Colloquium on Mathematics and Computer Science

Versailles, September 2000

An Algebra for Proper Generating Trees

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- Proper Riordan Arrays
- Generating trees

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BASIC DEFINITIONS

A proper Riordan Array is defined by a couple of fps $(d(t), h(t))$, $d(0) \neq 0, h(0) \neq 0$, and the generic element of the two dimensional sequence induced by the pRA is

$$d_{n,k} = [t^n]d(t)(th(t))^k$$

A monic, integer fps is a formal power series $f(t) = \sum_{k=0}^{\infty} f_k t^k$ such that $f_0 = 1$ and $f_k \in \mathbf{Z}$, for every $k > 0$.

A monic, integer pRA is a proper Riordan Arrays whose elements are in **Z** and those in the main diagonal are 1.

Many combinatorial triangles are examples of monic, integer pRA's, like the Pascal, Catalan, Motzkin and Schröder triangles.

The Pascal triangle

n/k	0	1	2	3	4
0	1				
1	1	1			
2	1	2	1		
3	1	3	3	1	
4	1	4	6	4	1

Here,

$$d(t) = h(t) = \frac{1}{1-t} \qquad d_{n,k} = [t^{n-k}]\frac{1}{(1-t)^{k+1}} = \binom{n}{k}$$

THEOREM: Let $D = (d(t), h(t))$ be a pRA: D is a monic, integer pRA if and only if $d(t)$ and $h(t)$ are monic, integer fps.

THE RIORDAN GROUP

If $\mathcal{R}$ denotes the set of pRA's, then $(\mathcal{R}, \star)$, where $\star$ is the usual row-by-column product, is a non-commutative group, called the Riordan group.

Let $D = (d(t), h(t))$ and $E = (f(t), g(t))$ be two pRA's:

PRODUCT

$$(d(t), h(t)) \star (f(t), g(t)) = (d(t)f(th(t)), h(t)g(th(t)))$$

NEUTRAL ELEMENT

$$(d(t), h(t)) = (1, 1)$$

INVERSE

$$(d(t), h(t)) \star (\bar{d}(t), \bar{h}(t)) = (1, 1)$$

$$\bar{d}(y) = \left[\frac{1}{d(t)} \middle| y = th(t) \right] \qquad \bar{h}(y) = \left[\frac{1}{h(t)} \middle| y = th(t) \right]$$

The existence and uniqueness of $\bar{d}$ and $\bar{h}$ is guaranteed by the fact that the functional equation $y = th(t)$ has a unique solution $t = t(y)$ with $t(0) = 0$ if and only if $h(0) \neq 0$.

THEOREM: If D is a monic integer pRA, then also its inverse $\bar{D}$ is such.

$\rightarrow$ monic integer pRA's is a non commutative group.

THE $A-$ AND $Z-$ SEQUENCES: some properties

In [Rog78] it is proved ($a_0 \neq 0$):

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \ldots = \sum_{j=0}^{\infty} a_j d_{n,k+j}.$$

The sequence $A = \{a_0, a_1, a_2, \ldots\}$ is called the $A-$sequence of the pRA and we have:

$$h(t) = A(th(t)) \qquad A(y) = [h(t)\,|y = th(t)]$$

In [MRSV97] it is proved that for column 0 another sequence $Z = \{z_0, z_1, z_2, \ldots\}$ exists, called the Z-sequence of the pRA, such that:

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \ldots = \sum_{j=0}^{\infty} z_j d_{n,j}$$

and we have:

$$d(t) = \frac{d_{0,0}}{1 - tZ(th(t))} \qquad Z(y) = \left[\frac{d(t) - d_0}{td(t)} \right| y = th(t) \bigg]$$

where $d_{0,0} = d_0$.

At this point, we have a new characterisation of pRA's by means of the triple

$$(d_{0,0}, A(t), Z(t))$$

It is now immediate to observe that a monic integer pRA corresponds to a monic-integer A-sequence, and vice versa if also $d(t)$ is monic integer.

Instead, the Z-sequence is integer, but it can be non-monic; in fact, we see that z_0 is related to d_1 and not to d_0.

