w)$:

Lemma 4. *Let d be a descent of w and suppose (i, j) is a pair $1 \leq i < j \leq n$ that indexes two entries of w included in the subword $v_d(w)$ of w. Let (i', j') be the corresponding indices in $\tilde{v}_d(w)$. Then $w(i \leftrightarrow j)$ covers w if and only if $\tilde{v}_d(w)(i' \leftrightarrow j')$ covers $\tilde{v}_d(w)$.*

Let d be a descent of w and suppose that

$$i_1 < i_2 < \ldots < i_f = a < \ldots < i_s = d < i_{s+1} = d + 1 < i_{s+2} < \ldots < i_g = b < \ldots < i_t$$

are the indices of the subword $v_d(w)$ of w, where $w(a \leftrightarrow b)$ covers w. By Lemmas 2 and 4 combined, any solution satisfies

$$1 = \sum_{r=a}^{b-1} \alpha_r = (s - f) + (g - s - 1) + \alpha_d = g - f - 1 + \alpha_d$$

Now, $g - f - 1$ is the inner corner distance of the corresponding inner corner of $\lambda(\tilde{v}_d(w))$ under the bijection of Lemma 3. Since α_d is fixed, $g - f - 1$ is independent of our choice of a and b. Hence, all of the inner corners of $\lambda(\tilde{v}_d(w))$ have the same inner corner distance, and therefore they must all lie on the same antidiagonal.

2.4. Sufficiency of the combinatorial conditions of Theorem 1. Assume that the combinatorial conditions of Theorem 1 hold. We will show that in fact

(4)
$$\alpha_r = \begin{cases} 1 - \Im(\tilde{v}_r(w)) & \text{if } r \text{ is a descent} \\ 1 & \text{otherwise.} \end{cases}$$

for $1 \le r \le n - 1$ solves (2).

It suffices to show that $\sum_{r=i}^{j-1} \alpha_r = 1$ whenever $w(i \leftrightarrow j)$ covers w. We prove this by induction on $j - i \ge 1$.

The base case $j - i = 1$ of the induction holds by our definition of α_r, since in this case, w does not have a descent at position i.

Now suppose that $j - i > 1$. Let k be chosen (if possible) so that $i < k < j$ and $w(k)$ is minimal such that $w(k) > w(j)$. Similarly, let ℓ be chosen (if possible) so that $i < \ell < j$ and $w(\ell)$ is maximal such that $w(\ell) < w(i)$. Notice that since $w(i \leftrightarrow j)$ covers w, at least one of k or ℓ must exist. We now separately examine the possible cases:

First suppose k exists but not ℓ. Observe that w has a descent at position $j - 1$, since, in fact $w(j) < w(m)$ for all $i \le m \le j - 1$. So we may consider the subword $v_{j-1}(w)$ of w. Notice that this necessarily includes $w(i)$, $w(j - 1)$, and $w(j)$. By Lemma 4, if f and g are indices between i and $j - 1$ in w which correspond to successive entries of $v_{j-1}(w)$, then $w(f \leftrightarrow g)$ covers w. So by induction,

(5)
$$\sum_{r=f}^{g-1} \alpha_r = 1.$$

Since by assumption, the inner corner distances of $\tilde{v}_{j-1}(w)$ are all the same, by (4):

$$\sum_{r=i}^{j-1} \alpha_r = \sum_{r=i}^{j-2} \alpha_r + \alpha_{j-1} = \Im(\tilde{v}_{j-1}(w)) + \alpha_{j-1} = 1$$

as desired. A similar argument works in the case that ℓ exists but not k, except that $v_i(w)$ must be used instead.

Next suppose that both k and ℓ exist. First consider the situation where $k > \ell$. Then by construction,

$$w(i \leftrightarrow k), \ w(\ell \leftrightarrow j), \ \text{and } w(\ell \leftrightarrow k) \text{ each cover } w.$$

Therefore, by the induction hypothesis, we have

$$\sum_{r=i}^{k-1} \alpha_r = 1, \ \sum_{r=\ell}^{j-1} \alpha_r = 1, \ \text{and } \sum_{r=\ell}^{k-1} \alpha_r = 1.$$

Hence,

$$\sum_{r=i}^{j-1} \alpha_r = \sum_{r=i}^{k-1} \alpha_r + \sum_{r=\ell-1}^{j-1} \alpha_r - \sum_{r=\ell}^{k-1} \alpha_r = 1$$

as desired.

