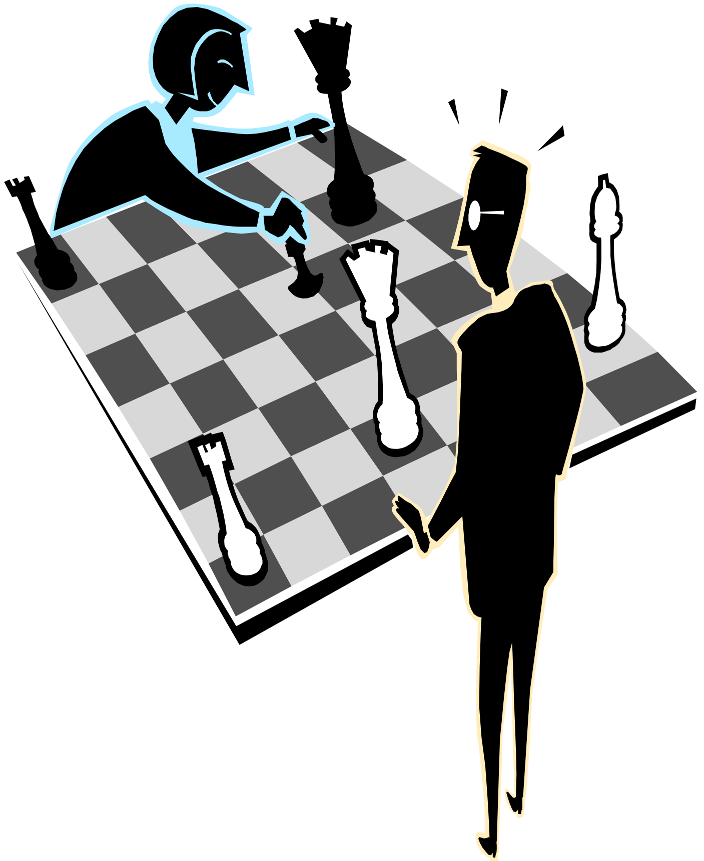
The N+1 Queens Problem

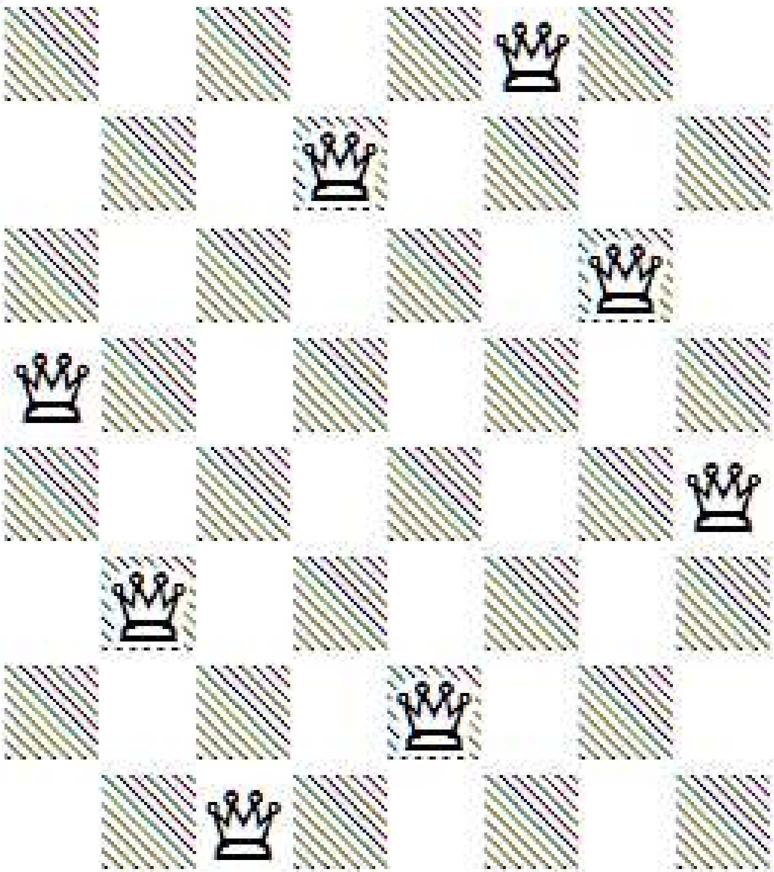
Doug Chatham
Morehead State University
November 5, 2004

Acknowledgements

- Joint work with Gerd Fricke and R. Duane Skaggs



The Eight Queens Problem



- Place eight queens on a standard chessboard so that no two attack each other.
- First posed in 1848.
- Generalized in 1850 to N queens on an N x N board.

Counting N-queens solutions

N	Number of solutions	N	Number of solutions
4	2	11	2,680
5	10	12	14,200
6	4	13	73,712
7	40	14	365,596
8	92	15	2,279,184
9	352	16	14,772,512
10	724	17	95,815,104

More counting

N	Number of solutions to N-queens problem
18	666,090,624
19	4,968,057,848
20	39,029,188,884
21	314,666,222,712
22	2,691,008,701,644
23	24,233,937,684,440
24	227,514,171,973,736

More about N-queens…

- Theorem: For N > 3, there is at least one solution to the N-queens problem.
- Proved first by Ahrens in 1910.
- Also proved by Hoffman, Loessi, and Moore in 1969. (Mathematics Magazine, March-April 1969, 66-72.)
- Also proved by others.