SOME OTHER PROPERTIES

THEOREM: Let $D = (d(t), h(t))$ be a pRA and $\bar{D} = (\bar{d}(t), \bar{h}(t))$ its inverse. Then $\bar{A}(t) = 1/h(t)$.

THEOREM: Let $D = (d(t), h(t))$, $E = (e(t), k(t))$ be two pRA and $F = (f(t), g(t))$ their product. If $A_D(t)$,$A_E(t)$ and $A_F(t)$ are the generating functions of the corresponding A-sequences, we have:

$$A_F(t) = A_E(t) \left[A_D(y) \middle| t = yk(y) \right].$$

THEOREM: Let $D = (d(t), h(t))$ and $\bar{D} = (\bar{d}(t), \bar{h}(t))$ its inverse pRA; then if $Z(t)$, $\bar{Z}(t)$ are the generating functions of the corresponding Z-sequences, we have

$$\bar{Z}(t) = \frac{d_0 - d(t)}{d_0 t h(t)} = \frac{-Z(th(t))}{h(t)(1 - tZ(th(t))}.$$

Let D be the Pascal triangle and E the Catalan triangle, so that we have

$$D = \left(\frac{1}{1-t}, \frac{1}{1-t}\right) \qquad E = \left(\frac{1-\sqrt{1-4t}}{2t}, \frac{1-\sqrt{1-4t}}{2t}\right)$$

and $A_D(t) = 1 + t$, $A_E(t) = 1/(1-t)$. Numerically we have:

$$\begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix} \star \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 2 & 2 & 1 & & \\ 5 & 5 & 3 & 1 & \\ 14 & 14 & 9 & -4 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & & & & \\ 2 & 1 & & & \\ 5 & 4 & 1 & & \\ 15 & 14 & 6 & 1 & \\ 51 & 50 & 27 & 8 & 1 \end{pmatrix} = F$$

In this case, the equation $t = yk(y)$ is $t = \left(1 - \sqrt{1-4y}\right)/2$ and its solution is $y = t - t^2$, so that $A_D(y) = 1 + y = 1 + t - t^2$ and

$$A_F(t) = \frac{1 + t - t^2}{1-t} = 1 + 2t + t^2 + t^3 + t^4 + \cdots.$$

We also have:

$$F = \left(\frac{1 - \sqrt{\frac{1-5t}{1-t}}}{2t}, \frac{1 - \sqrt{\frac{1-5t}{1-t}}}{2t}\right)$$

GENERATING TREES: some bibliography

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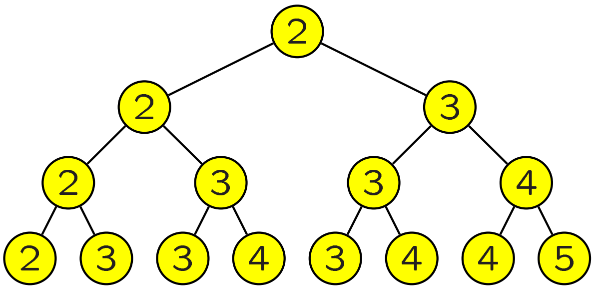
In particular, in [MV00] it is shown that a particular subset of generating trees has a correspondence with some pRA's.

DEFINITION An infinite matrix $\{d_{n,k}\}_{n,k\in\mathbb{N}}$ is *associated to a generating tree* with root (c) (AGT matrix for short) if $d_{n,k}$ is the number of nodes at level n with label $k + c$.

Under suitable conditions, this matrix corresponds to a pRA, and vice versa.

$$\begin{cases} root: & (2) \\ rule: & (k) \quad \rightarrow (k)(k+1) \end{cases} \qquad (1)$$



The Pascal triangle corresponds to the AGT matrix associated to the previous generating tree specification.