Finally, we have the case where $k < \ell$. Observe that the values of w between k and ℓ must consist of numbers larger than $w(k)$ followed by numbers smaller than $w(\ell)$, since otherwise it is easy to see that there must exist a $\{(1 \leftrightarrow 5), (2 \leftrightarrow 3)\}$-restricted embedding of 31524 or a $\{(1 \leftrightarrow 5), (3 \leftrightarrow 4)\}$-restricted embedding of 24153, contradicting the assumptions. Similarly, the values of w between i and k are necessarily smaller than $w(i)$.

Let q be the last index $k \leq q < \ell$ such that $w(q) \geq w(k)$; hence w has a descent at q. Consider the subword $v_q(w)$ of w and observe that $w(i)$ and $w(j)$ are in $v_q(w)$, as, otherwise, we would find a bad 31524 or 24153 pattern. We are now ready to employ a similar argument as above. By Lemma 4, if f and g are indices of w, with either both f and g in the interval $[i, q]$ or both in the interval $[q + 1, j]$, and f and g correspond to consecutive entries of $v_q(w)$, then $w(f \leftrightarrow g)$ covers w; now the induction hypothesis implies (5) as before. Therefore, by (4) and our assumptions about $\lambda(\widetilde{v}_q(w))$, we have

$$\sum_{r=i}^{j-1} \alpha_r = \Im(\widetilde{v}_q(w)) + \alpha_q = 1$$

as required.

Theorem 1 follows immediately from the discussion above.

2.5. **Conclusion of the proof of Theorem 2.** In order to complete the above arguments to prove Theorem 2, we need two facts about the Borel-Weil construction; see, for example, [8, Section 1.4] and the references therein. First, we note that since $\mathcal{L}_{\Lambda_{n-r}} \cong \bigwedge^{n-r} \mathcal{Q}_r$ describes the Borel-Weil line bundle associated to the fundamental weight Λ_{n-r}, where $\mathcal{Q}_r$ is the tautological quotient bundle whose fiber at a flag $F_\bullet = (\langle 0 \rangle \subseteq F_1 \subseteq \ldots \subseteq F_n = \mathbb{C}^n)$ is $\mathbb{C}^n/F_r$, then $\mathrm{div}(\mathcal{L}_{\Lambda_{n-r}}) = [X_{(r \leftrightarrow r+1)}] \in \mathrm{CaCl}(\mathrm{Flags}(\mathbb{C}^n))$; therefore,

$$\mathrm{div}(\mathcal{L}_{\Lambda_{n-r}}\big|_{X_w}) = \sum_{\substack{a \leq r < b \\ \ell(w(a \leftrightarrow b)) = \ell(w)+1}} [X_{w(a \leftrightarrow b)}] \in \mathrm{CaCl}(X_w).$$

Secondly, addition of weights corresponds to tensor product of line bundles, so that, for any weights α and β, the line bundle $\mathcal{L}_{\alpha+\beta} = \mathcal{L}_\alpha \otimes \mathcal{L}_\beta$.

We have shown that, when X_w is Gorenstein,

$$\sum_{r=1}^{n-1} \alpha_r \, \mathrm{div}(\mathcal{L}_{\Lambda_{n-r}}\big|_{X_w}) = \sum_{\substack{v \succ w \\ \ell(v) = \ell(w)+1}} [X_v] = \mathrm{div}(\mathcal{I}(\partial X_w)).$$

Therefore, we have that $\mathcal{I}(\partial X_w) \cong \mathcal{L}_\alpha\big|_{X_w}$, where $\alpha = \sum_{r=1}^{n-1} -\alpha_r \Lambda_{n-r}$. Since $\rho = \sum_{r=1}^{n-1} \Lambda_r$, and we have set $\widetilde{\alpha}_r = -1 - \alpha_r$ in (1), this proves Theorem 2. $\qquad \square$

2.6. **Proof of Corollary 1.** We prove Corollary 1 by comparing Theorem 1 with a description of the generic singularities of a Schubert variety given in [12, 25]. [3]

Suppose a Schubert variety X_w is not Gorenstein along its maximal singular locus. By the local definition of Gorensteinness given in Section 2.1, it is not Gorenstein. To prove the other direction, suppose X_w is not Gorenstein. Then w contains one of the two forbidden patterns, or violates the inner corner condition. If w contains a forbiddern pattern, then, in the language of Cortez [12], w has a configuration II with $r = 0$ and $s + t \geq 1$, and therefore has a generic singularity whose neighborhood is isomorphic to the product of $\mathbb{C}^k$ for some k and the variety of $(s+t+2) \times 2$ matrices of rank at most 1. It is well known that the variety of $p \times q$ matrices of rank at most 1 is Gorenstein if and only if $p = q$; see for example [9, Thm. 7.3.6]; this shows that X_w is not Gorenstein at a generic singularity. If w violates the inner corner condition, then w has a configuration I with $s \neq t$, yielding a

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[3]Note that our notation differs from these papers by right multiplication of a permutation w by w_0.

corresponding generic singularity, which, as it a neighborhood isomorphic to the product of $\mathbb{C}^k$ for some k and the variety of $s \times t$ matrices of rank at most 1, is not Gorenstein.

