UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. 9 (1998), 29–33

SOME COMBINATORIAL ASPECTS OF DIFFERENTIAL OPERATION COMPOSITION ON THE SPACE $\mathbf{R^n}$

Branko J. Malešević

In this paper we present a recurrent relation for counting meaningful compositions of the higher-order differential operations on the space $\mathbf{R^n}$ (n=3,4,...) and extract the non-trivial compositions of order higher than two.

1. DIFFERENTIAL FORMS AND OPERATIONS ON THE SPACE $\mathbf{R^3}$

It is well known that the first-order differential operations grad, curl and div on the space $\mathbf{R}^3$ can be introduced using the operator of the exterior differentiation d of differential forms [**1**]:

$$\Omega^0(\mathbf{R}^3) \xrightarrow{d} \Omega^1(\mathbf{R}^3) \xrightarrow{d} \Omega^2(\mathbf{R}^3) \xrightarrow{d} \Omega^3(\mathbf{R}^3),$$

where $\Omega^i(\mathbf{R}^3)$ is the space of differential forms of degree $i = 0, 1, 2, 3$ on the space $\mathbf{R}^3$ over the ring of functions $\mathbf{A} = \{f : \mathbf{R}^3 \to \mathbf{R} \,|\, f \in C^\infty(\mathbf{R}^3)\}$. In the consideration, which follows, we give definitions of the first-order differential operations.

Let us notice that one-dimensional spaces $\Omega^0(\mathbf{R}^3)$ and $\Omega^3(\mathbf{R}^3)$ are isomorphic to $\mathbf{A}$ and let $\varphi_0 : \Omega^0(\mathbf{R}^3) \to \mathbf{A}$, $\varphi_3 : \Omega^3(\mathbf{R}^3) \to \mathbf{A}$ be the corresponding isomorphisms. Next, the set of vector functions $\mathbf{B} = \{\boldsymbol{f} = (f_1, f_2, f_3) : \mathbf{R}^3 \to \mathbf{R}^3 \,|\, f_1, f_2, f_3 \in C^\infty(\mathbf{R}^3)\}$, over the ring $\mathbf{A}$, is three-dimensional. It is isomorphic to $\Omega^1(\mathbf{R}^3)$ and $\Omega^2(\mathbf{R}^3)$. Let $\varphi_1 : \Omega^1(\mathbf{R}^3) \to \mathbf{B}$, $\varphi_2 : \Omega^2(\mathbf{R}^3) \to \mathbf{B}$ be the corresponding isomorphisms. In that case, the compositions $\varphi_0^{-1} \circ \varphi_3 : \Omega^3(\mathbf{R}^3) \to \Omega^0(\mathbf{R}^3)$ and $\varphi_1^{-1} \circ \varphi_2 : \Omega^2(\mathbf{R}^3) \to \Omega^1(\mathbf{R}^3)$ are isomorphisms of the corresponding spaces of differential forms. The first-order differential operations are defined via the operator of the exterior differentiation d of differential forms in the following form:

$$\nabla_1 = \varphi_1 \circ d \circ \varphi_0^{-1} : \mathbf{A} \to \mathbf{B}, \;\; \nabla_2 = \varphi_2 \circ d \circ \varphi_1^{-1} : \mathbf{B} \to \mathbf{B}, \;\; \nabla_3 = \varphi_3 \circ d \circ \varphi_2^{-1} : \mathbf{B} \to \mathbf{A}.$$

Therefore we obtain explicit expressions for the first order differential operations ∇_1, ∇_2, ∇_3 on the space $\mathbf{R}^3$ in the following form:

(1) $\quad \operatorname{grad} f = \nabla_1 f = \dfrac{\partial f}{\partial x_1} \boldsymbol{e_1} + \dfrac{\partial f}{\partial x_2} \boldsymbol{e_2} + \dfrac{\partial f}{\partial x_3} \boldsymbol{e_3} : \mathbf{A} \to \mathbf{B},$

