The Taylor Series Coefficients of the Jacobi Elliptic Functions

“Le calcul des coefficients par les dérivées successives est impraticable.” [Briot and Bouquet 1875]

Gregynog 23rd May 2000

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1. The Jacobi Elliptic Functions
2. A Formal Definition — The Dumont-Schett Method
3. Myung’s Method
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$$\int_0^{\mathrm{sn}(u,k)} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = u \qquad \text{just like} \qquad \int_0^{\sin(u)} \frac{dt}{\sqrt{1-t^2}} = u$$

So $\mathrm{sn}(u,0) = \sin(u)$ and $\mathrm{sn}(u,1) = \tanh(u)$

$$\frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}} = \sum_{n\geq 0} k^n\, P_n\!\left(\frac{k^2+1}{2k}\right) t^{2n},$$

where $P_n(x)$ is the Legendre polynomial

So Taylor series for $\mathrm{sn}(u,k)$ is the inverse series (Lagrange inversion) of

$$\sum_{n\geq 0} k^n\, P_n\!\left(\frac{k^2+1}{2k}\right) \frac{t^{2n+1}}{2n+1}$$

$$\operatorname{sn}(u,k) = u - (1+k^2)\frac{u^3}{3!} + (1+14k^2+k^4)\frac{u^5}{5!}$$
$$- (1+135k^2+135k^4+k^6)\frac{u^7}{7!} + \cdots$$

$$\operatorname{cn}(u,k) = \sqrt{1-\operatorname{sn}^2(u,k)} \quad \text{and} \quad \operatorname{dn}(u,k) = \sqrt{1-k^2\operatorname{sn}^2(u,k)}$$

$$\operatorname{cn}(u,k) = 1 - \frac{u^2}{2!} + (1+4k^2)\frac{u^4}{4!} - (1+44k^2+16k^4)\frac{u^6}{6!} + \cdots,$$
$$\operatorname{dn}(u,k) = 1 - k^2\frac{u^2}{2!} + (4k^2+k^4)\frac{u^4}{4!} - (16k^2+44k^4+k^6)\frac{u^6}{6!} - \cdots.$$

Dumont's 3-parameter functions (1981)

$\mathfrak{D} = yz\frac{\partial}{\partial x} + xz\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z}$ is a derivation, so $e^{u\mathfrak{D}}$ is multiplicative

$$X = X(u;x,y,z) = e^{u\mathfrak{D}}x = \sum_{n\geq 0} X_n(x,y,z)\frac{u^n}{n!}$$
$$= x + yzu + x(y^2+z^2)\frac{u}{2} + yz(4x^2+y^2+z^2)\frac{u^3}{6} + \cdots$$

$$\mathfrak{D}X_n = X_{n+1}$$

X_n is a homogeneous integer polynomial in x, y, z of degree $n+1$

Similarly $Y = e^{u\mathfrak{D}}y$ and $Z = e^{u\mathfrak{D}}z$

Let $X' = \frac{\partial}{\partial u} X$

The key calculation

$$X' = \frac{\partial}{\partial u} X = \frac{\partial}{\partial u} e^{u\mathfrak{D}} x = e^{u\mathfrak{D}} \mathfrak{D} x = e^{u\mathfrak{D}} yz = \left(e^{u\mathfrak{D}} y\right)\left(e^{u\mathfrak{D}} z\right) = YZ$$

Similarly $Y' = XZ$ and $Z' = XY$

Consequences

Chain rule and $X' = YZ$ imply

$$(X^2)' = 2XYZ$$

Similarly $(Y^2)' = 2XYZ$ and $(Z^2)' = 2XYZ$

Thus $X^2 - Y^2$ is constant, i.e., independent of u

$$X^2 - Y^2 = x^2 - y^2, \qquad X^2 - Z^2 = x^2 - z^2, \quad \text{and} \quad Y^2 - Z^2 = y^2 - z^2$$

Now

$$(X')^2 = Y^2 Z^2 = (x^2 - y^2 - X^2)(x^2 - z^2 - X^2)$$

Let $x = 0$, $y = i$ and $z = ik$, then

$$\frac{X'}{\sqrt{(1 - X^2)(k^2 - X^2)}} = 1,$$

so $X(u; 0, i, ik) = -k\,\mathrm{sn}(u, k)$

Similarly

$$Y(u; 0, i, ik) = i\,\mathrm{dn}(u, k)$$

$$Z(u; 0, i, ik) = ik\,\mathrm{cn}(u, k)$$

The identity $X' = YZ$ leads to

$$X_{n+1} = \sum_{j=0}^{n} \binom{n}{j} Y_j Z_{n-j},$$

and, using the symmetry between X, Y and Z, we obtain a quadratic, unbounded recursion for the coefficients