3. REMARKS AND APPLICATIONS

3.1. **Extension to partial flag varieties.** More generally, let $\text{Flags}(i_1 < i_2 < \ldots < i_k, \mathbb{C}^n)$ denote the variety of partial flags $F_\bullet : \langle 0 \rangle \subseteq F_{i_1} \subseteq F_{i_2} \subseteq \ldots \subseteq F_{i_k} \subseteq \mathbb{C}^n$ in $\mathbb{C}^n$ where here $\dim(F_{i_k}) = i_k$. By convention let $i_0 = 0$ and $i_{k+1} = n$. Now let $S = S_{i_1 - i_0} \times S_{i_2 - i_1} \times \cdots \times S_{i_{k+1} - i_k} \subseteq S_n$ denote the Young subgroup where the $S_{i_j - i_{j-1}}$ factor is generated by the simple reflections $s_{i_{j-1}+1}, \ldots, s_{i_j - 1}$ for all j such that $1 \le j \le k$. The Schubert varieties of $\text{Flags}(i_1 < i_2 < \ldots < i_k, \mathbb{C}^n)$ are indexed by cosets of S. The natural "forgetting subspaces" projection $\pi : \text{Flags}(\mathbb{C}^n) \twoheadrightarrow \text{Flags}(i_1 < i_2 < \ldots < i_k, \mathbb{C}^n)$ is a smooth fiber bundle. It follows that a Schubert variety X_{wS} in $\text{Flags}(i_1 < i_2 < \ldots < i_k, \mathbb{C}^n)$ indexed by a coset wS is Gorenstein if and only if the Schubert variety $X_{\widetilde{w}} = \pi^{-1}(X_{wS})$ in $\text{Flags}(\mathbb{C}^n)$ is Gorenstein, where $\widetilde{w}$ is the minimal length element of wS. In particular, our main result implies the Grassmannian case as presented in the introduction.

3.2. **Uniqueness of (4).** It is worthwhile to note that the induction in Section 2.4 implies that (4) is a solution to (2) if and only if X_w is Gorenstein. Moreover, this solution is essentially unique. The only exception to uniqueness arises for those r where

$$\left[X_{(r \leftrightarrow r+1)}\right] [X_w] = \sum_{\substack{a \le r < b \\ \ell(w(a \leftrightarrow b)) = \ell(w)+1}} \left[X_{w(a \leftrightarrow b)}\right] = 0$$

because the sum on the right hand side is vacuous. In these cases, we can arbitrarily assign a value to α_r in order to arrive at a solution. (This is also apparent from the bar diagrams of Section 2.2, as in these cases no horizontal bars cross the rth vertical bar.) Consequently, the expression for ω_{X_w} given in Theorem 2 is unique, up to tensoring by bundles which are trivial when restricted to X_w. Furthermore:

$\mathbb{Q}$-*Gorensteinness:* A variety is said to be $\mathbb{Q}$-**Gorenstein** if it is Cohen-Macaulay and some multiple of the canonical divisor is Cartier. Consequently, a Schubert variety X_w is $\mathbb{Q}$-Gorenstein if (2) has a rational solution. However, since if any solution exists, an integral solution exists, Gorensteinness and $\mathbb{Q}$-Gorensteinness are equivalent. This will not hold in general for flag varieties of other Lie types.

Computational efficiency: In order to check if a permutation w corresponds to a Gorenstein Schubert variety, it is typically more computationally efficient solve for (2) than to use Theorem 1. In particular, it is enough to check if (4) works.

3.3. **Is it pattern avoidance?** In view of [23], it is natural to wonder if it is possible to reformulate Theorem 1 in terms of "classical pattern avoidance", that is, if there is a finite list of permutations $w_1, w_2, \ldots, w_n$ such that X_w is Gorenstein if and only if w pattern avoids these permutations.