(2) $\quad \operatorname{curl} \boldsymbol{f} = \nabla_2 \boldsymbol{f} = \Big(\dfrac{\partial f_3}{\partial x_2} - \dfrac{\partial f_2}{\partial x_3}\Big) \boldsymbol{e_1} + \Big(\dfrac{\partial f_1}{\partial x_3} - \dfrac{\partial f_3}{\partial x_1}\Big) \boldsymbol{e_2} + \Big(\dfrac{\partial f_2}{\partial x_1} - \dfrac{\partial f_1}{\partial x_2}\Big) \boldsymbol{e_3} : \mathbf{B} \to \mathbf{B},$

(3) $\quad \operatorname{div} \boldsymbol{f} = \nabla_3 \boldsymbol{f} = \dfrac{\partial f_1}{\partial x_1} + \dfrac{\partial f_2}{\partial x_2} + \dfrac{\partial f_3}{\partial x_3} : \mathbf{B} \to \mathbf{A}.$

1991 Mathematics Subject Classification: 26B12, 58A10

Let us count meaningful compositions of differential operations $\nabla_1, \nabla_2, \nabla_3$. Consider the set of functions $\Theta = \{\nabla_1, \nabla_2, \nabla_3\}$. Let us define a binary relation ρ *"to be in composition"* with $\nabla_i \rho \nabla_j = \top$ iff the composition $\nabla_j \circ \nabla_i$ is meaningful $(\nabla_i, \nabla_j \in \Theta)$. The CAYLEY's table of this relation reads:

(4)

ρ	∇_1	∇_2	∇_3
∇_1	$\perp$	$\top$	$\top$
∇_2	$\perp$	$\top$	$\top$
∇_3	$\top$	$\perp$	$\perp$

.

We form the graph of relation ρ as follows. If $\nabla_i \rho \nabla_j = \top$ then we put the node ∇_j under the node ∇_i. Let us mark ∇_0 as nowhere-defined function ϑ, with domain and range being the empty set [**2**]. We shall consider $\nabla_0 \rho \nabla_i = \top$ $(i = 1, 2, 3)$. For the set of functions $\Theta \cup \{\nabla_0\}$ our graph is the tree with the root in the node ∇_0.

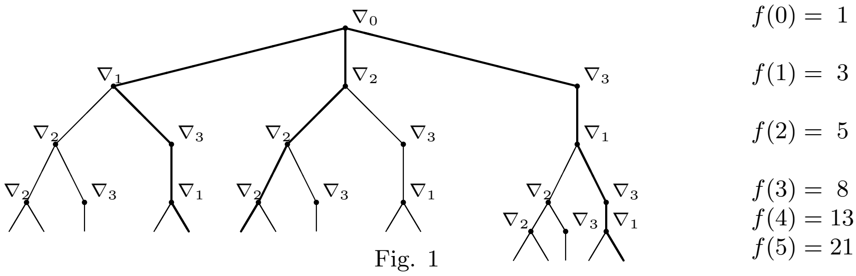


Fig. 1

Let $f_i(k)$ be a number of meaningful compositions of the k^{th}-order beginning with ∇_i. Let $f(k)$ be a number of meaningful composition of the k^{th}-order of operations over Θ. Then $f(k) = f_1(k) + f_2(k) + f_3(k)$. Based on partial self similarity of the tree (Fig. 1), which is formed according to CAYLEY's table (4), we get equalities:

$$f_1(k) = f_2(k-1) + f_3(k-1) \quad \wedge \quad f_2(k) = f_2(k-1) + f_3(k-1) \quad \wedge \quad f_3(k) = f_1(k-1).$$

Now, a recurrent relation for $f(k)$ can be derived as follows:

$$\begin{aligned} f(k) &= f_1(k) + f_2(k) + f_3(k) \\ &= \big(f_1(k-1) + f_2(k-1) + f_3(k-1)\big) + \big(f_3(k-1) + f_2(k-1)\big) \\ &= f(k-1) + \big(f_1(k-2) + f_2(k-2) + f_3(k-2)\big) = f(k-1) + f(k-2). \end{aligned}$$

Based on the initial values: $f(1) = 3$, $f(2) = 5$, $f(3) = 8$ we conclude that $f(k) = F_{k+3}$, where is FIBONACCI's number of order $k + 3$.

