# Partitions of Integers

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## 1 Introduction

A partition of an integer, $n$, is one way of writing $n$ as the sum of positive integers where the order of the addends (terms being added) does not matter. For the integer, $n$, the function giving the number of partitions is denoted by $p(n)$. As shown below, there are 7 partitions of 5. Thus $p(5) = 7$.

$$\begin{aligned} 5 &= 5 \\ &= 4+1 \\ &= 3+2 \\ &= 3+1+1 \\ &= 2+2+1 \\ &= 2+1+1+1 \\ &= 1+1+1+1+1 \end{aligned}$$

The subject of partitioning integers is very rich and quite deep. Covering every aspect of partitions would take hundreds of pages. This article provides a quick overview of partitions, introduces a few techniques for dealing with partitions, and explores some interesting problems. This article will hopefully shed some light on the beauty of partitions, combinatorics, and mathematics in general for you.

# 2 Tools for Dealing With Partitions

## 2.1 Ferrers Diagrams

A Ferrers diagram is a way of visualizing partitions with dots. Each row represents one addend in the partition. The number of dots in a row represents the value of that total addend. For example, the partition of 10 into $5 + 3 + 1 + 1$ is shown below.

$$\begin{array}{cccccc}
5 & \bullet & \bullet & \bullet & \bullet & \bullet \\
3 & \bullet & \bullet & \bullet & & \\
1 & \bullet & & & & \\
1 & \bullet & & & &
\end{array}$$

**Example 1.** Show that the number of partitions of an integer $n$ into parts the largest of which is $r$ is equal to the number of partitions of $n$ into exactly $r$ parts.

*Solution.* We are trying to find a way to relate two different types of partitions of $n$ both in terms of $r$. Perhaps a Ferrers diagram could lead us in the right direction. So let us try some examples.

Suppose $n = 10$ and $r = 3$. Then one partition of $n$ in which $r$ is the largest part is 3+3+2+1+1. In a Ferrers diagram this looks like:

$$\begin{array}{cccc}
3 & \bullet & \bullet & \bullet \\
3 & \bullet & \bullet & \bullet \\
2 & \bullet & \bullet & \\
1 & \bullet & & \\
1 & \bullet & &
\end{array}$$

The numbers on the left side of the diagram are found by counting up the dots in the row next to it. However, consider what would happen if we put numbers at the top and counted the dots in the columns. This would look like:

$$\begin{array}{ccc}
5 & 3 & 2 \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \\
\bullet & & \\
\bullet & &
\end{array}$$

We now have a partition of 10 with 3 parts. Now we can see that if we start off with a diagram of a partition whose largest part is $r$ and count by the columns instead of by the rows we will end up with a partition of $n$ with exactly $r$ parts.

This now suggests a way to show a one-to-one correspondence between the two sets. Take a partition of $n$ into exactly $r$ parts, i.e. $p_1 + p_2 + \cdots + p_r$. Now make the partition $q_1 + q_2 + \cdots + q_k$ where each $q_i$ represents the number of $(p_j)$'s as large as $i$. For example, if we started with 5+3+2+2 we would end up with $4 + 4 + 2 + 1 + 1$. But now notice that we start off with a partition into exactly $r$ parts meaning that the new partition will have addends which are at most $r$. Thus every partition of $n$ into exactly $r$ parts directly corresponds to one partition of $n$ whose largest addend is $r$.

Similarly, if we start off we a partition of $n$ into parts whose largest addend is $r$, $s_1+s_2+\cdots+s_m$, then we can make the new partition, $t_1+t_2+\cdots+t_h$, where each $t_i$ represents the number of $(s_j)$'s as large as $i$. Since the largest addend is $r$, we will have exactly $r$ terms in this partition.

We have thus shown a one-to-one correspondence and we are done.

It should also be noted that if we take a Ferrers diagram of a partition and reflect it about the 45° downward slanting line through the upper left dot we would obtain the same partition as when we count the dots by the columns rather than the rows. Let's try this with $3 + 3 + 2 + 1 + 1$. We start off with the diagram on the left, reflect it about the 45° line, and end up with the diagram on the right:

$$\begin{array}{llcl}
3 & \bullet\;\bullet\;\bullet & & \\
3 & \bullet\;\bullet\;\bullet & & 5\;\; \bullet\;\bullet\;\bullet\;\bullet\;\bullet \\
2 & \bullet\;\bullet & \longrightarrow & 3\;\; \bullet\;\bullet\;\bullet \\
1 & \bullet & & 2\;\; \bullet\;\bullet \\
1 & \bullet & &
\end{array}$$

When we reflect a Ferrers diagram like this the resulting diagram is called the conjugate of the original diagram.

