

# Counting Divisors

by Harold Reiter

The purpose of this note is to find and study a method for determining and counting all the positive integer divisors of a given positive integer. Let  $N$  be a given positive integer. We say  $d$  is a divisor of  $N$  and write  $d|N$  if  $N/d$  is a positive integer. Thus, for example,  $3|15$ . Denote by  $D_N$  the set of all positive integer divisors of  $N$ . For example  $D_6 = \{1, 2, 3, 6\}$ . We discuss a method for finding  $D_N$  and we study its ‘geometric’ structure. There are four parts to this note. In the first part, we count the divisors of a given positive integer  $N$  based on its prime factorization. In the second part, we construct all the divisors, and in the third part we discuss the ‘geometry’ of the divisors. In part four, we discuss applications to contest problems.

1. Counting the divisors of  $N$ . First consider the example  $N = 72$ . To find the number of divisors of 72, note that the prime factorization of 72 is given by  $72 = 2^3 3^2$ . Each divisor  $d$  of 72 must be of the form  $d = 2^i 3^j$  where  $0 \leq i \leq 3$  and  $0 \leq j \leq 2$ . Otherwise,  $2^3 3^2/d$  could not be an integer, by the Fundamental Theorem of Arithmetic (the theorem that guarantees the unique factorization into primes of each positive integer). So there are 4 choices for the exponent  $i$  and 3 choices for  $j$ . Hence there are  $4 \cdot 3 = 12$  divisors of 72. Reasoning similarly, we can see that for any integer

$$N = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k},$$

the number of divisors is

$$\prod_{i=1}^k (e_i + 1) = (e_1 + 1)(e_2 + 1) \cdots (e_k + 1).$$

2. Constructing the divisors of  $N$ . In part 1 we found the number of members of  $D_N$  for any positive integer  $N$ . In this part, we seek the list of divisors themselves. Again we start with the prime factorization of  $N$ . Suppose  $N = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ . If  $k = 2$  the listing is straightforward. In this case, we build a table by first listing the powers of  $p_1$  across the top of the table and the powers of  $p_2$  down the side, thus obtaining an  $(e_1 + 1) \cdot (e_2 + 1)$  matrix of divisors. Again we use  $N = 72$  as an

example.

	$2^0$	$2^1$	$2^2$	$2^3$
$3^0$	1	2	4	8
$3^1$	3	6	12	24
$3^2$	9	18	36	72

What do we do when the number of prime factors of  $N$  is more than 2? If  $k = 3$  we can construct  $e_3 + 1$  matrices of divisors, one for each power of  $p_3$ . For example, if  $N = 360 = 2^3 3^2 5$ , we construct one  $4 \times 3$  matrix for  $5^0$  and one for  $5^1$ . The result is a  $4 \times 3 \times 2$  matrix of divisors. The two  $4 \times 3$  matrices are shown below.

$5^0$	$2^0$	$2^1$	$2^2$	$2^3$	and	$5^1$	$2^0$	$2^1$	$2^2$	$2^3$
$3^0$	1	2	4	8		$3^0$	5	10	20	40
$3^1$	3	6	12	24		$3^1$	15	30	60	120
$3^2$	9	18	36	72		$3^2$	45	90	180	360

For larger values of  $k$  we can create multiple copies of the matrix associated with the number  $N = p_1^{e_1} p_2^{e_2} \cdots p_{k-1}^{e_{k-1}}$ .

3. The geometry of  $D_N$ . To do this, we first explore the relation ‘divides’. Recall that  $a|b$  means that  $a$  and  $b$  are positive integers for which  $b/a$  is an integer. The relation ‘|’ has several important properties, three of which are crucial to our discussion.

1. Reflexive. For any positive integer  $a$ ,  $a|a$ .
2. Antisymmetry. For any pair of positive integers  $a, b$ , if  $a|b$  and  $b|a$ , then  $a = b$ .
3. Transitivity. For any three positive integers,  $a, b, c$ , if  $a|b$  and  $b|c$  then  $a|c$ .

