

PALINDROMIC PRIME PYRAMIDS

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Have you ever seen the great stone pyramids of ancient Egypt or Central America? For over 5000 years, mankind has been building, visiting, and even sleeping in pyramids. When Memphis, Tennessee, decided to build a new arena in 1991, they chose the shape of a pyramid—this time constructed of steel and glass rather than rock and rubble. In this paper we also build pyramids, ours built of the unbreakable stones of mathematics: the primes. But not just any primes, we have chosen the symmetry of nested palindromes as mortar. For example, beginning with the prime 2, we can build two pyramids of height five. (Unlike the ancients, we build our pyramids from the top down.)

2	2
929	929
39293	39293
7392937	3392933
373929373	733929337

Here each step is a palindromic prime with the previous step as its central digits. These two pyramids are the tallest that can be built beginning with the prime 2.

The tallest such pyramids that can be built from the other one-digit primes are as follows:

		5	5	5	
3	3	151	353	757	7
131	131	31513	33533	37573	373
11311	71317	3315133	1335331	9375739	93739

Like many that have come before us, we ask how can we build them higher? For example, if instead of just one-digit primes, we begin with larger palindromic primes, can they be taller? If instead of adding just one digit to each side, we allow two or more, how much taller can we get? Are these pyramids always finite? Join us on a quick tour as we seek answers to these questions, and pose others for our readers.

Simple Step Pyramids

Starting with a single digit prime and at each level adding just one digit to each side, we found the tallest possible prime-pyramids (using nested palindromes) had height five. This is because there are only four possible digits we can add at each step: 1, 3, 7, and 9. Starting with larger primes is unlikely to help much, but there are so many to choose from that we might get lucky. For example, Felice Russo [10, 11 seq. A046210] found the following truncated palindromic prime pyramid of height nine.

```
7159123219517
371591232195173
33715912321951733
7337159123219517337
973371591232195173379
39733715912321951733793
3397337159123219517337933
933973371591232195173379339
39339733715912321951733793393
```

However, if instead we add two digits on each side, there are forty pairs of digits we can add to each end (and still avoid our steps being divisible by 2 or 5). Starting with the prime 2, the tallest that can be built (with step two) has height 26. In fact, there are two pyramids of this height. One of these is shown in figure 1. The other is the pyramid ending in the following 101-digit prime:

[Insert figure 1 on a page near here](#)

```
1 3189272993 3733012747 5151938943 3901197127
2339635702 0753693327 2179110933 4983915157 4721033733 9927298131
```

How do we know these are the tallest? Using UBASIC [3] we started with 2 and built *every possible pyramid*--at each step discarding those for which the new number was not a Fermat probable-prime [7, pg. 140]. Then for those pyramids of maximum height, we used UBASIC's application program APRT-CL [5] to complete primality proofs for every step. We also applied this approach to pyramids starting with the other one-digit primes. There are three pyramids tied for tallest starting with the prime 3, each of height 28. There is one each starting with the primes 5 and 7, both of height 29. Further information is available on-line [4].

Surely increasing the step size to three (or more) should increase the height, but by how much? How many pyramids would we have to check for an exhaustive search? We address these questions in the next section.

Heights and Heuristics

First, let $l(n)$ be the number of digits (the length) of n . Let $f(n,h,d)$ be the number of palindromic primes pyramids with height h (not necessarily the maximal height), beginning with n and with step size d . For example, $f(2,1,d) = 1$ (there is only one pyramid starting with 2 and

height 1, that is just “2”). However, $f(101,2,2) = 4$ since there are four pyramids starting with 101 of height 2 and step 2:

