Historical Overview of Pell Equations

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Abstract

A Pell Equation with variables x and y in parameter D is given by the following expression:

$$x^2 = 1 + Dy^2$$

The search for a solution to such an equation involves finding the least integer solution $(x, y) \neq (1, 0)$. Through the years it has been noted that such an equation has no solutions if D is a perfect square and an infinitude of solutions otherwise. The first known mention of these equations appears in Ancient Greece. This paper traces the search for an exhaustive solution beginning with Brahmagupta's method of composition, picking up again with Fermat's independent work and subsequent challenge finally put to rest by Lagrange in his *Additions* to Euler's *Elements of Algebra*. Lagrange proved that a method given earlier produces all solutions given the least positive initial solution. The author aims to provide an introduction to Pell equations by placing selected results in a historical context.

1. Introduction

A Pell Equation with variables x and y in parameter D is given by the following expression:

$$x^2 = 1 + Dy^2 \tag{1.1}$$

By a solution to this equation, we mean the least positive integer solution $(x, y) \neq (1, 0)$. The requirement of a least pair is a hint of a method to come that produces an exhaustive list of solutions given some intial solution. We require positive solutions to eliminate the trivial solutions $(-x, -y), (-x, y), (x, -y)$ given solution (x, y). Note that these equations have only the trivial solution $(1, 0)$ when D is a perfect square. We begin with a quick proof of this fact.

Proposition 1.1. If $D = k^2$ for some integer k, then the equation $x^2 - Dy^2 = 1$ has only the trivial solution $(1, 0)$.

Proof. Assume, to the contrary, that $x^2 - Dy^2 = 1$ has some other least positive solution, $(a, b) \neq (1, 0)$. Upon substitution, $x^2 - Dy^2 = (a + kb)(a - kb) = 1$, where a, b, k are all integers. But the product of two integers cannot equal one unless they both are one. We already know we are not in the case $(a, b) = (1, 0)$, so the assumption that the equation has some other solution must be incorrect.

A solution (a, b) to (1.1) provides a good rational approximation to $\sqrt{D}$. Dividing both sides of (1.1) by y^2 and taking the square root gives $a/b = \sqrt{1/y^2 + D}$. For large values of y, the approximation becomes better. For example

$$x^2 = 1 + 2y^2$$

has solutions:

$$(a_1, b_1) = (17, 12) \Rightarrow a/b = 1.416667$$

$$(a_2, b_2) = (577, 408) \Rightarrow a/b = 1.414216$$

This brings us to the next point, a solution (a, b) to (1.1) leads to infinitely many solutions. Rearranging and factoring (1.1) produces the following useful forms:

$$x^2 - Dy^2 = 1 \tag{1.2}$$

$$(x + y\sqrt{D})(x - y\sqrt{D}) = 1 \tag{1.3}$$

Proposition 1.2. The Pell equation in parameter D, not a perfect square, has an infinite number of solutions.

Proof.

The Pell Equation has a long and broken history. It was first inadvertantly studied by Diophantus and Archimedes. Diophantus solved equations of this form in specific cases. Archimedes' Cattle Problem can be reduced to a Pell equation, although it is not known whether he intended for this. In addition to his many other significant contributions, the Indian mathematician Brahmagutpa provided the first known general solution method, though not exhaustive. Fermat brought these equations into modern Western mathematics with a single example as one of his famous challenge problems. Euler mistakenly named the equations after Pell. Lagrange set the solution to this class of equations in stone by further proving the continued fraction method given by Euler.

This problem is purported "to be one proposed by Archimedes, in a letter to Eratosthenes, to the mathematicians of Alexandria." [Dickson 342] This in formation was put forth in a manuscript published in 1773 by Gotthold Lessing.

Shouldn't be name after John Pell... misnomer.

2. Greek Contributions

Archimedes Cattle Problem. Started with a large system of linear equations. Quote line that leads to Pell Equation. Show Pell equation. Quote saying it was not known whether Archimedes knew how to solve this. Diophantus also solved some specific cases.