These properties are easy to prove. The first says that each integer is a divisor of itself; that is,  $a/a$  is an integer. The second says that no two different integers can be divisors of one another. This is true since a larger integer can never be a divisor of a smaller one. The third property follows from the arithmetic  $b/a \cdot c/b = c/a$  together with the property that the product of two positive integers is a positive integer. Any set  $S$  with a relation  $\preceq$  defined on it that satisfies all three of the properties above is called a *partially ordered set*, or a *poset*. A branch of discrete mathematics studies the properties of posets,  $(S, \preceq)$ .

Each finite poset has a unique directed graph representation. This pictorial representation is what we mean by the geometry of  $D_N$ . To construct the directed graph of a poset  $(S, \preceq)$ , draw a vertex (dot) for each member of  $S$ . Then connect two vertices  $a$  and  $b$  with a directed edge (an arrow) if  $a \preceq b$ . Of course, in our case

$D_N$  this means we connect  $a$  to  $b$  if  $a|b$ . The case  $D_6$  is easy to draw:

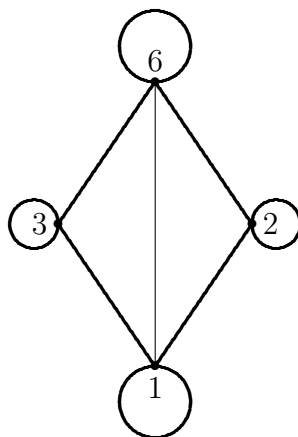


Fig. 1 The digraph of  $D_6$

The circles at each of the four vertices are called loops. They are there because each number is a divisor of itself. The reader should imagine that all the non-loop edges are upwardly directed. The directed edge from 1 to 6 indicates that  $1|6$ . But since we know that the vertices of  $D_6$  satisfied all three properties required of a poset, we can leave off both (a) the loops, which are implied by the reflexive property, and (b) the edges that are implied by the transitivity condition. The ‘slimmed down’ representation, called the Hasse diagram, is much easier to understand. It captures all the essential information without cluttering up the scene. The Hasse diagram of  $D_6$  is shown below.

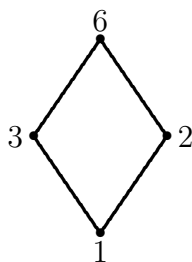


Fig. 2, the Hasse diagram of  $D_6$

In general, the Hasse diagram for  $D_N$  has only those non-loop edges which are not implied by transitivity, that is, those edges from  $a$  to  $b$  for which  $b$  is a prime

number multiple of  $a$ . The Hasse diagrams of  $D_{72}$ ,  $D_{30}$ , and  $D_{60}$  are shown below

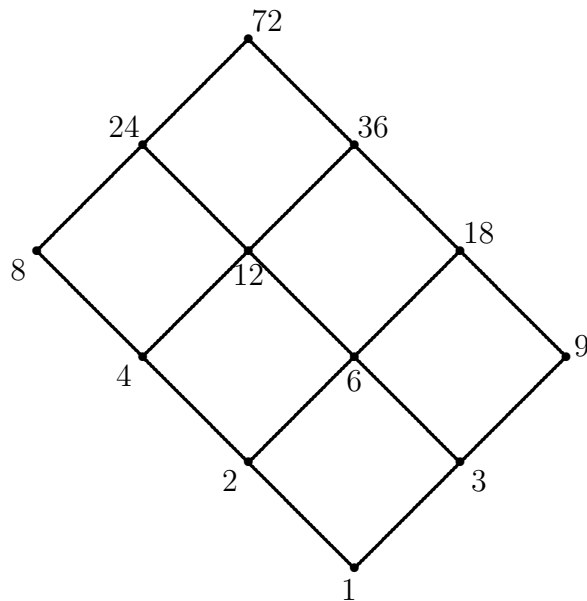


Fig. 3 The Hasse diagram of  $D_{72}$

Notice that each prime divisor of 30 can be considered a direction, 2 is left ( $\nwarrow$ ), 3 is up ( $\uparrow$ ) and 5 is right ( $\nearrow$ ). Also note that if  $a, b$  are divisors of 30 then  $a|b$  if and only if there is a sequence of upwardly directed edges starting at  $a$  and ending at  $b$ . For example  $1|30$  and  $(1, 3), (3, 15), (15, 30)$  are all directed edges in the digraph of  $D_{30}$ . On the other hand, we say 2 and 15 are incomparable because neither divides the other, and indeed there is no upwardly directed sequence of edges from either one to the other.