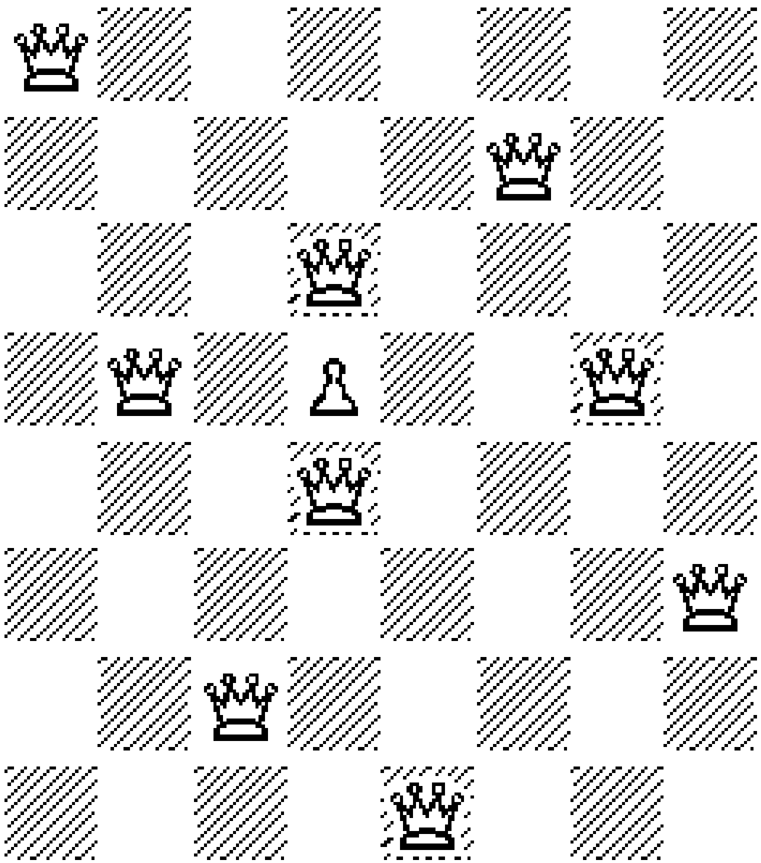
At most N queens, right?

- **Clearly**, each row can only have one queen.
- **Clearly**, an N x N chessboard can have at most N non-attacking queens placed on it.

The Nine Queens Contest

- January-March 2004, The Chess Variant Pages at chessvariants.org.
- If we place a pawn between two queens on the same row (or column or diagonal), the queens no longer attack each other.
- Question: How many pawns do we need in order to put 9 nonattacking queens on a standard chessboard?

Solution to 9 Queens Contest



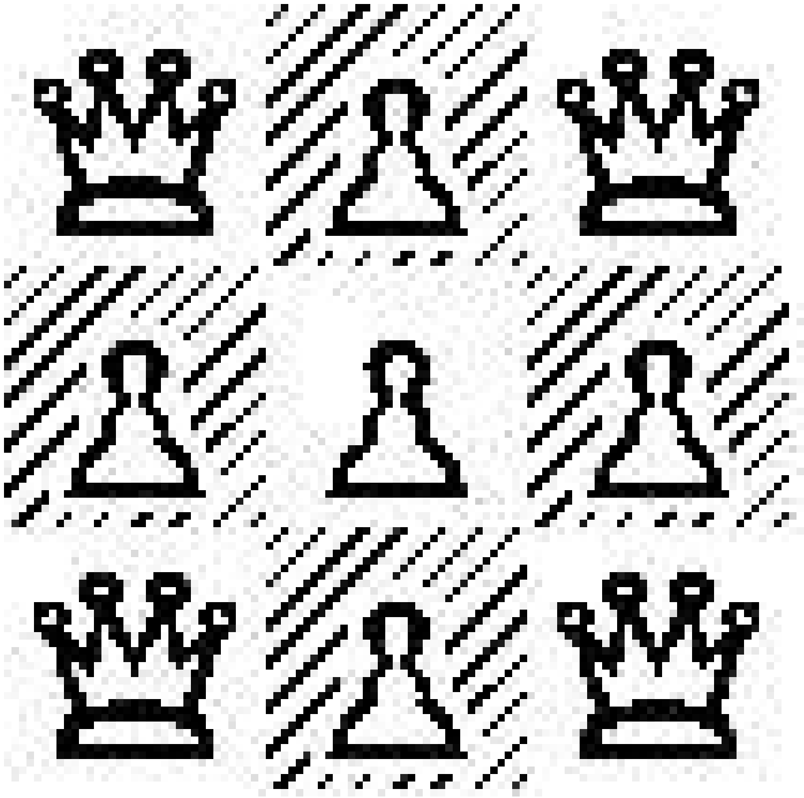
- Answer: One pawn.
- A solution:
  - QUEENS at a8, b5, c2, d4, d6, e1, f7, g5, h3
  - PAWN at d5

The N+k Queens Problem

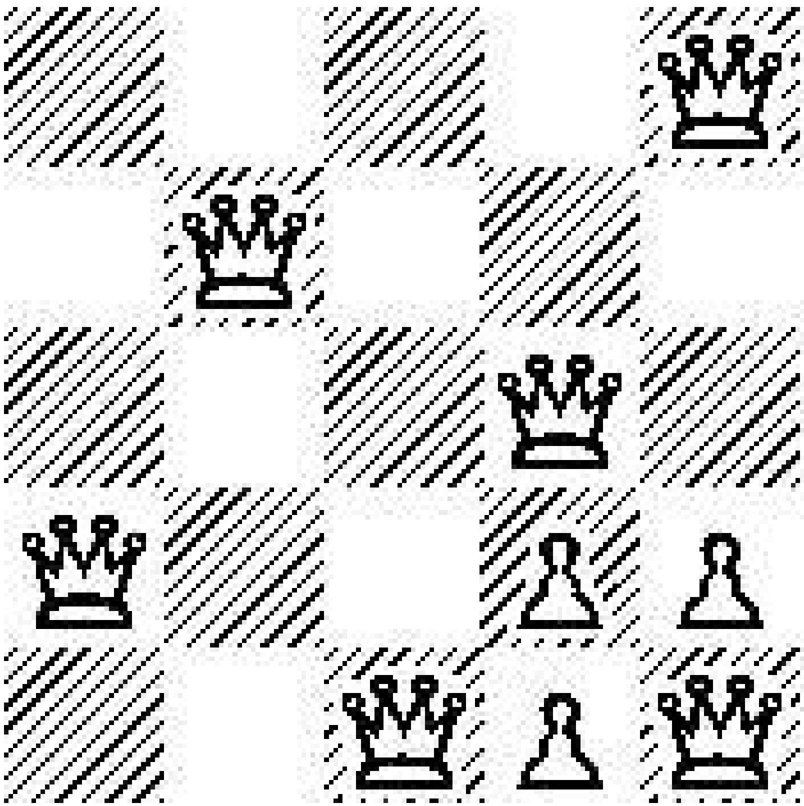
- Given N+k queens, how many pawns need to be put on an N x N board to allow the queens to be placed on the board so they don't attack each other?
- This is the **queens' separation number** $s_Q(N+k,N)$.

