

ON THE MEAN VALUE OF THE ADDITIVE ANALOGUE OF SMARANDACHE FUNCTION*

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ABSTRACT. For any positive integer n , let $S(n)$ denotes the Smarandache function, then $S(n)$ is defined the smallest $m \in N^+$, where $n|m!$. In this paper, we study the mean value properties of the additive analogue of $S(n)$, and give an interesting mean value formula for it.

1. INTRODUCTION AND RESULTS

For any positive integer n , let $S(n)$ denotes the Smarandache function, then $S(n)$ is defined the smallest $m \in N^+$, where $n|m!$. In paper [2], Jozsef Sandor defined the following analogue of Smarandache function:

$$(1) \quad S_1(x) = \min\{m \in N : x \leq m!\}, \quad x \in (1, \infty),$$

which is defined on a subset of real numbers. Clearly $S_1(x) = m$ if $x \in ((m-1)!, m!]$ for $m \geq 2$ (for $m = 1$ it is not defined, as $0! = 1! = 1!$), therefore this function is defined for $x > 1$.

About the arithmetical properties of $S(n)$, many people had studied it before (see reference [3]). But for the mean value problem of $S_1(n)$, it seems that no one have studied it before. The main purpose of this paper is to study the mean value properties of $S_1(n)$, and obtain an interesting mean value formula for it. That is, we shall prove the following:

Theorem. For any real number $x \geq 2$, we have the mean value formula

$$\sum_{n \leq x} S_1(n) = \frac{x \ln x}{\ln \ln x} + O\left(\frac{x(\ln x)(\ln \ln \ln x)}{(\ln \ln x)^2}\right).$$

Key words and phrases. Smarandache function; Additive Analogue; Mean Value formula.

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2. PROOF OF THE THEOREM

In this section, we shall complete the proof of the theorem. First we need following one simple Lemma. That is,

Lemma. *For any fixed positive integers m and n , if $(m-1)! < n \leq m!$, then we have*

$$m = \frac{\ln n}{\ln \ln n} + O\left(\frac{(\ln n)(\ln \ln \ln n)}{(\ln \ln n)^2}\right).$$

Proof. From $(m-1)! < n \leq m!$ and taking the logistic computation in the two sides of the inequality, we get

$$(2) \quad \sum_{i=1}^{m-1} \ln i < \ln n \leq \sum_{i=1}^m \ln i.$$

Then using the Euler's summation formula we have

$$(3) \quad \sum_{i=1}^m \ln i = \int_1^m \ln t dt + \int_1^m (t - [t])(\ln t)' dt = m \ln m - m + O(\ln m)$$

and

$$(4) \quad \sum_{i=1}^{m-1} \ln i = \int_1^{m-1} \ln t dt + \int_1^{m-1} (t - [t])(\ln t)' dt = m \ln m - m + O(\ln m).$$

Combining (2),(3) and (4), we can easily deduce that

$$(5) \quad \ln n = m \ln m - m + O(\ln m).$$

So

$$(6) \quad m = \frac{\ln n}{\ln m - 1} + O(1).$$

Similarly, we continue taking the logistic computation in two sides of (6), then we also have

$$(7) \quad \ln m = \ln \ln n + O(\ln \ln m),$$

and

$$(8) \quad \ln \ln m = O(\ln \ln \ln n).$$

Hence, by (6), (7) and (8) we have

$$m = \frac{\ln n}{\ln \ln n} + O\left(\frac{(\ln n)(\ln \ln \ln n)}{(\ln \ln n)^2}\right).$$

This completes the proof of Lemma.

Now we use Lemma to complete the proof of Theorem. For any real number $x \geq 2$, by the definition of $S_1(n)$ and Lemma we have

$$\begin{aligned}
 \sum_{n \leq x} S_1(n) &= \sum_{\substack{n \leq x \\ (m-1)! < n \leq m!}} m \\
 &= \sum_{n \leq x} \left(\frac{\ln n}{\ln \ln n} + O\left(\frac{(\ln n)(\ln \ln \ln n)}{(\ln \ln n)^2}\right) \right) \\
 (8) \qquad &= \sum_{n \leq x} \frac{\ln n}{\ln \ln n} + O\left(\frac{x(\ln x)(\ln \ln \ln x)}{(\ln \ln x)^2}\right).
 \end{aligned}$$

By the Euler's summation formula, we deduce that

$$\begin{aligned}
 \sum_{n \leq x} \frac{\ln n}{\ln \ln n} &= \int_2^x \frac{\ln t}{\ln \ln t} dt + \int_2^x (t - [t]) \left(\frac{\ln t}{\ln \ln t}\right)' dt + \frac{\ln x}{\ln \ln x} (x - [x]) \\
 (9) \qquad &= \frac{x \ln x}{\ln \ln x} + O\left(\frac{x}{\ln \ln x}\right).
 \end{aligned}$$

So, from (8) and (9) we have

$$\sum_{n \leq x} S_1(n) = \frac{x \ln x}{\ln \ln x} + O\left(\frac{x(\ln x)(\ln \ln \ln x)}{(\ln \ln x)^2}\right).$$

This completes the proof of Theorem.

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