Let us note that $\nabla_2 \circ \nabla_1 = 0$ and $\nabla_3 \circ \nabla_2 = 0$, because $d^2 = 0$. On the other hand, the compositions $\nabla_1 \circ \nabla_3$, $\nabla_2 \circ \nabla_2$ and $\nabla_3 \circ \nabla_1$ are not annihilated, because of $\varphi_0^{-1} \circ \varphi_3 \neq i$ and $\varphi_1^{-1} \circ \varphi_2 \neq i$. Thus, as in the paper [**2**], we conclude that the non-trivial compositions are of the following form:

$$\begin{array}{c} (\nabla_1 \circ)\nabla_3 \circ \cdots \circ \nabla_1 \circ \nabla_3 \circ \nabla_1, \\ \nabla_2 \circ \nabla_2 \circ \cdots \circ \nabla_2 \circ \nabla_2 \circ \nabla_2, \\ (\nabla_3 \circ)\nabla_1 \circ \cdots \circ \nabla_3 \circ \nabla_1 \circ \nabla_3. \end{array} \tag{5}$$

As non-trivial compositions we consider those which are not identical to the zero function. Terms in parentheses are included in for an odd number of terms and are left out otherwise.

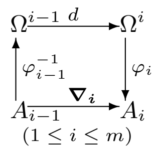
2. DIFFERENTIAL FORMS AND OPERATIONS ON THE SPACE $\mathbf{R^n}$

Let us present a recurrent relation for counting meaningful compositions of the higher-order differential operations on the space $\mathbf{R}^n$ $(n = 3, 4, \ldots)$ and extract the non-trivial compositions of order higher than two. Let us form the following sets of functions:

$$\mathbf{A}_i = \{\boldsymbol{f} : \mathbf{R}^n \to \mathbf{R}^{\binom{n}{i}} | f_1, \ldots, f_{\binom{n}{i}} \in C^\infty(\mathbf{R}^n)\}$$

for $i = 0, 1, \ldots, m$ where $m = [n/2]$. Let $\Omega^i(\mathbf{R}^n)$ be a set of differential forms of degree $i = 0, 1, \ldots, n$ on the space $\mathbf{R}^n$. Let us notice that $\Omega^i(\mathbf{R}^n)$ and $\Omega^{n-i}(\mathbf{R}^n)$, over ring $\mathbf{A}_0$, are spaces of the same dimension $\binom{n}{i}$, for $i = 0, 1, \ldots, m$. They can be identified with $\mathbf{A}_i$, using the corresponding isomorphisms:

$$\varphi_i : \Omega^i(\mathbf{R}^n) \to \mathbf{A}_i \;\; (0 \le i \le m) \quad \text{and} \quad \varphi_{n-i} : \Omega^{n-i}(\mathbf{R}^n) \to \mathbf{A}_i \;\; (0 \le i < n - m).$$



We define the first-order differential operations on the space $\mathbf{R}^n$ via the operator of the exterior differentiation d as follows:

$$\nabla_i = \varphi_i \circ d \circ \varphi_{i-1}^{-1} \;\; (1 \le i \le n).$$

Therefore, we obtain the first order differential operations on the space $\mathbf{R}^n$, depending on pairity of dimension n, in the following form:

$n = 2m :$

$$\begin{array}{l} \nabla_1 : \mathbf{A}_0 \to \mathbf{A}_1 \\ \nabla_2 : \mathbf{A}_1 \to \mathbf{A}_2 \\ \vdots \\ \nabla_i : \mathbf{A}_i \to \mathbf{A}_{i+1} \\ \vdots \\ \nabla_m : \mathbf{A}_{m-1} \to \mathbf{A}_m \\ \nabla_{m+1} : \mathbf{A}_m \to \mathbf{A}_{m-1} \\ \vdots \\ \nabla_{n-j} : \mathbf{A}_{j+1} \to \mathbf{A}_j \\ \vdots \\ \nabla_{n-1} : \mathbf{A}_2 \to \mathbf{A}_1 \\ \nabla_n : \mathbf{A}_1 \to \mathbf{A}_0, \end{array}$$

