Identities inspired from Ramanujan Notebooks II

by Simon Plouffe, July 21, 1998

These identities where found using an implementation of LLL algorithm in Pari-Gp (2.0.10) + MapleV.5
see also these other identities found in 1993 about Catalan. They are not knew, the presentation is.

$$\zeta(3) = \frac{7}{180}\pi^3 - 2\left(\sum_{n=1}^{\infty} \frac{1}{n^3\,(e^{(2\pi n)} - 1)}\right)$$

$$\zeta(3) = 14\left(\sum_{n=1}^{\infty} \frac{1}{n^3 \sinh(\pi n)}\right) - \frac{11}{2}\left(\sum_{n=1}^{\infty} \frac{1}{n^3\,(e^{(2\pi n)} - 1)}\right) - \frac{7}{2}\left(\sum_{n=1}^{\infty} \frac{1}{n^3\,(e^{(2\pi n)} + 1)}\right)$$

$$\zeta(5) = \frac{1}{294}\pi^5 - \frac{72}{35}\left(\sum_{n=1}^{\infty} \frac{1}{n^5\,(e^{(2\pi n)} - 1)}\right) - \frac{2}{35}\left(\sum_{n=1}^{\infty} \frac{1}{n^5\,(e^{(2\pi n)} + 1)}\right)$$

$$\zeta(5) = -\frac{39}{20}\left(\sum_{n=1}^{\infty} \frac{1}{n^5\,(e^{(2\pi n)} - 1)}\right) + \frac{1}{20}\left(\sum_{n=1}^{\infty} \frac{1}{n^5\,(e^{(2\pi n)} + 1)}\right) + 12\left(\sum_{n=1}^{\infty} \frac{1}{n^5 \sinh(\pi n)}\right)$$

$$\zeta(7) = \frac{19}{56700}\pi^7 - 2\left(\sum_{n=1}^{\infty} \frac{1}{n^7\,(e^{(2\pi n)} - 1)}\right)$$

They are all inspired by Ramanujan formulas in Bruce Berndt's book (**Ramanujan Notebooks II**, chapter 14 formulas 25.1 and 25.3. Formula 25.1 is

$$\sum_{n=1}^{\infty} \frac{\coth(\pi n)}{n^3} = \frac{7}{180}\pi^3$$

This sum suggests that (by splitting the sum) there could be one with Zeta(3), which is the case. The coefficients are the same. Formula 25.2 confirm that observation. Formula 25.3 gives a clue on a general formula as well with Bernoulli numbers.

NOTE :The formulas for Zeta(4*n+3) appeared in "On the Khintchine Constant" by Bailey, Borwein and Crandall, Math of Computation, 66 #217, page 413 (1997). There is one for Zeta(4*n+1) but that one on Zeta(5) is simpler.

I could compute **39000** digits of Zeta(5) and **50000** digits of Zeta(7) with it. The convergence rate is rather good since 1/exp(2*Pi) = 1/535.49, it gives 2.72 digits/term. Those formulas can probably be programmed to get 1 million digits of Zeta(2*n+1), Zeta(3) is already known to 32 million digits.

The computation is rather straightforward since we already have Pi to many billion of digits, the harder part is the evaluation of exp(2*Pi) but it can be done only once.

See also these new ones in the same vein (August 7, 1998).

Here are the other identities found with Zeta(9) to Zeta(21) and the results of Joerg Arndt as computed on July 27 up to Zeta(163).

$$37122624 \left(\sum_{n=1}^{\infty} \frac{1}{n^{9} (\exp(2 \pi n) - 1)} \right) + 74844 \left(\sum_{n=1}^{\infty} \frac{1}{n^{9} (\exp(2 \pi n) + 1)} \right) + 18523890 \zeta(9) - 625 \pi^{9} = 0$$

$$-851350500 \left(\sum_{n=1}^{\infty} \frac{1}{n^{11} (\exp(2 \pi n) - 1)} \right) - 425675250 \zeta(11) + 1453 \pi^{11} = 0$$

$$514926720 \left(\sum_{n=1}^{\infty} \frac{1}{n^{13} (\exp(2 \mathrm{Pi}\, n) - 1)} \right) + 62370 \left(\sum_{n=1}^{\infty} \frac{1}{n^{13} (\exp(2 \mathrm{Pi}\, n) + 1)} \right) + 257432175 \,\mathrm{Zeta}(13) - 89 \,\mathrm{Pi}^{13} = 0$$

$$-781539759000 \left(\sum_{n=1}^{\infty} \frac{1}{n^{15} (\exp(2 \mathrm{Pi}\, n) - 1)} \right) - 390769879500 \,\mathrm{Zeta}(15) + 13687 \,\mathrm{Pi}^{15} = 0$$

$$3808863131673600 \left(\sum_{n=1}^{\infty} \frac{1}{n^{17} (\exp(2 \mathrm{Pi}\, n) - 1)} \right) + 29116187100 \left(\sum_{n=1}^{\infty} \frac{1}{n^{17} (\exp(2 \mathrm{Pi}\, n) + 1)} \right) + 1904417007743250 \,\mathrm{Zeta}(17) - 6758333 \,\mathrm{Pi}^{17} = 0$$

$$-42877225028137500 \left(\sum_{n=1}^{\infty} \frac{1}{n^{19} \left(\exp(2 \pi n) - 1\right)} \right) - 21438612514068750\, \mathrm{Zeta}(19) + 7708537\, \pi^{19} = 0$$

$$-3762129424572110592000 \left(\sum_{n=1}^{\infty} \frac{1}{n^{21} \left(\exp(2 \pi n) - 1\right)} \right) - 1793047592085750 \left(\sum_{n=1}^{\infty} \frac{1}{n^{21} \left(\exp(2 \pi n) + 1\right)} \right) - 1881063815762259253125\, \mathrm{Zeta}(21) + 68529640373\, \pi^{21} = 0$$