n/k	0	1	2	3	4
0	1				
1	1	1			
2	1	2	1		
3	1	3	3	1	
4	1	4	6	4	1

MARKED GENERATING TREES

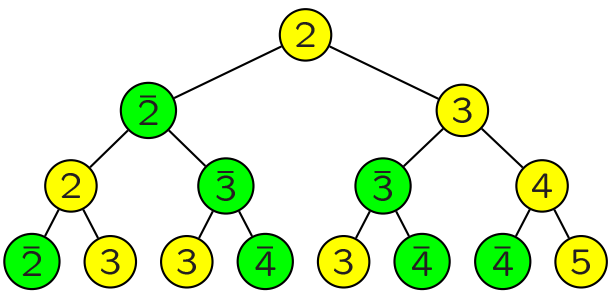
DEFINITION: A marked generating tree is a rooted labelled tree (the labels can be marked or non-marked) with the property that if v_1 and v_2 are any two nodes with the same label then, for each label l, v_1 and v_2 have exactly the same number of children with label l. To specify a generating tree it therefore suffices to specify:

1) the label of the root;

2) a set of rules explaining how to derive from the label of a parent the labels of all of its children.

A simple example is given by the following generating tree specification:

$$\begin{cases} root: & (2) \\ rule: & (k) \to (\overline{k})(k+1) \\ & (\overline{k}) \to (k)(\overline{k+1}) \end{cases} \tag{2}$$



n/k	0	1	2	3	4
0	1				
1	-1	1			
2	1	-2	1		
3	-1	3	-3	1	
4	1	-4	6	-4	1

The idea is that marked labels kill or annihilate the non-marked labels with the same number, i.e. the count relative to an integer j is the difference between the number of non-marked and marked labels j at a given level. This gives a negative count if marked labels are more numerous than non-marked ones.

DEFINITION: An infinite matrix $\{d_{n,k}\}_{n,k\in\mathbb{N}}$ is said to be "associated" to a marked generating tree with root (c) (AGT matrix for short) if $d_{n,k}$ is the difference between the number of nodes at level n with label $k+c$ and the number of nodes with label $\overline{k+c}$. By convention, the level of the root is 0.

SOME NOTATIONS

$(x) = (\bar{\bar{x}})$;
$(x)^p = \underbrace{(x)\cdots(x)}_{p},\ p \geq 0$
$(x)^p = \underbrace{(\bar{x})\cdots(\bar{x})}_{-p},\ p < 0$
$\overline{(x)}^p = (\bar{x})^p,\ p > 0$
$\overline{(x)}^p = (x)^{-p},\ p < 0$
$\prod_{j=0}^{i}(k-j)^{\alpha_j} = (k)^{\alpha_0}(k-1)^{\alpha_1}\cdots(k-i)^{\alpha_i}$

THE MAIN RESULT

THEOREM: Let $c \in \mathbf{N}$, $a_j, b_k \in \mathbf{Z}$, $\forall j \geq 0$ and $k \geq c$, $a_0 = 1$, and let

$$\begin{cases} root: & (c) \\ rule: & (k) \quad \rightarrow (c)^{b_k} \prod_{j=0}^{k+1-c} (k+1-j)^{a_j} \\ & (\bar{k}) \quad \rightarrow \overline{(c)}^{b_k} \prod_{j=0}^{k+1-c} \overline{(k+1-j)}^{a_j} \end{cases} \qquad (3)$$

be a marked generating tree specification. Then, the AGT matrix associated to (3) is a monic integer pRA D defined by the triple (d_0, A, Z), such that

$$d_0 = 1, \quad A = (a_0, a_1, a_2, \ldots),$$

$$Z = (b_c + a_1, b_{c+1} + a_2, b_{c+2} + a_3, \ldots).$$

Vice versa, if D is a monic integer pRA defined by the triple $(1, A, Z)$ with $a_j, z_j \in \mathbf{Z}$, $\forall j \geq 0$ and $a_0 = 1$, then D is the AGT matrix associated to the generating tree specification (3) with $b_{c+j} = z_j - a_{j+1}$, $\forall j \geq 0$.

A generating tree corresponding to the specification (3) will be called a *proper generating tree*.

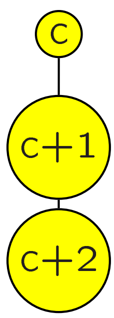
AN ALGEBRA FOR PROPER GENERATING TREES

Definition: Given two generating tree specifications t_1 and t_2 of type (3) and the corresponding AGT matrices T_1 and T_2, we define the *generating tree specification product of t_1 and t_2* as the specification t_3 having $T_3 = T_1 \star T_2$ as AGT matrix.