The N+1 Queens Problem

- $s_Q(4,3) = 5$.
- $s_Q(5,4)$ doesn't exist.
  - When we put 5 queens on a 4 x 4 board, at least two queens will be on adjacent squares.



The N+1 Queens Problem



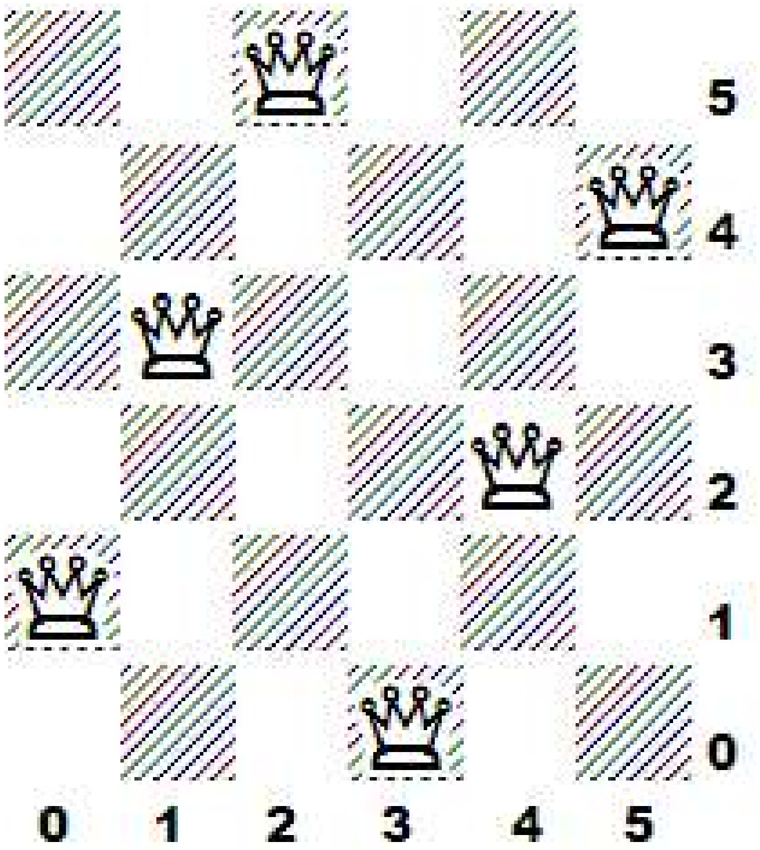
- $s_Q(6,5) = 3$.
  - Verified by computer search.
- Theorem: For $N \geq 6$, $s_Q(N+1,N) = 1$

Sketch of Proof of Theorem

- We take known solutions to the n-queens problem and add extra rows, columns, queens, and a pawn.
- We have four main patterns and a few exceptional cases.

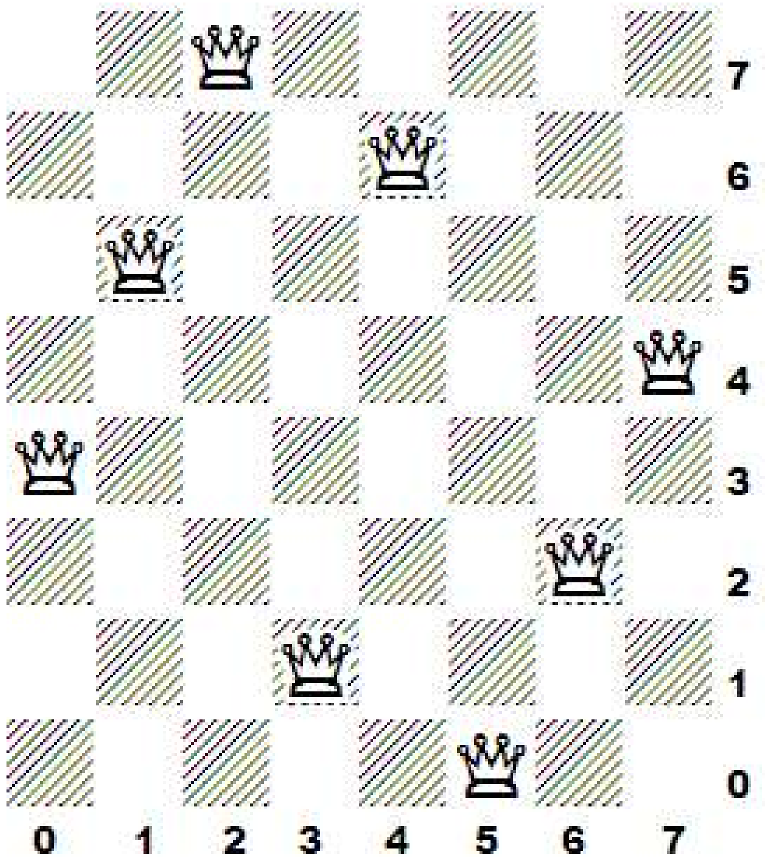
N-Queens Construction A

- $n \geq 4$
- $n \equiv 0$ or 4 (mod 6)
- Number rows and columns 0, 1,…, n-1
- Queens at (2i+1,i) for i=0,…, n/2-1
- Queens at (2i-n,i) for i=n/2,…,n-1.