The first conditions lead to a large, but easily solved set of linear equations. The second part of the problem statement is open to some interpretation. "G. H. F. Nesselmann argued that the final part of the epigram leading to conditions [of square and triangular numbers] was a later addition." [Dickson 344] Of course this is the part of the problem that is most troublesome. It leads to the following Pell equation:

$$x^2 = 1 + 4,729,494y^2$$

It is not known whether Archimedes possessed the capabilities to solve such an equation. Brahmagutpa was the first mathematician to put forth a method for solving exactly these types of equations. Not only did he develop methods for generating a single solution, but he realized that a single solution could be modified to produce a large number of solutions.

3. Indian Contributions

Composition and stuff...

Proposition 3.1. If (a, b) and (c, d) are solutions to the Pell equation $x^2 - Dy^2 = 1$, then $(ac+Dbd, ad+bc)$ and $(ac-Dbd, ad-bc)$ are also solutions to the same equation.

Proof. Brahmagupta's method of composition relies on the following identities:

$$\begin{aligned}(a^2 - Db^2)(c^2 - Dd^2) &= a^2c^2 - Da^2d^2 - Db^2c^2 + D^2b^2d^2 \\ &= a^2c^2 + D^2b^2d^2 - D(a^2d^2 + b^2c^2) \\ &= a^2c^2 + 2Dabcd + D^2b^2d^2 - D(a^2d^2 + 2abcd + b^2c^2) \\ &= (ac + Dbd)^2 - D(ad + bc)^2\end{aligned}$$

$$\begin{aligned}(a^2 - Db^2)(c^2 - Dd^2) &= a^2c^2 - Da^2d^2 - Db^2c^2 + D^2b^2d^2 \\ &= a^2c^2 + D^2b^2d^2 - D(a^2d^2 + b^2c^2) \\ &= a^2c^2 - 2Dabcd + D^2b^2d^2 - D(a^2d^2 - 2abcd + b^2c^2) \\ &= (ac - Dbd)^2 - D(ad - bc)^2\end{aligned}$$

If (a,b) and (c,d) are solutions to $x^2 - Dy^2 = 1$, then

$$a^2 - Db^2 = 1$$

$$c^2 - Dd^2 = 1$$

By substitution into the left hand sides of the previous two identities, we arrive at

$$(ac + Dbd)^2 - D(ad + bc)^2 = 1$$

$$(ac - Dbd)^2 - D(ad - bc)^2 = 1$$

4. European Contributions

Fermat's Challenge, Euler's Nomenclature and Lagrange the Keystone.

From Lenstra we have that "the n-th solution x_n, y_n can be expressed in terms of the first one, x_1, y_1 by $x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n$." We now state this more formally.

Proposition 4.1. If the integers x,y satisfy a Pell equation with parameter D, then the integers also satisfy the Pell equation in D.

Proof. This proof will be by induction.

Given (x, y) such that $x^2 = 1 + Dy^2$, show that $P(n)$ holds for all $n \in N$.

5. Lagrange's Magical Complete Solution

6. Final Comments

No mention of how to find initial solution. Here we outline the general solution method with an example.

Solve $x^2 = 1 + 17y^2$.

We begin with the continued fraction expansion of $\sqrt{17}$.

$$\begin{aligned} x = \sqrt{17} &= 4 + \frac{1}{x} \\ \Rightarrow \frac{1}{x} &= \sqrt{17} - 4 \\ \Rightarrow x &= \frac{1}{\sqrt{17} - 4} \cdot \frac{\sqrt{17} + 4}{\sqrt{17} + 4} = \sqrt{17} + 4 \end{aligned}$$

Substituting yields:

$$\begin{aligned} \sqrt{17} = 4 + \frac{1}{x} &= 4 + \frac{1}{4 + \sqrt{17}} \\ &= 4 + \frac{1}{8 + \frac{1}{8 + \dots}} \\ &= [4; 8, 8, 8, \dots] \end{aligned}$$

Truncating this pattern at the first repetition, we get

$$\frac{x}{y} = 4 + \frac{1}{8} = \frac{33}{8}$$

Therefore, the pair $(33, 8)$ is the least positive solution to the given Pell equation.

$$33^2 - 17 \cdot 8^2 = 1$$

Polynomial time algorithm? Ineresting topic that leads into a number of algebraic and number theoretical topics.

References

[1] Uspensky, J. V. and Heaslet, M. A. (1939) *Elementary Number Theory.* McGraw-Hill Book Company, New York and London.

Proposition 1.1.

Proof.