In fact, this is already impossible for Grassmannian permutations. For example, we know $1346 \mid 25 \in S_6$ does not correspond to a Gorenstein Schubert variety. But $w' = 12569 \mid 3478 \in S_9$ does. Note that w' contains w as a subpattern, so if a classical pattern avoidance permutation reformulation of Theorem 1 existed, it would imply that $X_{w'}$ is not Gorenstein, which is not true.

3.4. A characterization of Fano Schubert varieties.

A Gorenstein algebraic variety is **Fano** if its anticanonical divisor is ample. It follows from Theorem 2 that a Gorenstein Schubert variety X_w in $\mathrm{Flags}(\mathbb{C}^n)$ is Fano if and only if all of the inner corner distances of w are at most 1. This appears to give new examples of Fano varieties. It seems to have been previously unknown whether or not all smooth Schubert varieties of the flag variety are Fano. By the above remark, it is easy to find examples of Schubert varieties that are smooth but not Fano, in contrast to the case for Grassmannians, for which all smooth Schubert varieties are Fano.

3.5. Matrix Schubert varieties and ladder determinantal varieties.

Let $v \in S_n$ be a permutation, and Y_v the associated matrix Schubert variety; this was defined in [15] as the closure in $\mathbb{C}^{n^2}$, considered as the space of $n \times n$ matrices, of $p^{-1}(X_v)$, where $p : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})/B = \mathrm{Flags}(\mathbb{C}^n)$ is the quotient map. Now let $w = v \times \mathrm{id} \in S_n \times S_n \subseteq S_{2n}$ be the permutation agreeing with v on $1, \dots, n$ and fixing $n + 1, \dots, 2n$. The intersection of X_w with the opposite big cell of flags intersecting the canonical reference flag (whose i-th vector space is $\langle e_1, \dots, e_i \rangle$) generically is then isomorphic to $Y_v \times \mathbb{C}^{n^2-n}$. Every singularity of X_w is represented in this opposite big cell, so Y_v is Gorenstein if and only if X_w is. Identifying ladder determinantal varieties with the appropriate matrix Schubert varieties allows us to recover the characterizations of Gorenstein ladder determinantal varieties found in [11] and [18].

3.6. Theorem 2 and cohomology of line bundles on Gorenstein Schubert varieties.

Theorem 2 can be applied to obtain information about the sheaf cohomology groups $H^i(X_w, \mathcal{L}_\alpha|_{X_w})$ of the line bundle $\mathcal{L}_\alpha|_{X_w}$ on a Gorenstein Schubert variety X_w. The groups are classically known in the case $X_{\mathrm{id}} \cong \mathrm{Flags}(\mathbb{C}^n)$ and $\alpha \in \mathrm{Hom}(T, \mathbb{C}^n)$ is arbitrary (the classical Borel-Weil-Bott theorem [7]), and for arbitrary $w \in S_n$ when α is dominant [14]; see, for example, [21]. It is an open problem to compute these groups in most of the remaining cases; see [2] for some recent progress on this problem.

Serre duality (see, for example, [19, III.7]) states that, for any projective, equidimensional, d-dimensional, Cohen-Macaulay scheme X, and any coherent sheaf $\mathcal{F}$ on X, we have

$$H^i(X, \mathcal{F}) \cong \mathrm{Ext}^{d-i}(\mathcal{F}, \omega_X)^*.$$

Let α be the (non-dominant) weight defined in Theorem 2, and β any weight. Then:

$$\begin{aligned}
H^i\big(X_w, \mathcal{L}_{\alpha-\beta}\big|_{X_w}\big) &\cong \mathrm{Ext}^{n-\ell(w)-i}(\mathcal{L}_{\alpha-\beta}\big|_{X_w}, \omega_X)^* \\
&= \mathrm{Ext}^{n-\ell(w)-i}(\mathcal{L}_{\alpha-\beta}\big|_{X_w}, \mathcal{L}_\alpha\big|_{X_w})^* \\
&\cong \mathrm{Ext}^{n-\ell(w)-i}(\mathcal{O}_{X_w}, \mathcal{L}_\beta\big|_{X_w})^* \\
&\cong H^{n-\ell(w)-i}\big(X_w, \mathcal{L}_\beta\big|_{X_w}\big)^*.
\end{aligned}$$

When β is dominant, this relates the cohomology groups $H^i\big(X_w, \mathcal{L}_{\alpha-\beta}\big|_{X_w}\big)$ to the cohomology groups known by Demazure's theorem. For example, it follows that, when β is dominant, $H^i\big(X_w, \mathcal{L}_{\alpha-\beta}\big|_{X_w}\big) \cong 0$ for $i \neq n - \ell(w)$.

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