## 2.2 Generating Functions

Generating functions were first applied to partitions by Leonhard Euler. This technique can reduce the difficulty of otherwise complex problems. We use generating functions because they can be manipulated much more easily than combinatorial quantities. The whole idea of a generating function is that we have what is called a power series, $A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$, where the coefficient of $x^k$, $a_k$, represents the number of ways that the event $k$ can happen. As a quick example, consider the generating function $S(x) = \sum_{k=0}^{\infty} 2^k x^k = 1 + 2x + 2^2x^2 + \cdots + 2^n x^n + \cdots$. Out of the various possibilities, we could say that this is the generating function for the number of subsets of an $n$-element set. This is due to the fact that the number of subsets of an $n$-element set is equal to $2^n$ and in the the generating function the coefficient of $x^n$ is indeed $2^n$.

To find the generating function for the number of partitions of a number we need to consider how many ones there are in the partitions, how many twos, threes and so on. In each partition one can occur $0, 1, 2, 3 \ldots$ times; thus contributing a factor of $1 + x + x^2 + \cdots$ to the generating function. Similarily two can occur $0, 1, 2, 3 \ldots$ times; thus contributing a factor of $1 + x^2 + x^4 + \cdots$. Continuing this logic we find that the generating function for the number of partitions of an integer is:

$$(1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)\cdots$$

We can assume that $|x| < 1$ and use the formula for the sum of a geometric series to get:

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}$$

The logic behind building this generating function is very important. Similar logic is needed to build generating functions for problems which require that the partitions meet certain conditions. We can use generating functions to prove certain relationships between partitions even though we cannot find a closed formula. There are very good approximations for the number of partitions of the general integer, $n$, however.[1]

**Example 2.** Show that the number of partitions of a number into parts which have at most one of each distinct even part (e.g. $1+1+1+2+3+4$) equals the number of partitions of the number in which each part can appear at most three times (e.g. $1+1+1+2+2+4+4+4$).

*Solution.* We shall approach this problem with generating functions. To find the generating function for the number of ways to partition a number into parts which have at most one of each distinct even part let us look at the generating functions for even and odd numbers separately. Two contributes a factor of $1 + x^2$ because you can either have zero or one two's. Four contributes a factor of $1 + x^4$ because you can either have zero or one fours. Continuing this logic we find that even numbers

[1]The best approximation yet was given by Rademacher. The formula is far too advanced for this paper, however.

contribute a factor of:

$$(1+x^2)(1+x^4)(1+x^6)\cdots$$

to the overall generating function.

The odd parts can be treated in the usual fashion. One contributes a factor of

$$1+x+x^2+\cdots=\frac{1}{1-x}.$$

Three contributes a factor of

$$1+x^3+x^6+x^9+\cdots=\frac{1}{1-x^3}.$$

And so on with the odd parts.

Thus the generating function for the number of partitions of a number into parts which have at most one of each distinct even part is:

$$\frac{(1+x^2)(1+x^4)(1+x^6)\cdots}{(1-x)(1-x^3)(1-x^5)\cdots}.$$

The generating function for the number of partitions of a number in which each part can appear at most three times is:

$$(1+x+x^2+x^3)(1+x^2+x^4+x^6)(1+x^3+x^6+x^9)\cdots=\frac{(1-x^4)(1-x^8)(1-x^{12})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots}.$$

We now wish to show that these two are equal, i.e.:

$$\frac{(1+x^2)(1+x^4)\cdots}{(1-x)(1-x^3)\cdots}=\frac{(1-x^4)(1-x^8)(1-x^{12})\cdots}{(1-x)(1-x^2)(1-x^3)\cdots}.$$

Common to the denominator on both sides is $(1-x)(1-x^3)\cdots$ so we can cancel that out:

$$(1+x^2)(1+x^4)(1+x^6)\cdots=\frac{(1-x^4)(1-x^8)(1-x^{12})\cdots}{(1-x^2)(1-x^4)(1-x^6)\cdots}$$

Now, using the fact that $a^2-b^2=(a+b)(a-b)$ the $(1+x^2)(1+x^4)(1+x^6)\cdots$ cancels on the right hand side. Thus we are left with:

$$(1+x^2)(1+x^4)(1+x^6)\cdots=(1+x^2)(1+x^4)(1+x^6)\cdots.$$

This is indeed true so we are done.