Better

The definition of $\mathfrak{D}$ and $\mathfrak{D} X_n = X_{n+1}$ give a three-term, linear recurrence for the coefficients, hence we can obtain the expansions of sn, cn and dn

Myung (1998)

Let

$$S = \alpha X + \beta Y + \gamma Z$$

Since $X' = YZ$, $Y' = XZ$ and $Z' = XY$,

$$S' = \alpha YZ + \beta XZ + \gamma XY$$

On the other hand

$$S^2 = 2\beta\gamma YZ + 2\alpha\gamma XZ + 2\alpha\beta XY$$
$$+ \alpha^2 X^2 + \beta^2 Y^2 + \gamma^2 Z^2$$

Assume that α, β, γ commute with each other and with x, y, z, and that

$$\alpha^2 = \beta^2 = \gamma^2 = 0, \quad \alpha\beta = \tfrac{1}{2}\gamma, \quad \alpha\gamma = \tfrac{1}{2}\beta, \quad \beta\gamma = \tfrac{1}{2}\alpha,$$

then

$$S' = S^2$$

If $S = \sum_{n \geq 0} S_n \frac{u^n}{n!}$ so that $S_n = \alpha X_n + \beta Y_n + \gamma Z_n$, this differential equation corresponds to the recursion

$$S_{n+1} = \sum_{j=0}^{n} \binom{n}{j} S_j S_{n-j}$$

Since $S_0 = \alpha x + \beta y + \gamma z$,

$$S_1 = S_0^2 \qquad = 2\alpha\beta xy + 2\alpha\gamma xz + 2\beta\gamma yz$$

$$= \alpha yz + \beta xz + \gamma xy$$

$$S_2 = 2S_0 S_1 \qquad = 2\alpha\beta(x^2 z + y^2 z) + 2\alpha\gamma(x^2 y + yz^2) + 2\beta\gamma(xy^2 + xz^2)$$

$$= \alpha x(y^2 + z^2) + \beta y(x^2 + z^2) + \gamma z(x^2 + y^2)$$

$S_3 = 2S_0 S_2 + 2S_1^2 \qquad =$ *exercise*

The coefficient of α is X , etc.

Problem?

Letting $\delta = \beta + \gamma$, we have $\alpha\delta = \frac{1}{2}\delta$ so that

$$\alpha(\alpha\delta) = \tfrac{1}{4}\delta \qquad \text{but} \qquad (\alpha\alpha)\delta = 0$$

A non-associative algebra

Hermite's observation (1863)

$$\mathrm{cn}(u,k) = 1 - \frac{u^2}{2!} + (1+4k^2)\frac{u^4}{4!} - (1+44k^2+16k^4)\frac{u^6}{6!}$$
$$+ (1+408k^2+912k^4+64k^6)\frac{u^8}{8!} + \cdots$$

Multiply coefficients by k and substitute $k = \cos\theta$:

$$k(1+4k^2) = 4\cos\theta + \cos 3\theta$$

$$k(1+44k^2+16k^4) = 44\cos\theta + 16\cos 3\theta + \cos 5\theta$$

$$k(1+408k^2+912k^4+64k^6) = 912\cos\theta + 408\cos 3\theta + 64\cos 5\theta + \cos 7\theta$$

The Landen transformation (1775)

Substituting

$$z = \frac{b+c}{2}, \qquad y = \sqrt{\frac{a^2 + bc + \sqrt{(b^2 - a^2)(c^2 - a^2)}}{2}}, \qquad x = \frac{az}{y}$$

in $\frac{XY}{Z}$ gives $X(u; a, b, c)$

If $a = 0$,

$$\frac{XY}{Z}\Big(u; 0, \sqrt{bc}, \tfrac{b+c}{2}\Big) = X(u; 0, b, c)$$

A consequence of the Landen transformation is

$$e^{i\theta}\,\mathrm{cn}(ue^{-i\theta},e^{2i\theta})+e^{-i\theta}\,\mathrm{cn}(ue^{i\theta},e^{-2i\theta})=2\cos\theta\;\mathrm{cn}(u,\cos\theta)$$

With

$$\mathrm{cn}(u,k)=\sum_{n,j}(-1)^n C_{n,j}k^{2j}\,\frac{u^{2n}}{(2n)!},$$

$$\sum_{q=0}^{n-1} C_{n,q}\cos\big((2n-4q-1)\theta\big)=\sum_{p=0}^{n-1} C_{n,p}\cos^{2p+1}\theta$$

Thus linear equations for the $C_{n,j}$

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