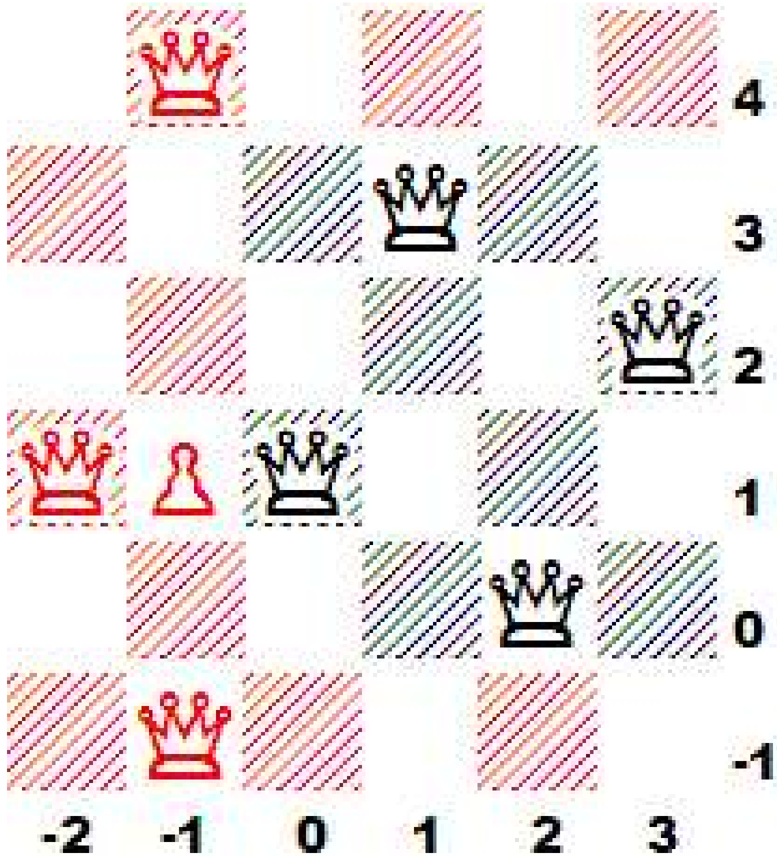
N-Queens Construction B

- $n \geq 4$, $n \equiv 2$ or $4 \pmod 6$
- Number rows and columns 0, 1,…, n-1
- Queens at (n/2+2i-1 (mod n),i) for i=0,1,…n/2-1 and at (n/2+2i+2 (mod n), i) for i=n/2-1,…,n-1



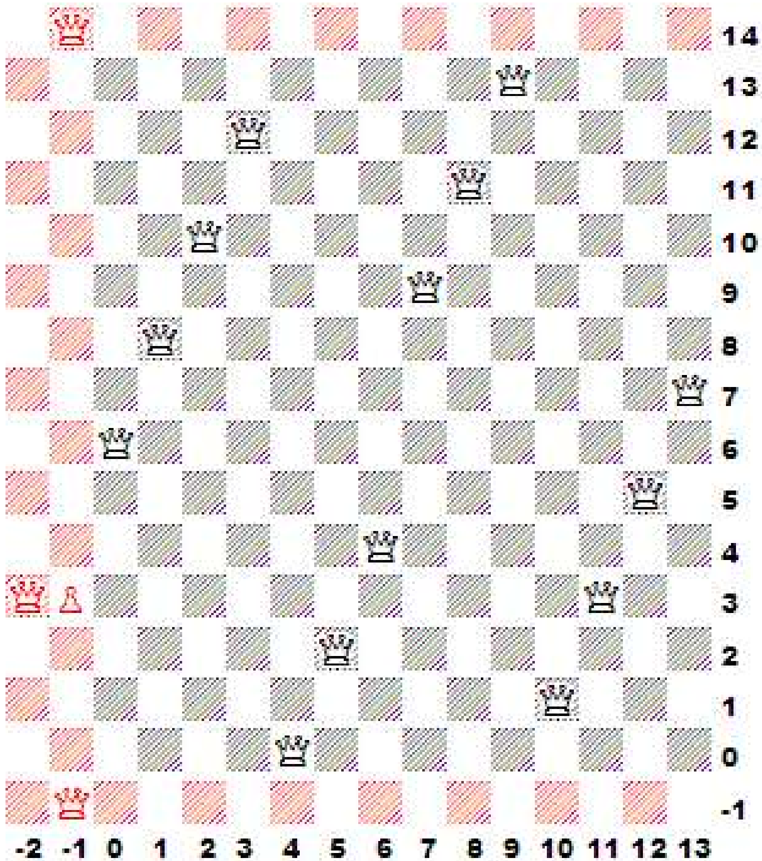
Pattern I: N≥6, N ≡ 0 or 2 (mod 6)

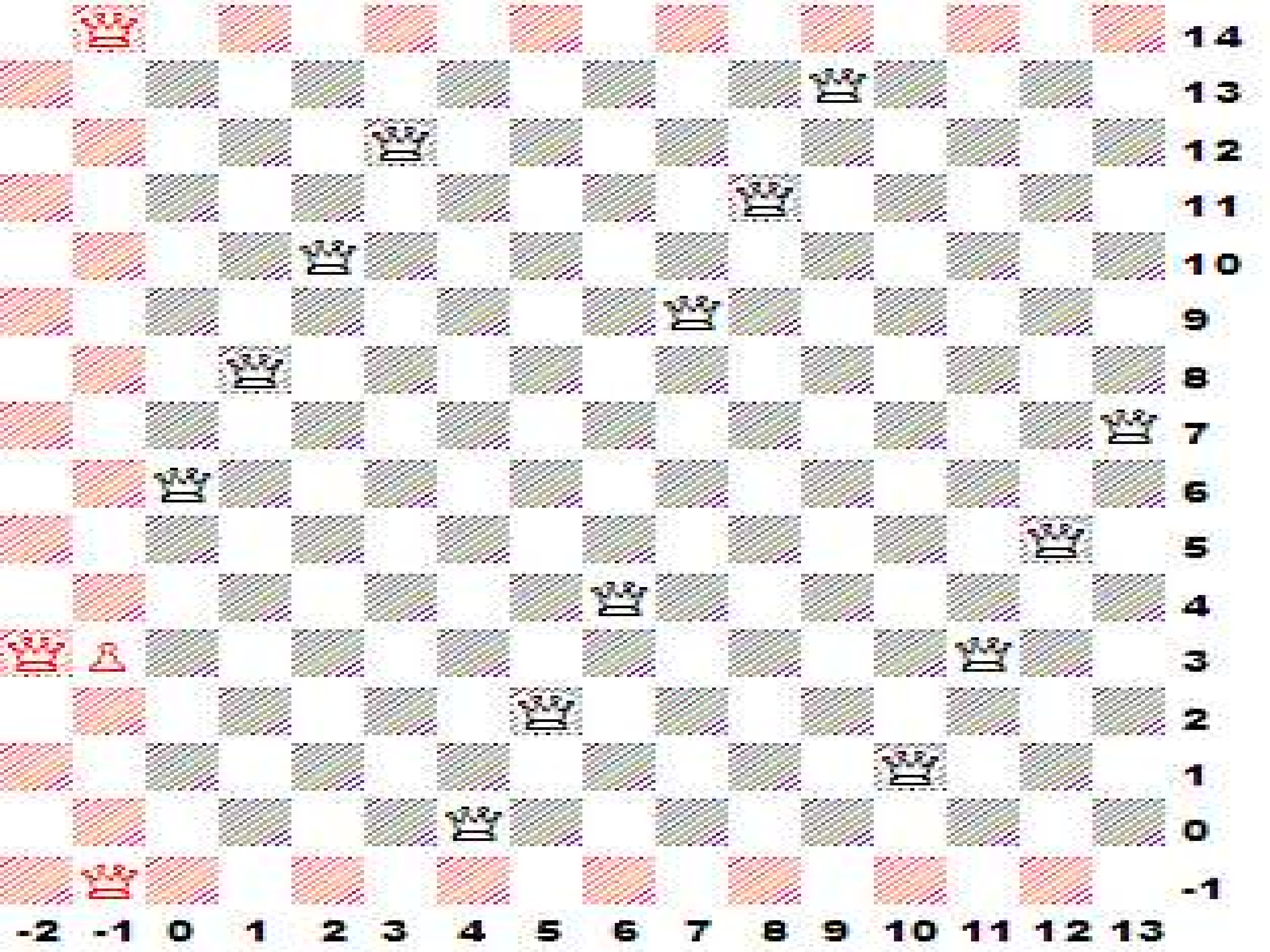
- Take Construction A solution to (n-2)-Queens.
- Add two columns to left and one row to top and bottom
- Put pawn to the left of the queen in column 0.
- Put extra queens at top and bottom of the pawn's column and to the left of the pawn.