Definition: Given a generating tree specification t_1 of type (3) and the corresponding AGT matrix T_1, we define the *generating tree specification inverse of t_1* as the specification t_2 having $T_2 = T_1^{-1}$ as AGT matrix.

Definition: The *identity generating tree specification* t_I is the one having the identity matrix I as AGT matrix. The specification and the corresponding generating trees are shown below:

$$\begin{cases} root: & (c) \\ rule: & (k) \quad \rightarrow (k+1) \end{cases} \tag{4}$$



PASCAL GENERATING TREES

The specification (6) is the inverse of (5), as can be easily verified.

$$\begin{cases} root: & (2) \\ rule: & (k) \rightarrow (k)(k+1) \end{cases} \tag{5}$$

$$\begin{cases} root: & (2) \\ rule: & (k) \rightarrow (\bar{k})(k+1) \\ & (\bar{k}) \rightarrow (k)(\overline{k+1}) \end{cases} \tag{6}$$

In fact, for the Pascal triangle we have $d_0 = 1$, $A = \{1, 1, 0, 0, \ldots\}$ and $Z = \{1, 0, 0, \ldots\}$ and for its inverse $\bar{d}_0 = 1$, $\bar{A} = \{1, -1, 0, 0, \ldots\}$ and $\bar{Z} = \{-1, 0, 0, \ldots\}$.

The Pascal triangle and its inverse

n/k	0	1	2	3	4
0	1				
1	1	1			
2	1	2	1		
3	1	3	3	1	
4	1	4	6	4	1

n/k	0	1	2	3	4
0	1				
1	-1	1			
2	1	-2	1		
3	-1	3	-3	1	
4	1	-4	6	-4	1

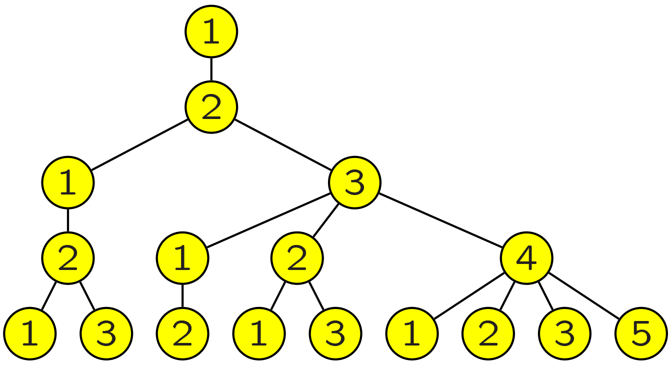
MOTZKIN GENERATING TREES

The first specification is related to Motzkin numbers $M_j = \{1, 1, 2, 4, 9, \ldots\} = [t^j](1 - t - \sqrt{1 - 2t - 3t^2})/(2t)$:

$$\left\{ \begin{array}{lll} root: & (1) & \\ rule: & (k) & \rightarrow (1) \ldots (k-1)(k+1) \end{array} \right. \qquad (7)$$

and a partial generating tree is:

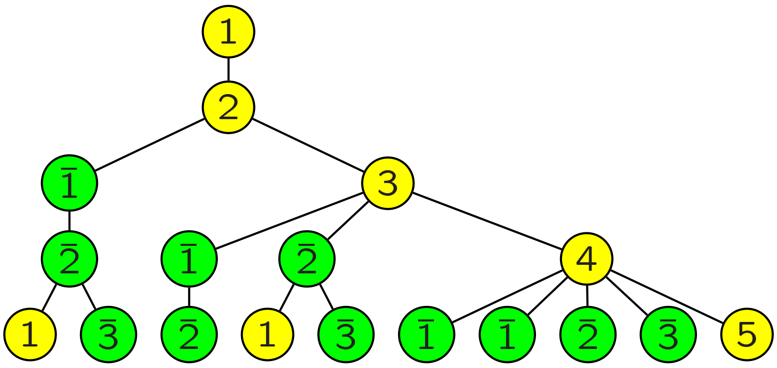
The Motzkin generating tree



The inverse specification is the following:

$$\left\{ \begin{array}{lll} root: & (1) & \\ rule: & (k) & \rightarrow (k+1) \prod_{j=2}^{k} (\overline{k+1-j})^{M_{j-2}} \\ & (\overline{k}) & \rightarrow (\overline{k+1}) \prod_{j=2}^{k} (k+1-j)^{M_{j-2}} \end{array} \right. \qquad (8)$$