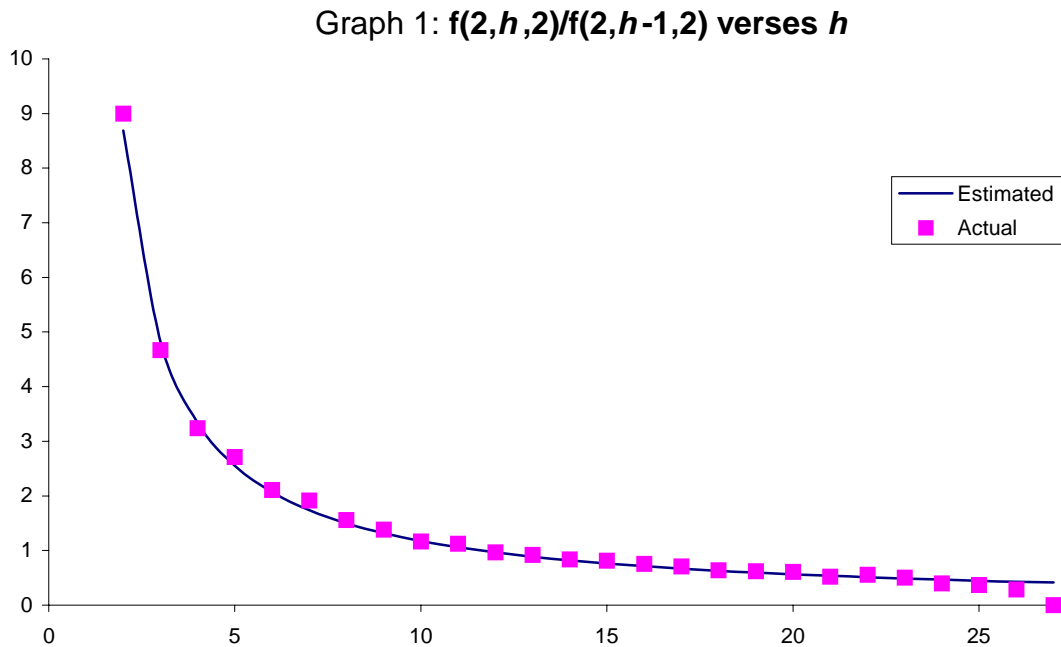
$$\begin{array}{cccc} 101 & 101 & 101 & 101 \\ 3310133 & 7310137 & 9110119 & 9610169 \end{array}$$

We will attempt to estimate $f(n,h,d)$ and use this to estimate the maximum height.

Recall that the *prime number theorem* [7, pp. 225-227] states that the number of primes less than x is approximately $x/\ln x$. (Technically, the theorem says these quantities are asymptotic--so the larger x is, the better this estimate is). One interpretation of this theorem is that the probability of a random integer the size of the integer x being prime is about $1/\ln x$. When we move to the next step of a pyramid, there are 10^d integers to try, so if these new numbers behave as a random sample we would expect

$$\frac{f(n,h,d)}{f(n,h-1,d)} \approx \frac{10^d}{(l(n) + 2(h-1)d) \ln 10} \tag{1}$$

In graph 1 we see this rough estimate (graphed as a solid curve) compared to the actual ratios (graphed as individual squares) for $n=2$ and $d=2$.



These match surprisingly well! The drop off at the end is caused by the low numbers—we might expect 40% of two numbers to be prime, but in actuality only 0, 1, or 2 of them can be, not 0.80 of them. This is typical since this type of heuristic estimate (educated guess) works best for large numbers (see, for example, [2] or [12]).

As h grows, the ratio in (1) soon becomes one and then decreases to zero, so we would expect the number of pyramids to start decreasing at that point and then drop off to zero. From this we make the following conjecture.

Conjecture: All palindromic prime pyramids with fixed step size are finite.

We can predict more with this heuristic model. Repeatedly using the estimate (1), we have

$$f(n,h,d) \approx \left(\frac{10^d}{\ln 10}\right)^{h-1} \frac{f(n,1,d)}{(l(n) + 2d(h-1)) (l(n) + 2d(h-2)) \cdots (l(n) + 2d)}. \quad (2)$$

The denominator of the second term is Pochhammer's symbol and can be expressed via the gamma function¹ [1, eq. 6.1.22]. This yields the following.

$$f(n,h,d) \approx \left(\frac{10^d}{2d \ln 10}\right)^{h-1} \frac{\Gamma\left(\frac{l(n)}{2d} + 1\right)}{\Gamma\left(\frac{l(n)}{2d} + h\right)}. \quad (3)$$

This estimate is one when $h=1$ (the top of the pyramid). Just past where this estimate is one again (for some larger h), we would expect to have the pyramid with greatest height. Using a computer program such as Maple [13] it is easy to solve for this value. Sadly, we can only test our estimates for small values of d where we expect the greatest relative error, but the comparisons still are heartening—see Table 1.

Table 1: **The average maximum height of pyramids**

length $l(n)$	number*	step $d = 1$		step $d = 2$		step $d=3$	step $d=4$
		predicted	actual	predicted	actual	predicted	predicted
1	4	3.55	3.75	26.8	28.0	193	1471
3	15	1.31	2.53	25.0	25.8	191	1469
5	93	1**	2.10	23.3	24.3	190	1467
7	668	1**	1.79	21.7	22.1	188	1466
9	5172	1**	1.58	20.1	20.2	186	1464

* The number of starting values (palindromic primes) n of the given length $l(n)$

** The estimate (3) is only one once for these values of $l(n)$ (when $h=1$)

We found the actual values in table one by exhaustive search: for each palindromic prime of the given length, we found the pyramid of maximum height (by finding all pyramids beginning with this prime). We then averaged over all the palindromic primes of this length.