Pattern II: $N \geq 10$, $N \equiv 0$ or $4 \pmod 6$

- Take Construction B solution to (N-2)-Queens. Add rows and columns as in Pattern I.
- If N=10, w=4.
- If N=12, w=7.
- If N>12, w= $\lfloor (N-1)/4 \rfloor$
- Pawn: (w, -1)
- Extra Queens: (w, -2), (-1,-1), (N-2, -1)

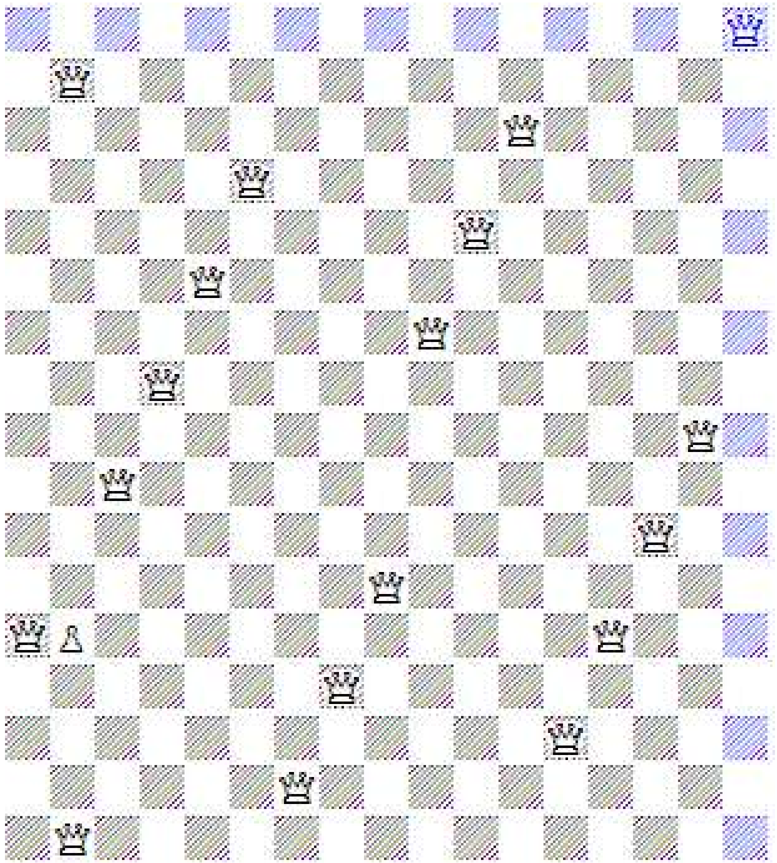




14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
-1
-2 -1 0 1 2 3 4 5 6 7 8 9 10 11 12 13

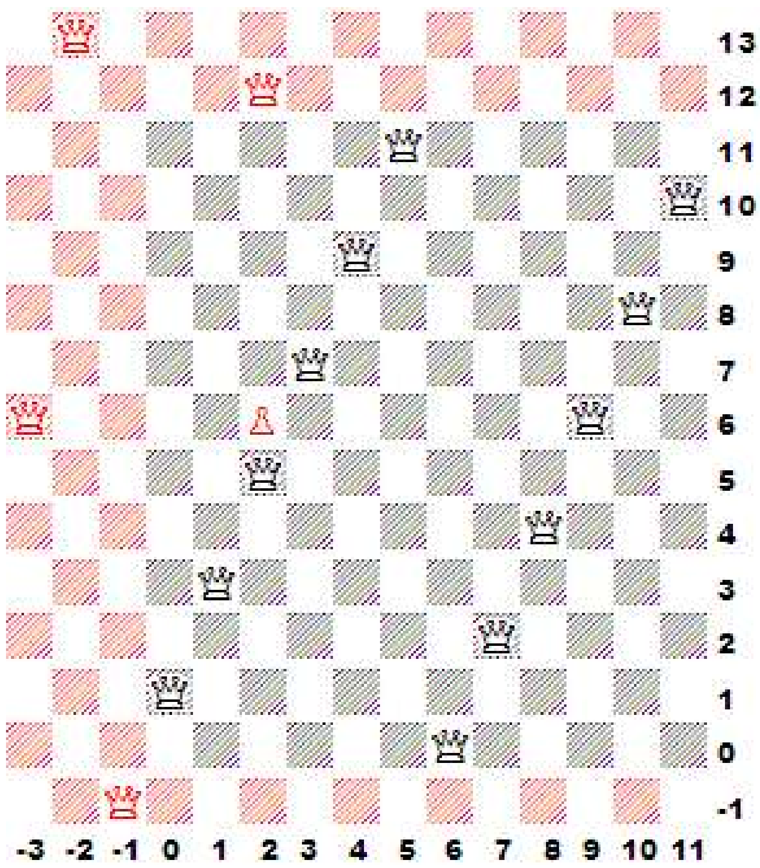
Pattern III: $N \geq 11$, $N \equiv \pm 1 \pmod 6$