The *inverse* of Motzkin generating tree



The Motzkin triangle and its inverse

n/k	0	1	2	3	4
0	1				
1	0	1			
2	1	0	1		
3	1	2	0	1	
4	3	2	3	0	1

n/k	0	1	2	3	4
0	1				
1	0	1			
2	-1	0	1		
3	-1	-2	0	1	
4	0	-2	-3	0	1

$$d_0 = 1, \quad A = (1, 0, 1, 1, \ldots)$$

$$Z = (0, 1, 1, \ldots)$$

$$\bar{d}_0 = 1, \quad \bar{A} = (1, 0, -1, -1, -2, -4 \ldots)$$

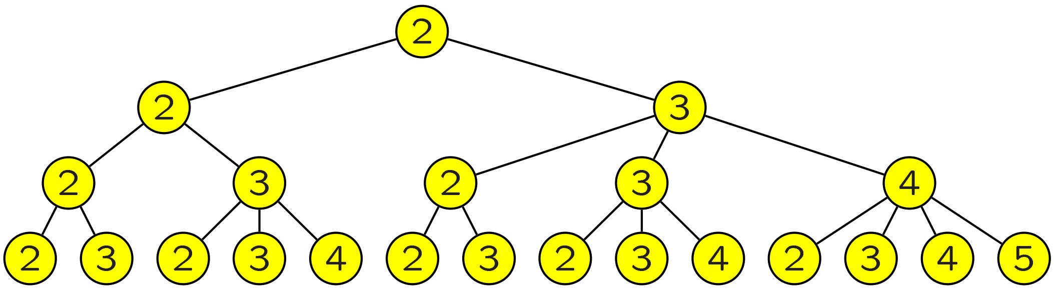
$$\bar{Z} = (0, -1, -1, -2, -4, \ldots)$$

CATALAN GENERATING TREES

The following example is related to the Catalan numbers $C_j = \{1, 1, 2, 5, 14, \ldots\} = \frac{1}{j+1}\binom{2j}{j}$.

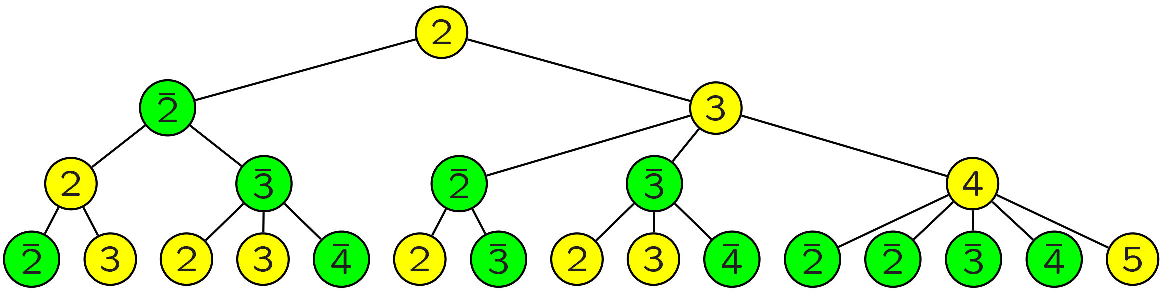
$$\begin{cases} root: & (2) \\ rule: & (k) \quad \rightarrow (2)\cdots(k)(k+1) \end{cases} \tag{9}$$

The Catalan generating tree: this is the *product* between the Pascal and the Motzkin trees



$$\begin{cases} root: & (2) \\ rule: & (k) \quad \rightarrow (k+1)\prod_{j=1}^{k-1}(\overline{k+1-j})^{C_{j-1}} \\ & (\bar{k}) \quad \rightarrow (\overline{k+1})\prod_{j=1}^{k-1}(k+1-j)^{C_{j-1}} \end{cases} \tag{10}$$