Notice that even for $d=3$ and $n=2$, this exhaustive approach would be beyond the world's current computing ability because when the height was 73, (2) predicts there should be almost 10^{30} pyramids to deal with. However, we can still test this heuristic by keeping a fixed number

¹ $\Gamma(n)$ is the analytic continuation of the familiar factorial function: $\Gamma(n+1) = n!$ for positive integers n .

of pyramids at each step. In our test we kept a maximum of 160 pyramids at each height, so beginning at $h=74$ (when the ratio (2) is one) and continuing until we get a product less than one, we predict we should find a maximum height of about 103. Starting with the primes 2, 3, 5 and 7 we found maximal pyramids of heights 94, 101, 102 and 100 respectively. This is reasonable agreement for the relatively small number of pyramids (a maximum of 160) involved. (Again, these prime pyramids are available on the web [4])

Related Sequences

Keeping the step size fixed (apparently) forever binds our pyramids to a finite height. But suppose we instead allow any step size? An argument similar to the one above suggests that for any starting prime we should be able to build as high as we like, though the taller the pyramids get the larger our step size must be (on the average).

There is one case that is especially interesting: Suppose we ask that each row be the smallest prime that can be used. Then our pyramid would begin as follows:

```

      2
     727
    37273
   333727333
  93337273339
 309333727333903
1830933372733390381
92183093337273339038129
3921830933372733390381293
1333921830933372733390381293331
18133392183093337273339038129333181

```

When the first author built this pyramid, he was able to verify the primality of the first 33 rows.

This pyramid has also been presented as a sequence a_1, a_2, a_3, \dots . To do this let $a_1=2$ and then for each positive integer n , let a_{n+1} be the smallest palindromic prime with a_n as the central digits [11, seq. A053600]. We can condense this sequence by writing a_1 , followed by the digits added on the left at each stage. Carlos Rivera [8] extended the first author's 33 terms using a probabilistic primality test and found the condensed sequence [11, seq. A052091] (most likely) begins:

```

2, 7, 3, 33, 9, 30, 18, 92, 3, 133, 18, 117, 17, 15, 346, 93, 33,
180, 120, 194, 126, 336, 331, 330, 95, 12, 118, 369, 39, 32, 165,
313, 165, 134, 13, 149, 195, 145, 158, 720, 18, 396, 193, 102,
737, 964, 722, 156, 106, 395, 945, 303, 310, 113, 150, 303, 715,
123

```

Finally, Russo took a different approach to palindromic prime pyramids, and asked what was the smallest palindromic prime a_n that generates a prime pyramid of maximum height n ? This sequence [11, seq. A046210] begins 11, 131, 2, 929, 10301, 16361, 10281118201, 35605550653, 7159123219517...

Conclusion

As we look around in the world, we see many variations on the basic pyramids of Egypt. Above we have mentioned just a few of the variations on our pyramids that have appeared since the first author proposed the idea. For even more variations, look on-line [4, 6, 9, 11].

We have left many open questions and leave the reader with the most basic of challenges: build them higher! Perhaps you can develop a way of finding the tallest pyramids with fixed step sizes--something far better than exhaustive search. Or perhaps can you prove (rather than just heuristically suggest) that fixed step size pyramids are finite. We built pyramids in decimal (base 10), why not try another base (e.g., binary)?

In all cases, we would be glad to hear of your results.

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Figure 1: A palindromic prime pyramid of step size two

2
30203
903020309
3790302030973
98379030203097389
969837903020309738969
9996983790302030973896999
72999698379030203097389699927
997299969837903020309738969992799
9099729996983790302030973896999279909
94909972999698379030203097389699927990949
779490997299969837903020309738969992799094977
7977949099729996983790302030973896999279909497797
17797794909972999698379030203097389699927990949779771
751779779490997299969837903020309738969992799094977977157
7375177977949099729996983790302030973896999279909497797715737
72737517797794909972999698379030203097389699927990949779771573727
987273751779779490997299969837903020309738969992799094977977157372789
3098727375177977949099729996983790302030973896999279909497797715737278903
70309872737517797794909972999698379030203097389699927990949779771573727890307
397030987273751779779490997299969837903020309738969992799094977977157372789030793
3539703098727375177977949099729996983790302030973896999279909497797715737278903079353
36353970309872737517797794909972999698379030203097389699927990949779771573727890307935363
333635397030987273751779779490997299969837903020309738969992799094977977157372789030793536333
3433363539703098727375177977949099729996983790302030973896999279909497797715737278903079353633343
99343336353970309872737517797794909972999698379030203097389699927990949779771573727890307935363334399