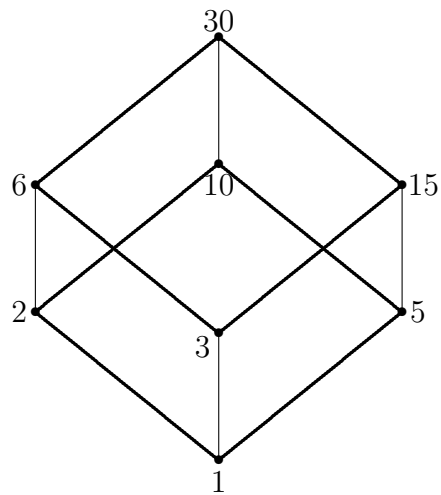


Fig. 4 The Hasse diagram of  $D_{30}$

What would the divisors of 60 look like if we build such a diagram for them?  
 Try to construct it before you look at  $D_{60}$ .

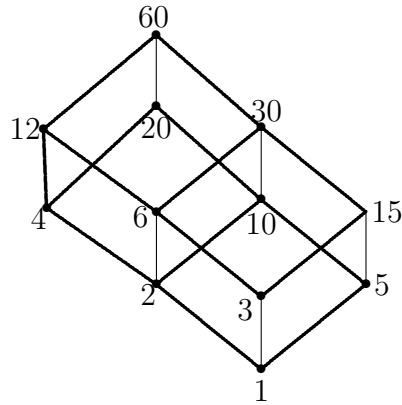


Fig. 5 The Hasse diagram of  $D_{60}$

Consider the same (lattice/Hasse) diagram for the divisors of 210. We can draw this in several ways. The first one (Fig. 6a) places each divisor of 210 at a level determined by its number of prime divisors. The second one (Fig. 6b) emphasizes the 'degrees of freedom'. These two diagrams are representations of a four dimensional cube, not surprising since the Hasse diagram for  $D_{30}$  is a three-dimensional cube. A mathematical way to say the two digraphs are the same is to say they are *isomorphic*. This means that they have the same number of vertices and the same number of edges and that a correspondence between the vertices also serves as a correspondence between the edges. Note that the digraphs in 6a and 6b have the required number of vertices (16) and the required number of edges (32). Can you find an  $N$  such that the Hasse diagram of  $D_N$  is a representation of a five-dimensional cube. Such a digraph must have  $2^5 = 32$  vertices, and  $2 \cdot 32 + 16 = 80$  edges.

