NUMBER THEORY TRIVIA: AMICABLE NUMBERS

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Once upon a time there was a sultan who considered himself a great problem solver. The guards told him that one of his prisoners was a mathematician. The very next day he visited the captive and offered him the following challenge: "Either you remain in prison for the rest of your life or you give me a problem to solve and I will let you go free until I find the answer, but as soon as I discover the solution, off comes your head". The clever captive did not hesitate in accepting the deal. He gave the sultan the following problem: "The sum of all the natural divisors of 220, except for the number itself, equals

$$1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$$

and the sum of all the natural divisors of 284, except for the number itself, is equal to

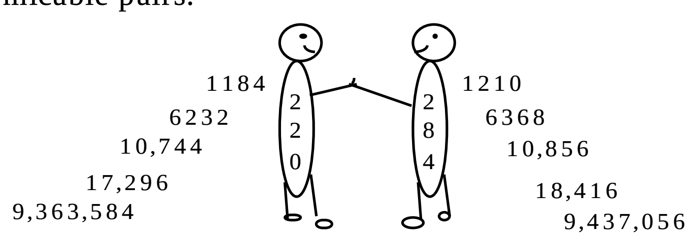
$$1 + 2 + 4 + 71 + 142 = 220.$$

Find another pair of such numbers".

The numbers 220 and 284 are called *amicable*. In general, we say that two positive integers are *amicable* or *friendly* if each of them is equal to the sum of all the natural proper divisors of the other, including 1.

The prisoner of this story went free and eventually died of old age because the sultan was never able to solve the problem given to him. The numbers 220 and 284 form the first pair of amicable numbers. This pair was originally found by Pythagoras. In antiquity, the amicable numbers were thought by mystics to possess magical powers. Astrologers used these numbers for preparing talismans and horoscopes. They believed that amicable numbers had the power to create special ties between individuals.

Eventually the mystical notoriety of amicable numbers caused them to be studied more carefully by number theorists. The amicable pair (17,296 ; 18,416) is often attributed to Fermat but was actually discovered by the Arab, al-Banna in the late thirteenth or fourteenth century. The pair consisting of 9,363,584 and 9,437,056 was found by Descartes. A remarkable number of pairs of amicable numbers, 59 in all, were found by Euler, among them the pair (6,232 ; 6,368) and the pair (10,744 ; 10,856). Long after Euler, a sixteen year old Italian youth found a smaller, overlooked pair of amicable numbers, 1184 and 1210. E. Escott wrote a long paper dedicated to the amicable numbers, offering an inventory of 390 amicable pairs.



P. Poulet brought out another 43 amicable pairs, among them the pairs (122,368 ; 135,536) and (32,685,250 ; 34,538,270). Numerous other mathematicians devoted a considerable part of their time seeking for new pairs of amicable numbers.

There are various methods for discovering pairs of amicable numbers.

For example, if n is a positive integer such that

$$3 \cdot 2^{n} - 1, \quad 3 \cdot 2^{n+1} - 1 \quad \text{and} \quad 3^{2} \cdot 2^{2n+1} - 1 \quad \text{are all prime,}$$

$$\text{then} \quad 2^{n+1} \cdot (3 \cdot 2^{n} - 1) \cdot (3 \cdot 2^{n+1} - 1) \quad \text{and} \quad 2^{n+1} \cdot (3^{2} \cdot 2^{2n+1} - 1) \quad \text{form an amicable pair}$$

Show this and verify that for $n=1$ you get Pythagoras' pair and for $n=3$ you obtain Fermat's pair.

AS YET UNANSWERED QUESTIONS

(1) It is not known whether there exist infinitely many pairs of amicable numbers.

(2) There exist pairs of odd-amicable numbers, like (12,285 ; 14,595) or (67,095 ; 71,145), but it is not known whether there exists any such pair with one of the numbers odd and one even.

(3) It is not known whether there are pairs of *relatively prime* amicable numbers, however, H. J. Kanold proved that **if** there existed a pair of relatively prime amicable numbers, then each of the numbers had to be greater than 10^{23} and they would have together more than 20 prime factors.

The concept of a pair of amicable numbers has been extended to the notion of a *k-tuplet* of amicable numbers by L. E. Dickson and Th. E. Mason: the positive integers $n_1, n_2, \ldots, n_k$, form a k-tuplet of amicable numbers if the sum of all the natural proper divisors, including 1, of any one of them is equal to the sum of the other k-1 numbers. The numbers 1,980 , 2,016 and 2,556 constitute a triplet of amicable numbers. Another such triplet is formed by the numbers 103,340,640 , 123,228,768 and 124,015,008.

(4) It is also unknown whether there exist infinitely many k-tuplets of amicable numbers for some positive integer k greater than 2.

We invite the reader to use computers in searching for pairs, triplets, and more of amicable numbers and to formulate their own conjectures about these numbers. ✍

References

1. O. Ore, *Number Theory and Its History*, Dover Publications, New York, 1988.
2. W. Sierpinski, *Elementary Theory of Numbers*, Warszawa, 1964.