$n = 2m + 1 :$

$$\begin{array}{l} \nabla_1 : \mathbf{A}_0 \to \mathbf{A}_1 \\ \nabla_2 : \mathbf{A}_1 \to \mathbf{A}_2 \\ \vdots \\ \nabla_i : \mathbf{A}_i \to \mathbf{A}_{i+1} \\ \vdots \\ \nabla_m : \mathbf{A}_{m-1} \to \mathbf{A}_m \\ \nabla_{m+1} : \mathbf{A}_m \to \mathbf{A}_m \\ \nabla_{m+2} : \mathbf{A}_m \to \mathbf{A}_{m-1} \\ \vdots \\ \nabla_{n-j} : \mathbf{A}_{j+1} \to \mathbf{A}_j \\ \vdots \\ \nabla_{n-1} : \mathbf{A}_2 \to \mathbf{A}_1 \\ \nabla_n : \mathbf{A}_1 \to \mathbf{A}_0. \end{array}$$

Consider the set of functions $\Theta = \{\nabla_1, \nabla_2, \ldots, \nabla_n\}$. Let us define a binary relation ρ *"to be in composition"* with $\nabla_i \rho \nabla_j = \top$ iff the composition $\nabla_j \circ \nabla_i$ is meaningful $(\nabla_i, \nabla_j \in \Theta)$. It is not difficult to check that CAYLEY's table of this relation is determined with:

$$\nabla_i \rho \nabla_j = \left\{ \begin{array}{lcl} \top & : & (j = i + 1) \vee (i + j = n + 1), \\ \bot & : & (j \ne i + 1) \wedge (i + j \ne n + 1). \end{array} \right. \tag{6}$$

Let us form an adjacency matrix $\mathtt{A} = [a_{ij}] \in \{0, 1\}^{n \times n}$ of the graph, determined by relation ρ. Let $f_i(k)$ be a number of meaningful compositions of the k^{th}-order beginning with ∇_i (notice that $f_i(1)=1$ for $i=1,\ldots,n$). Let $f(k)$ be a number of meaningful composition of the $k^{\rm th}$-order of operations over Θ. Then $f(k) = f_1(k)+\ldots+f_n(k)$. Notice that the following is true:

$$f_i(k) = \sum_{j=1}^{n} a_{ij} \cdot f_j(k-1), \tag{7}$$

for $i = 1,\ldots,n$. Based on (7) we form the system of recurrent equations:

$$\begin{bmatrix} f_1(k) \\ \vdots \\ f_n(k) \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} f_1(k-1) \\ \vdots \\ f_n(k-1) \end{bmatrix}. \tag{8}$$

If $v_n = [\,1 \;\; \cdots \;\; 1\,]_{1\times n}$ then:

$$f(k) = v_n \cdot \begin{bmatrix} f_1(k) \\ \vdots \\ f_n(k) \end{bmatrix}. \tag{9}$$

So, the expression:

$$f(k) = v_n \cdot \mathtt{A}^{k-1} \cdot v_n^T. \tag{10}$$

follows from (8) and (9). Reducing the system of the recurrent equations (8), for any of the functions $f_i(k)$ we have:

$$\alpha_0 f_i(k) + \alpha_1 f_i(k-1) + \cdots + \alpha_n f_i(k-n) = 0 \qquad (k > n), \tag{11}$$

where $\alpha_0,\ldots,\alpha_n$ are coefficients of the characteristic polynomial $P_n(\lambda) = |\mathtt{A}-\lambda\mathtt{I}| = \alpha_0\lambda^n+\ldots+\alpha_n$. Thus, we conclude that the function $f(k) = \sum\limits_{i=1}^{n} f_i(k)$ also satisfies:

$$\alpha_0 f(k) + \alpha_1 f(k-1) + \cdots + \alpha_n f(k-n) = 0 \qquad (k > n). \tag{12}$$

Hence, the following theorem holds.