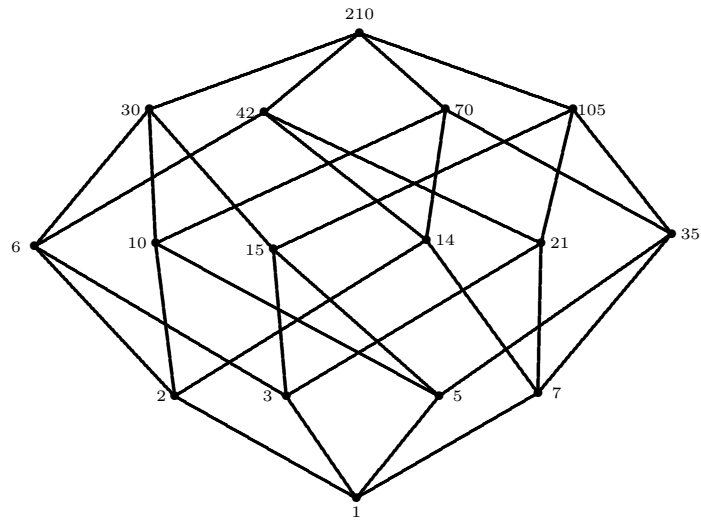


Fig 6a The Hasse diagram of  $D_{210}$

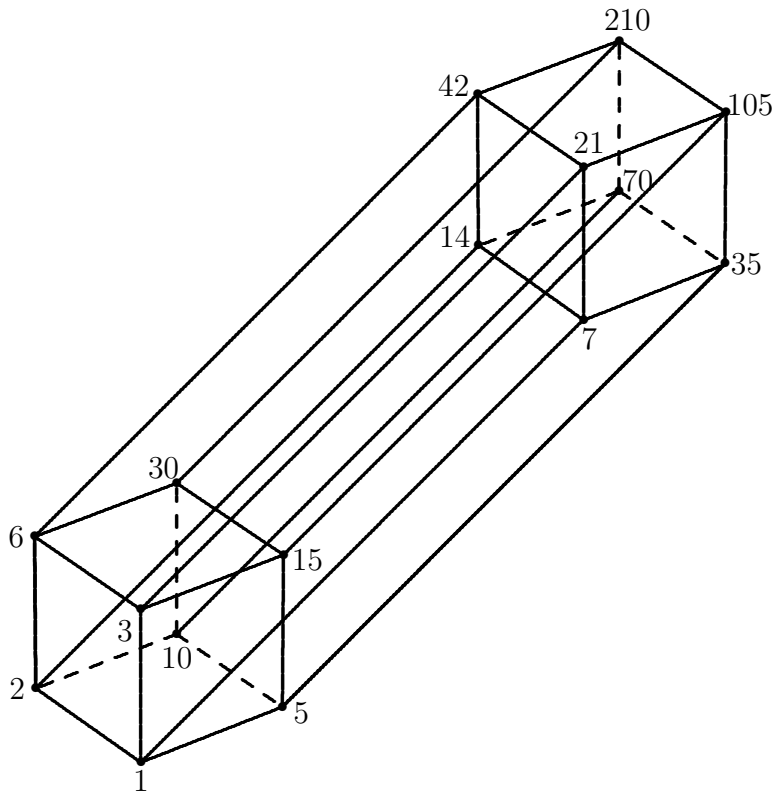


Fig 6b The Hasse diagram of  $D_{210}$

How can the geometry help us do number theory? One way to use the geometry is in the calculation of the  $GCF$  and  $LCM$  of two members of  $D_N$ . Note that each element  $d$  of  $D_N$  generates a downward ‘cone’ of divisors and an upward cone of multiples. We can denote these cones by  $F(d)$  and  $M(d)$  respectively. Then the  $GCF(d, e) = \max\{F(d) \cap F(e)\}$  and  $LCM(d, e) = \min\{M(d) \cap M(e)\}$ .

4. Problems from competitions. The following problems come from MathCounts and the American Mathematics Competitions.

1. How many positive integers less than 50 have an odd number of positive integer divisors?

**Solution:** A number has an odd number of divisors if and only if it is a perfect square. Therefore, there are exactly seven such numbers, 1, 4, 9, 16, 25, 36, and 49.

2. (The Locker Problem) A high school with 1000 lockers and 1000 students tries the following experiment. All lockers are initially closed. Then student number 1 opens all the lockers. Then student number 2 closes the even numbered



lockers. Then student number 3 changes the status of all the lockers numbered with multiples of 3. This continues with each student changing the status of all the lockers which are numbered by multiples of his or her number. Which lockers are closed after all the 1000 students have done their jobs?

**Solution:** Build a table for the first 20 lockers, and notice that the lockers that end up open are those numbered 1, 4, 9, and 16. This looks like the squares. Think about what it takes to make a locker end up open. It takes an odd number of changes, which means an odd number of divisors. We know how to count the number of divisors of a number  $N = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$ .  $N$  has  $D_N = (e_1 + 1)(e_2 + 1) \cdots (e_n + 1)$  divisors. So the issue is how can this last number be odd. It's odd if each factor  $e_i + 1$  is odd, which means all the  $e_i$  have to be even. This is true precisely when  $N$  is a perfect square. So the lockers that are closed at the end are just the non-square numbered lockers.

3. If  $N$  is the cube of a positive integer, which of the following could be the number of positive integer divisors of  $N$ ?

(A) 200    (B) 201    (C) 202    (D) 203    (E) 204

**Solution:** C. The number  $N$  must be of the form  $N = p^{3e} q^{3f} \cdots$ . So the number of divisors must be of the form  $D = (3e+1)(3f+1) \cdots$ . Such a number must be one greater than a multiple of 3. The number  $N = (2^{67})^3 = 2^{201}$  has exactly 202 divisors.