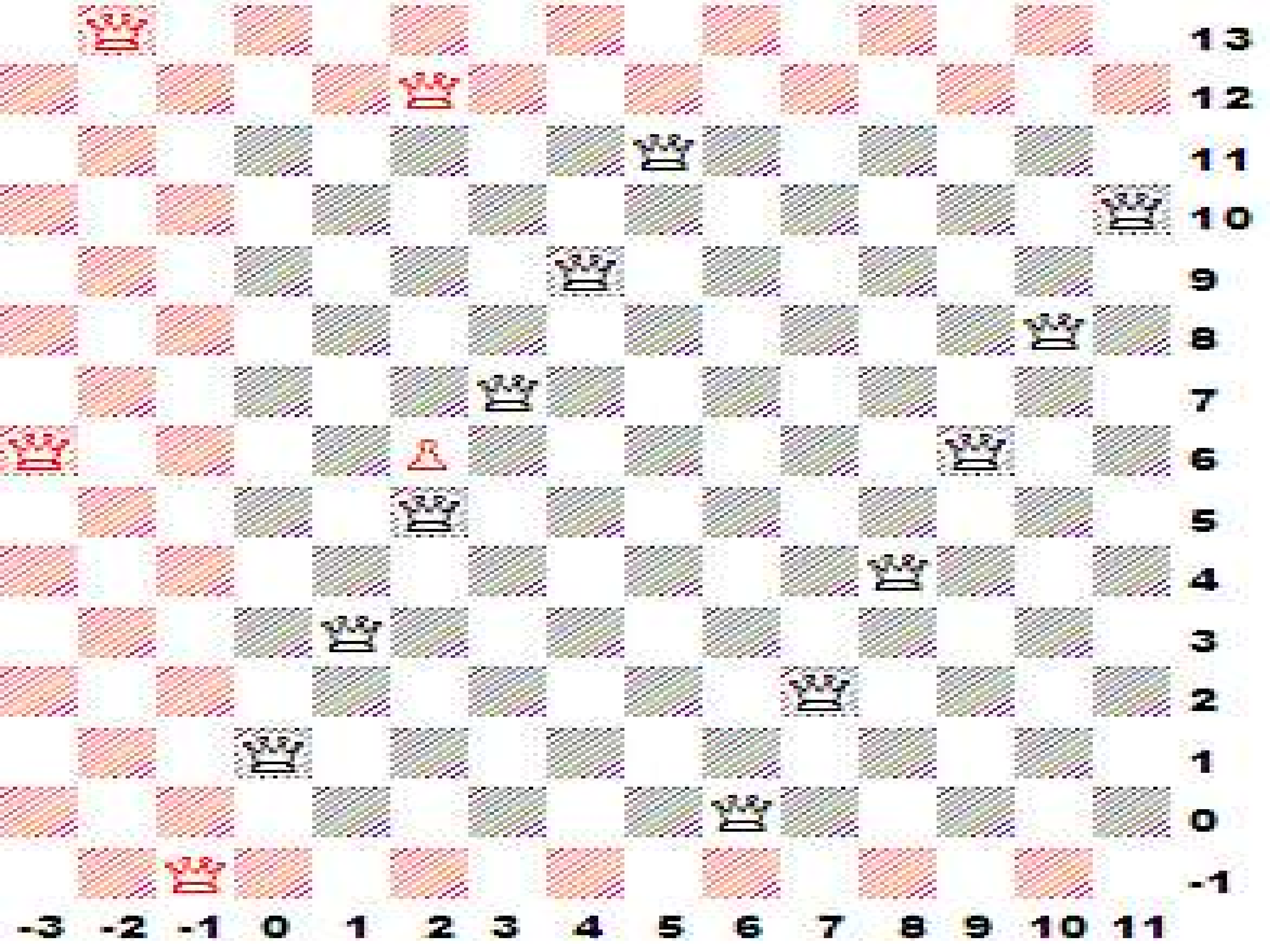
- Add row on top and column to right of Pattern II solution to (N-1)-Queens.
- Pattern II leaves main diagonal open, so add a queen to the upper right corner.



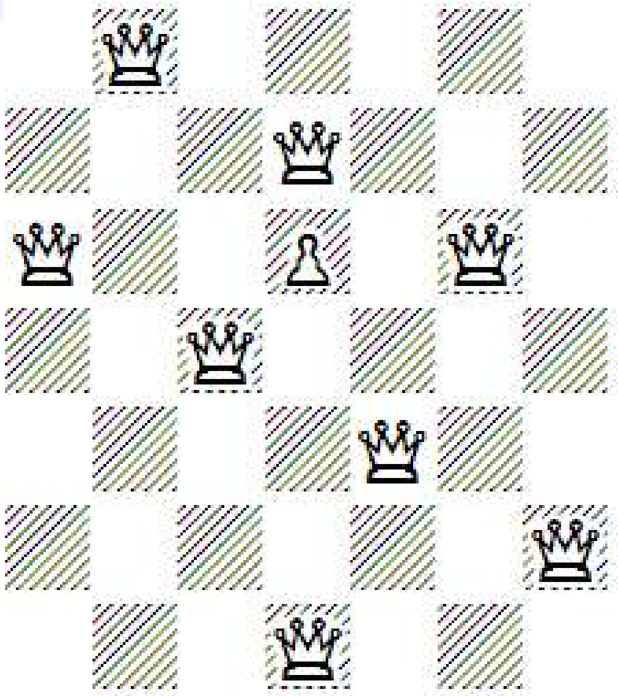
Pattern IV: $N \geq 15$, $N \equiv 3 \pmod 6$