## 3 Other Interesting Facts, Problems etc.

This section contains various problems concerning the partitioning of integers.

**Example 3.** Find a quick method to determine how many ways an integer, $n$, can be expressed as the sum of consecutive positive integers.

*Solution.* The problem can be expressed as:

$$n = r + (r+1) + \cdots + (r+k)$$

where we are trying to find the total number of possible $r$. Writing the sum in closed form gives:

$$n = \frac{(2r+k)(k+1)}{2} \Rightarrow (2r+k)(k+1) = 2n.$$

By inspection $2r+k$ and $k+1$ have opposite parity, i.e. if one is odd the other is even and vice versa. So we are looking for the number of ways $2n$ can be expressed as the product of an even number and an odd number. It is not too hard to see that for every odd factor of $n$ we will be able to find distinct $r$ and $k$.

Thus the number of ways to express a positive integer, $n$, as the sum of consecutive positive integers is equal to the number of odd factors of $n$.

**Example 4.** Let $s(n)$ be the number of partitions of $n$ whose parts all are greater than 1. Prove that $s(n) = p(n) - p(n-1)$. ( $p(n)$ is still the normal partition function ).

*Solution.* We can break $p(n)$ into the sum of two smaller functions. The first function will be $s(n)$ as defined in the problem. Now this means that the second function must be the number of ways to partition $n$ such that the smallest part is equal to one. Let us call this function $c(n)$. So know we know that:

$$s(n) + c(n) = p(n) \Rightarrow s(n) = p(n) - c(n).$$

Comparing this to the formula we wish to prove, i.e. $s(n) = p(n) - p(n-1)$, we see that if we prove that $c(n) = p(n-1)$ we will be done.

Now, $c(n)$ represents the partitions of $n$ with the smallest addend equal to one. If we start with one of the partitions from $p(n-1)$ we can attach an addend equal to one and we will have a partition of the form of $c(n)$. Doing this for each partition generates a distinct partition of the form of $c(n)$. Likewise, we can start off with a partition of $c(n)$ and remove one of the addends equal to one and we will have a partition of the form of $p(n-1)$. We have established a one-to-one correspondence and have shown that the two sets have equal magnitude. Thus, we are done.

**Example 5.** How many solutions in non-negative integers are there to the equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 32$$

*Solution. Method 1*

Imagine that there were 32 pebbles lined up in a row. Then if we put 5 sticks between them we will have partitioned them into 6 groups of pebbles each with a non-negative amount of marbles. For example suppose that the 6 groups had 1, 1, 3, 5, 10, 12 marbles, respectively. This can be expressed as:

$$\bullet | \bullet | \bullet\bullet\bullet | \bullet\bullet\bullet\bullet\bullet | \bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet | \bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet$$

So basically we have 37 spots and we are choosing 5 of them to be $|$'s. The number of ways to choose 5 objects from 37 is $\binom{37}{5} = 435897$.

*Method 2*

Each $x_i$ can contribute $0, 1, 2, 3 \ldots$ to the sum. So the generating function for each of the $x_i$ is $1 + x + x^2 + \cdots$. Thus the total generating function is:

$$(1 + x + x^2 + \cdots)^6 = \frac{1}{(1-x)^6} = \binom{5}{5} + \binom{6}{5}x + \binom{7}{5}x^2 + \binom{8}{5}x^3 + \cdots$$

We are looking for the coefficient of $x^{32}$, which is just $\binom{37}{5} = 6,724,520$.

This result can be modified to find the total number of non-negative solutions to the equation $x_1 + x_2 + \cdots + x_n = r$. Using either of the two aforementioned methods results in the answer of $\binom{n+r-1}{r-1}$.

**Example 6.** Show that if the function $p(k)$ gives the number of partitions of $k$ then $p(2n) \geq p(n) + p(n-1) + \cdots + p(2) + p(1)$.

