BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 36, Number 4, Pages 533–538
S 0273-0979(99)00795-8
Article electronically published on July 28, 1999

Representations and invariants of the classical groups, by Roe Goodman and Nolan R. Wallach, Cambridge Univ. Press, Cambridge, 1998, xvi + 685 pp., \$100.00, ISBN 0-521-58273-3, paperback, \$39.95, ISBN 0-521-66348-2

This 685-page text by Goodman and Wallach constitutes a carefully organized exposition, not indeed of the entire finite-dimensional representation-theory over $\boldsymbol{R}$ or $\boldsymbol{C}$ of the classical groups (which would be hard to imagine in a single text three times the size of this one), but rather of a selection of some key topics therein. This selection of topics, especially as concerns invariant-theory, seems (as Goodman and Wallach indicate in the preface to their text) to have been in many ways inspired by the contents of H. Weyl's *The Classical Groups, Their Invariants and Representations* [31].

Approximately half of the text under review is purely introductory; the remaining half then utilizes this introductory material to develop the main themes.

The introductory material, covered in a total of approximately 280 pages in the first three chapters and in four appendices, introduces the reader to some necessary topics in elementary algebraic and differential geometry and to such basic concepts of representation theory as: linear algebraic groups and Lie groups, Lie algebras associated to such, Jordan decomposition, maximal tori, roots and weights, real forms of classical groups, the PBW theorem, invariant integration over compact Lie groups, etc.

This hefty collection of introductory topics is presented in the Goodman-Wallach text in a manner which is well-planned and carefully detailed (if in places a trifle lacking in motivation). Since these matters are fairly standard, the reader learning these introductory concepts for the first time may wish to shop around also in such sources as, for instance, [3], [4], [10], [19] (there are many others) for the treatment most to her or his taste.

With these elements assumed known, the Goodman-Wallach text then makes available a rich selection of important and more advanced material, which it will be convenient to divide into representation-theoretical material proper and invariant-theoretical material.

The purely representation-theoretical material in the Goodman-Wallach text is also available in other texts, though it is valuable to have it collected into a single volume. This includes, for instance:

- Material (fairly standard) on homogeneous spaces for the classical groups.
- Two distinct proofs (presented in Chapter 7) of the Weyl character formula (cf. [31], [29] or [4] for the first (analytic) proof; [11] or [20] for the second (algebraic) proof).
- Chapter 8, dedicated to the Branching Rules for the irreducible representations of the classical groups, including the spin groups; most of this material can also be found in the texts [2], [33] and (for tables) [27].
- An excellent discussion of symmetric spaces for the classical groups (a topic developed more fully in Wallach's text [30]).

1991 *Mathematics Subject Classification.* Primary 20G20, 22E46.

(N.B. This is not an exhaustive list of the material in representation theory proper in the present text, but it covers much of this material.)

However, it is the material on invariant-theory which (in the reviewer's opinion) constitutes the most valuable and individual portion of the text under review. Much of this material is not available elsewhere (outside of journals). It will be convenient to subdivide the greater part of this material into 4 main topics:

Topic I

The First Fundamental Theorem of Classical Invariant Theory. The content of this theorem will be discussed further below. In Weyl's *The Classical Groups* [31] this theorem is proved (for $SL_n(\boldsymbol{C})$) using machinery centered about the Capelli identity; in the 1970's and 1980's a number of proofs were given (in arbitrary characteristic) using instead the machinery of double standard tableaux, developed by Turnbull in 1945 (in [28]) in the context of invariant theory. In Goodman-Wallach, the First and Second Fundamental Theorems of Classical Invariant Theory (to use the terminology perhaps invented by H. Weyl) are proved for the classical groups by a direct route, using neither of these auxiliary results. (For yet another approach, cf. [18].)