The *inverse* of Catalan generating tree



The Catalan triangle and its inverse

n/k	0	1	2	3	4
0	1				
1	1	1			
2	2	2	1		
3	5	5	3	1	
4	14	14	9	4	1

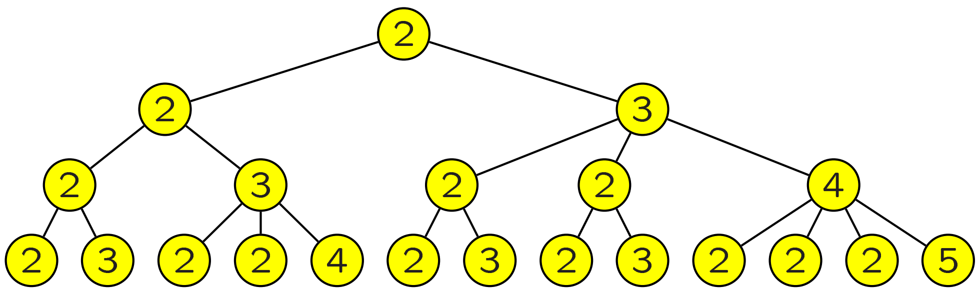
n/k	0	1	2	3	4
0	1				
1	-1	1			
2	0	-2	1		
3	0	1	-3	1	
4	0	0	3	-4	1

FIBONACCI GENERATING TREES

The last example is related to the Fibonacci numbers $F_j = \{1, 1, 2, 3, 5, 8, \ldots\} = [t^j]1/(1 - t - t^2)$.

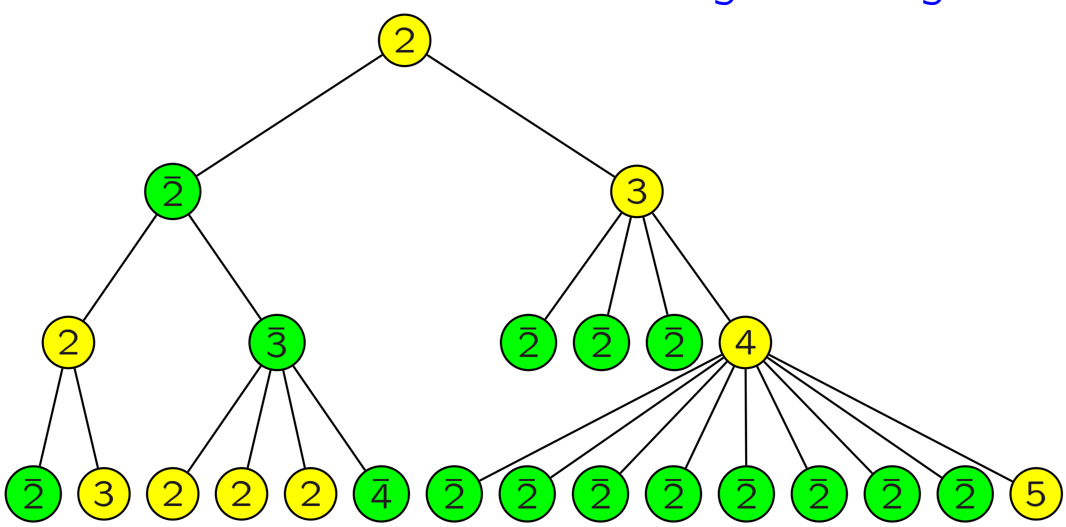
$$\begin{cases} root: & (2) \\ rule: & (k) \quad \to (2)^{k-1}(k+1) \end{cases} \tag{11}$$

The odd Fibonacci generating tree



$$\begin{cases} root: & (2) \\ rule: & (k) \quad \to (\bar{2})^{F_{2k-3}}(k+1) \\ & (\bar{k}) \quad \to (2)^{F_{2k-3}}(\overline{k+1}) \end{cases} \tag{12}$$

The *inverse* of odd Fibonacci generating tree



The odd Fibonacci triangle and its inverse

n/k	0	1	2	3	4
0	1				
1	1	1			
2	3	1	1		
3	8	3	1	1	
4	21	8	3	1	1

n/k	0	1	2	3	4
0	1				
1	-1	1			
2	-2	-1	1		
3	-3	-2	-1	1	
4	-4	-3	-2	-1	1

CONCLUSIONS

- Some combinatorial interpretations?
- What about fractional numbers?