4. Let

$$N = 69^5 + 5 \cdot 69^4 + 10 \cdot 69^3 + 10 \cdot 69^2 + 5 \cdot 69 + 1.$$

How many positive integers are factors of  $N$ ?

(A) 3    (B) 5    (C) 69    (D) 125    (E) 216

**Solution:** The expression given is the binomial expansion of  $N = (69 + 1)^5 = 70^5 = 2^5 \cdot 5^5 \cdot 7^5$ . So  $N$  has  $(5 + 1)(5 + 1)(5 + 1) = 216$  divisors.

5. How many of the positive integer divisors of  $N = 2^3 \cdot 3^3 \cdot 5^3 \cdot 7^3 \cdot 11^3$  have exactly 12 positive integer divisors?

**Solution:** Let  $n$  be a divisor of  $N$  with 12 divisors. Then  $n = p^3 q^2$  where  $p$  and  $q$  belong to  $\{2, 3, 5, 7, 11\}$  or  $n = p^2 q r$  where  $p, q, r \in \{2, 3, 5, 7, 11\}$ . There are  $5 \cdot 4 = 20$  of the former and  $5 \cdot \binom{4}{2} = 30$  of the later. So the answer is  $20 + 30 = 50$ .

6. How many ordered pairs  $(x, y)$  of positive integers satisfy

$$xy + x + y = 199?$$

**Solution:** Add one to both sides and factor to get  $(x+1)(y+1) = 200 = 2^3 5^2$ , which has  $4 \cdot 3$  factors, one of which is 1. Neither  $x+1$  nor  $y+1$  can be 1, so there are 10 good choices for  $x+1$ .

7. How many positive integers less than 400 have exactly 6 positive integer divisors?

**Solution:** Having exactly 6 divisors means that the number  $N$  is of the form  $p^5$  or  $p^2q$  where  $p$  and  $q$  are different prime numbers. Only  $2^5 = 32$  and  $3^5 = 243$  are of the first type. If  $p = 2$ , then  $q$  could be any prime in the range  $3, 5, \dots, 97$  of which there are 24 primes. If  $p = 3$ , then  $q$  could be 2 or any prime in the range  $5, \dots, 43$  and there are 13 of these. If  $p = 5$ , then  $q$  could be 2 or 3 or any prime in the range  $7, \dots, 13$ . If  $p = 7$  then  $q = 2, 3$ , or 5. There are two values of  $q$  for  $p = 11$  and one value of  $q$  for  $p = 13$ . Tally these to get  $2 + 24 + 13 + 5 + 3 + 2 + 1 = 50$ .

8. The product of four distinct positive integers,  $a, b, c$ , and  $d$  is  $8!$ . The numbers also satisfy

$$ab + a + b = 391 \tag{1}$$

$$bc + b + c = 199. \tag{2}$$

What is  $d$ ?

**Solution:** Add 1 to both sides and factor to get

$$(a+1)(b+1) = 392 = 2^3 7^2 \tag{3}$$

$$(b+1)(c+1) = 200 = 2^3 5^2. \tag{4}$$

Build a factor table for 392.

	$2^0$	$2^1$	$2^2$	$2^3$
$7^0$	1	2	4	8
$7^1$	7	14	28	56
$7^2$	49	98	196	392

Thus it follows that  $(a+1, b+1)$  is one of the following pairs:  $(1, 392), (2, 196), (4, 98), (8, 49), (7, 56)$ . Each of these lead to dead ends except  $(49, 8)$  and  $(8, 49)$ . For example,  $a+1 = 7, b+1 = 56$  leads to  $a = 6$  and  $b = 55 = 5 \cdot 11$ , which is not a factor of  $8!$ . Next note that  $b = 48$  cannot work because  $b+1 = 49$  is not a factor of 200. Therefore  $a = 48$  and  $b = 7$ . This implies that  $c+1 = 200/(b+1) = 25$  and  $c = 24$ . Finally,  $d = 5$ . Thus,  $a = 48, b = 7, c = 24$  and  $d = 5$ .