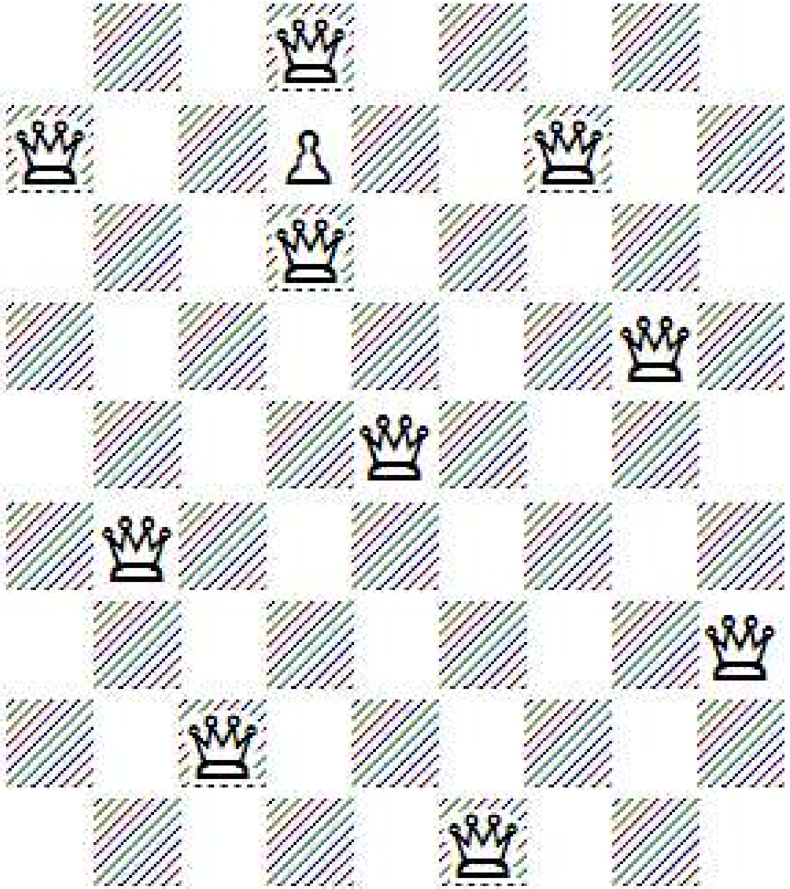
- Take Construction A solution to (N-3)-Queens.
- Add 3 columns to left, 2 rows on top, 1 row on bottom.
- Pawn: ((N-3)/2, 2)
- Extra Queens ((N-3)/2,-3), (N-2, -2), (-1,-1), (N-3,2)




13
12
11
10
9
8
7
6
5
4
3
2
1
0
-1
-3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11

Final Cases: N = 7 and N = 9



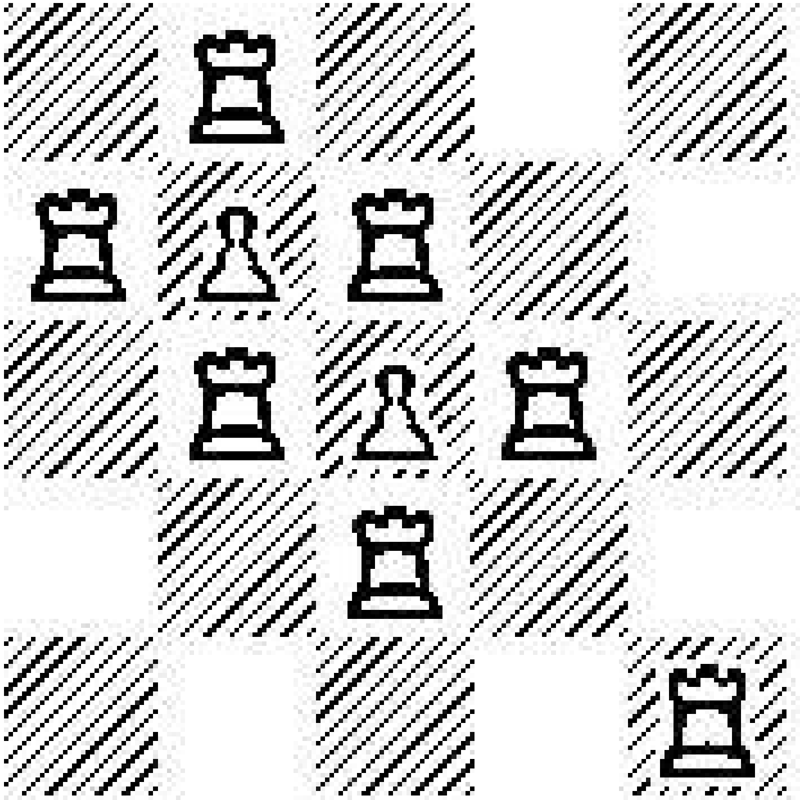


Counting N+1 Queens solutions

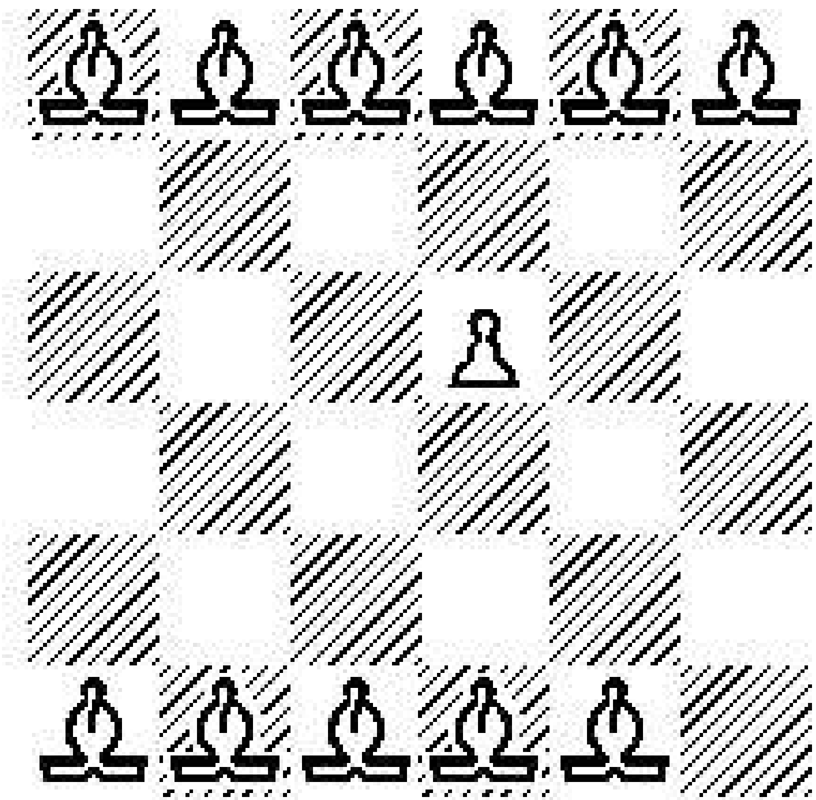
N	Number of solutions	N	Number of solutions
4	0	9	396
5	0	10	2,288
6	16	11	11,152
7	20	12	65,172
8	128		