Theorem 1. *The number of meaningful differential operations, on the space* $\mathbf{R}^n$ $(n = 3, 4, \ldots)$*, of the order higher than two, is determined by the formula* (10)*, i.e. by the recurrent formula* (12)*.*

In n-dimensional space $\mathbf{R}^n$, for dimensions $n = 3, 4, 5, \ldots, 10$, using the previous theorem we form a table of the corresponding recurrent formula:

Dimension:	Recurrent relations for the number of meaningful compositions:
$n = 3$	$f(i+2) = f(i+1) + f(i)$
$n = 4$	$f(i+2) = 2f(i)$
$n = 5$	$f(i+3) = f(i+2) + 2f(i+1) - f(i)$
$n = 6$	$f(i+4) = 3f(i+2) - f(i)$
$n = 7$	$f(i+5) = f(i+3) + 3f(i+2) - 2f(i+1) - f(i)$
$n = 8$	$f(i+4) = 4f(i+2) - 3f(i)$
$n = 9$	$f(i+5) = f(i+4) + 4f(i+3) - 3f(i+2) - 3f(i+1) + f(i)$
$n = 10$	$f(i+6) = 5f(i+4) - 6f(i+2) + f(i)$

Let us determine non-trivial higher-order meaningful compositions on the space $\mathbf{R}^n$. For isomorphisms φ_k we have:

$$\varphi_k^{-1} \circ \varphi_{n-k} \neq i, \tag{13}$$

for $k = 1, 2, \ldots, n$ and $2k \neq n$. Then, based on (6) and (13), all second-order compositions are given by the formula:

$$\nabla_j \circ \nabla_k = \begin{cases} 0 & : \ j = k+1, \\ g_{j,k} & : \ (k+j = n+1) \wedge (2k \neq n), \\ \vartheta & : \ (j \neq k+1) \wedge (k+j \neq n+1); \end{cases} \tag{14}$$

where 0 is a trivial composition, $g_{j,k}$ is a non-trivial second-order composition and ϑ is a nowhere-defined function for $j, k = 1, \ldots, n$. Notice that in $g_{j,k} = \nabla_j \circ \nabla_k = \varphi_{n+1-k} \circ d \circ \varphi_{n-k}^{-1} \circ \varphi_k \circ d \circ \varphi_{k-1}^{-1}$ $(j{=}n{+}1{-}k \ \wedge \ 2k{\neq}n)$ and switching the terms is impossible, because in that way we get nowhere-defined function ϑ. Hence, we conclude that the following theorem holds.

Theorem 2. *All meaningful non-trivial differential operations on the space* $\mathbf{R}^n$ $(n = 3, 4, \ldots)$, *of order higher than, two are given in the form of the following compositions*:

$$\begin{array}{l} (\nabla_k) \circ \nabla_j \circ \nabla_k \circ \cdots \circ \nabla_j \circ \nabla_k, \\ (\nabla_j) \circ \nabla_k \circ \nabla_j \circ \cdots \circ \nabla_k \circ \nabla_j, \end{array} \tag{15}$$

with to the condition $k + j = n + 1$ *and* $2k,\ 2j \neq n$ *for* $k,\ j = 1, 2, \ldots, n$. *Terms in parentheses are included in for an odd number of terms and are left out otherwise.*

Acknowledgment. I wish to express my gratitude to Professors M. Merkle and M. Prvanović who examined the first version of the paper and gave me their suggestions and some very useful remarks.

REFERENCES

1. R. Bott, L. W. Tu: *Differential forms in algebraic topology*, Springer, New York 1982.
2. B. J. Malešević: *A note on higher-order differential operations*, Univ. Beograd, Publ. Elektrotehn. Fak.,Ser. Mat. **7** (1996), 105-109.

University of Belgrade,
Faculty of Electrical Engineering,
P.O.Box 35-54, 11120 Belgrade,
Yugoslavia
malesevic@kiklop.etf.bg.ac.yu

(Received September 8, 1997)
(Revised October 30, 1998)