*Solution.* As an example let us look at $n = 4$. We want to show that:

$$p(8) \geq p(4) + p(3) + p(2) + p(1).$$

It would be easy enough to just crank out all the numbers but we want to find a way to relate the left and right hand sides. The partitions of 4 are:

$$\begin{aligned} 4 &= 4 \\ &= 3+1 \\ &= 2+2 \\ &= 2+1+1 \\ &= 1+1+1+1 \end{aligned}$$

We see that by appending on a 4 to each one of these partitions we will have a partition of 8, i.e.:

$$\begin{aligned} 8 &= 4+4 \\ &= 4+3+1 \\ &= 4+2+2 \\ &= 4+2+1+1 \\ &= 4+1+1+1+1 \end{aligned}$$

This gives us a good idea of why the desired inequality holds. Looking at

$$p(2n) \geq P(n) + p(n-1) + \cdots + p(2) + p(1)$$

again we see that for each partition represented by each $p(r)$ term on the right hand side we can append a $2n-r$ term to it and end up with a partition of $2n$. The greatest $2n-r$ can be is $n$ since $r$ is at most equal to $n$. Thus it follows that we can always produce as many partitions of $2n$ as there are of partitions represented by the right hand side by taking each of the latter and appending the required number so that we get a partition of $2n$. Thus the left hand side is at least as great as the right hand side.

## 4 Conclusion

Attacking partition problems usually requires an open mind and creativity. As with every branch of mathematics there is no clear-cut procedure that will solve every problem consistently. There are, however, certain strategies and tactics that will simplify problems. Always try the general techniques for solving problems: experiment, plug in numbers, try various cases, and look for patterns. This is one of the most critical parts of solving a problem and it takes a lot of practice and patience to be able to carry out these methods in a productive manner.

Do not forget how useful the two tools, generating functions and Ferrers diagrams, can be for partition problems. Generating functions are a vital component to solving many partition problems. They are particularly useful when showing that the number of a certain kind of partition is equal to the number of another kind of partition. If generating functions do not work, then do not sweat it. A lot of problems that can be solved with generating functions usually have alternate solutions. Also try to make use of visuals such as a Ferrers diagram. Do not be afraid to think outside of the box.

These are just a few tips for solving problems. Do not forget that strategy will not solve a problem alone. Experience, determination, and ingenuity are also important factors.

Assorted in no specific difficulty order below are a few problems dealing with partitions. They are here so that the reader can try his or her hand at these types of problems. If possible, try using Ferrers diagrams and generating functions on these problems. Have fun and enjoy!

**Problem 1.** Which is larger: the number of ways to partition 2004 into an odd number of odd integers or the number of ways to partition 2004 into an even number of odd integers?

**Problem 2.** Prove that every number of the form $n^k$ where $n$ is an integer greater than one and $k$ is a positive integer can be represented as the sum of $n$ positive odd integers.

**Problem 3.** Find the number of solutions to the inequality $a_1 + a_2 + a_3 + a_4 \leq 68$ where the $a_i$ are non-negative integers.

**Problem 4.** The number 999 is partitioned into four numbers $a$, $b$, $c$ and $d$. Find $a$, $b$, $c$, and $d$ if $a + 3 = b - 3 = 3c = \dfrac{d}{3}$.

**Problem 5.** Prove that the number of partitions of $n$ with no part greater than $k$ is equal to the number of partitions of $n + k$ with exactly $k$ parts

**Problem 6.** Show that the number of *ordered* partitions of a number $n$ into exactly $k$ parts is equal to $\dbinom{n-1}{k-1}$.

**Problem 7.** Prove that if $n > \dfrac{m(m+1)}{2}$, then the number of partitions of $n$ into $m$ distinct parts is equal to the number of partitions of $n - \dfrac{m(m+1)}{2}$ into at most $m$ (not necessarily distinct)parts.

**Problem 8.** In how many ways can $n$ be written as the sum of two positive integers? Representations differing in only the order are considered to be the same.

**Problem 9.** Prove that the number of partitions of any integer $n$ into distinct parts is equal to the number of partitions of $n$ into odd parts.

**Problem 10.** Let $n$ be a positive integer. Allie and Bob play a game constructing a partition $n = a_1 + a_2 + \cdots + a_k$ with $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$. Allie wins if there is an odd number of terms in the partition, i.e., if $k$ is odd; Bob wins otherwise. Allie begins by choosing a number $a_1$ between 1 and $n-1$ inclusive. Bob then chooses a number $a_2$ between $a_1$ and 1 inclusive such that $a_1 + a_2 \leq n$. Allie then chooses an $a_3$ between $a_2$ and 1 inclusive such that $a_1 + a_2 + a_3 \leq n$, and so on, with the game ending when the partition is complete. Determine with proof all $n > 1$ for which Bob has a winning strategy.