Other pieces?

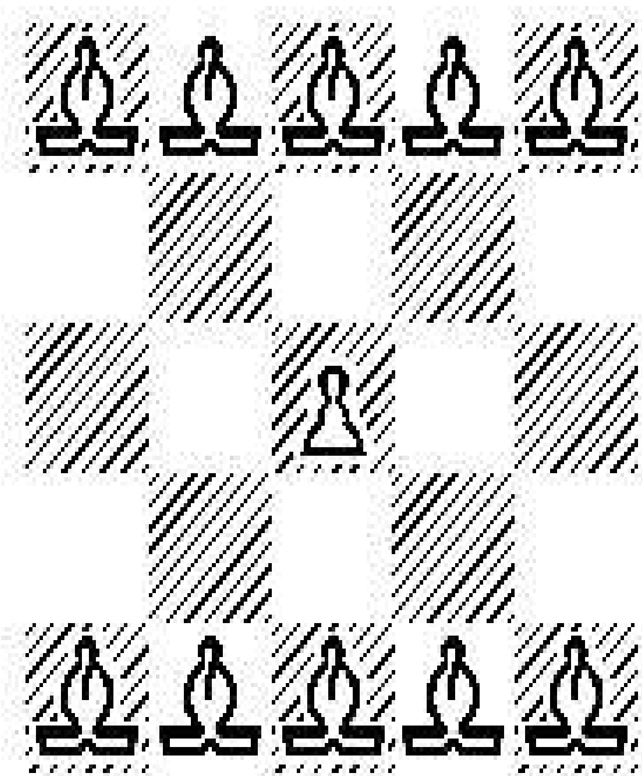
- What if we replace the queens with other pieces, such as rooks or bishops?
- For N ≥ k+2, $s_R(N+k, N) = k$.



Other pieces? (Continued)

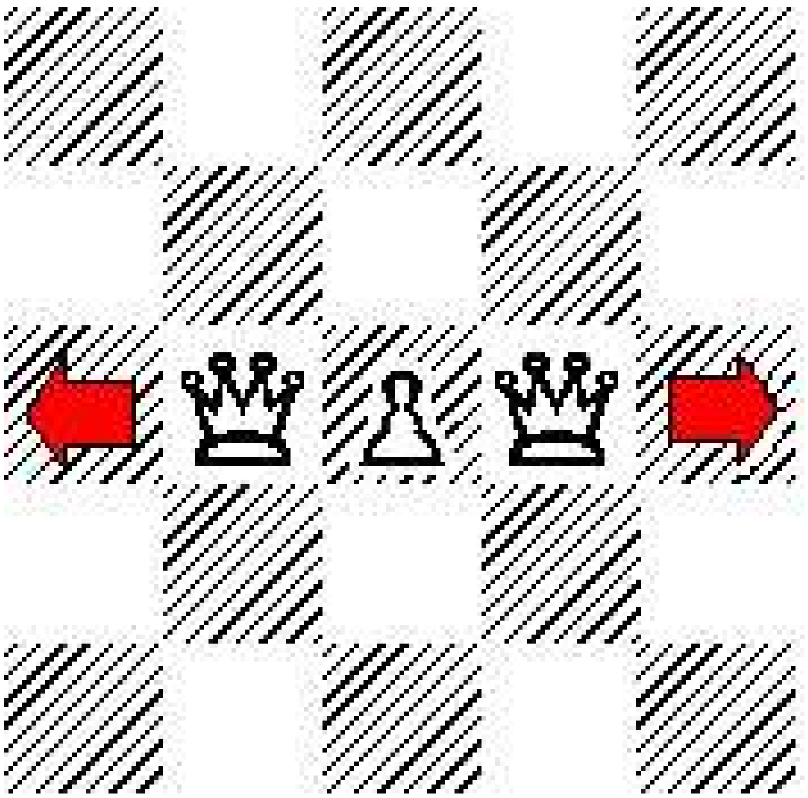


$s_B(2n-1,n) = 1$ for $n \geq 3$



$s_B(2n,n) = 1$ for $n \geq 3$ odd

Other boards?



- What if we look at rectangular or other-shaped boards?
  - For instance, if the board is donut-shaped, one pawn won't be enough.

Conjecture:
$s_Q(N+2,N) = 2$ for $N \geq 7$.

N	Number of solutions to "N+2 Queens"	N	Number of solutions to "N+2 Queens"
5	0	9	160
6	0	10	698
7	4	11	6,771
8	18		

Conjecture: $s_Q(N+3,N) = 3$ for $N \geq 8$.

N	Number of solutions to “N+3 Queens”
7	0
8	8
9	44
10	528

N+k Queens?

- Grand Conjecture: For each k, for large enough N we have $s_Q(N+k,N) = k$.
  - Is this true?
  - How much is “large enough”?
  - What happens when N isn’t “large enough”?

References

- http://www.chessvariants.org/problems.dir/9queens.html
  - Home of Nine Queens Contest.
- http://www.liacs.nl/~kosters/nqueens.html
  - Extensive bibliography on N-queens and related problems.
- Watkins, John J. (2004). Across the Board: The Mathematics of Chess Problems. Princeton: Princeton University Press. ISBN 0-691-11503-6.