Topic II

Schur's famous 'double centralizer' method showed how (in characteristic 0) to study the irreducible representations of $GL(V)$ by relating the decomposition into irreducible representations of $V^{\otimes m}$ to the character table of the symmetric group S_m. (An amusing variation from the usual treatment occurs in the Goodman-Wallach text: they instead derive (in Chapter 9, §1) the characters of the symmetric groups from those of the general linear groups.) Similar results hold with $GL(n, \boldsymbol{C})$ replaced by $O(n, \boldsymbol{C})$ and with the group algebras $C[S_m]$ replaced by a remarkable class of algebras first introduced by Brauer and certain quotients of these.

It is perhaps a question of taste whether this topic, treated in detail in Chapter 10 of the present text, belongs to representation theory proper or to invariant theory. The Goodman-Wallach text specifically assigns this topic to "classical invariant theory" (preface, p. xiv). The text gives interestingly novel treatments of these matters (also for the symplectic groups).

Topic III

As a further development of Topic II, the present text gives a valuable and detailed discussion of Howe's theory of "dual reductive pairs" (cf. [17]) which is not readily available in books elsewhere.

Topic IV

Yet another generalization of Topic II, of a rather specialized but quite interesting nature, may be derived (via the theory of symmetric spaces) from a theorem of Kostant and Rollis explained in Chapter 12.

The material just described (or at least Topics I, II, III) constitutes an expansion of the topics in invariant theory discussed in Weyl's *Classical Groups*; the Goodman-Wallach text describes this material as 'classical invariant theory' (preface, p. xiv). In his text, Weyl uses the phrase 'classical invariant theory' in rather a different way (2nd Edition, footnote 1 to Chapter VIII, p. 312), referring to such material as that covered in the earlier texts [13], [15]. To clarify this distinction, let us next briefly review some 19th century invariant theory.

Let F_n denote the set of all forms (homogeneous polynomials) of degree n, in the independent variables x, y and with complex coefficients. We have

$$\boldsymbol{C}[x,y] = \bigoplus_{n=0}^{\infty} F_n,$$

and there is a natural action of $SL_2(\boldsymbol{C})$ on the $(n+1)$-dimensional complex vector space F_n.

By a *covariant of binary $n-ic$ forms, of degree i and order w* will be meant an $SL_2(\boldsymbol{C})$-equivariant polynomial map

$$\Phi : F_n \longrightarrow F_w$$

homogeneous of degree i. If $w = 0$, one speaks of an *invariant.*

Example 1. The discriminant $F_2 \longrightarrow \boldsymbol{C} = F_0$, $a_0x^2 + 2a_1xy + a_2y^2 \mapsto a_1^2 - a_0a_2$ of a binary quadratic form, is an invariant of degree 2.

Example 2. The *Hessian* of a binary cubic form

$$f = a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3$$

is the quadratic form

$$H(f) = (a_0a_2 - a_1^2)x^2 + (a_0a_3 - a_1a_2)xy + (c_1c_3 - c_2^2)y^2,$$

and the resulting polynomial map $H : F_3 \longrightarrow F_2$ is a covariant of degree 2 and order 2.

There is an obvious generalization to the concept of a simultaneous covariant.

$$\Psi : F_{n_1} \times \cdots \times F_{n_s} \longrightarrow F_w \tag{1}$$

of s binary forms of degrees respectively $n_1, \dots n_s$. Let us denote by $Cov(n_1, \dots, n_s)$ the $\boldsymbol{C}$-algebra formed by all such Ψ, and by $Inv(n_1, \dots, n_s)$ the sub-algebra formed by the simultaneous *invariants* (for which $w = 0$).

In the second half of the 19th century, the phrase 'modern algebra' was used to denote the intensive study of these invariants and covariants (collectively called concomitants) – (cf. [24]). Perhaps one motivation for the detailed attention given these concomitants was the hope they (and their generalizations to forms in n variables) would furnish a basis-free geometric calculus for studying questions in projective algebraic geometry (cf. [6] and [23]); in this connection, Weyl says (in [31], 2nd Edition, p. 27), "The *theory of invariants* originated in England about the middle of the 19th century as the genuine analytic instrument for describing configurations and their inner geometric relations in projective geometry."

Let us record here what was perhaps considered the main object of these 19th century investigations, since it is a gorgeous open question of great antiquity. (One wonders whether the coming century will finally bring its solution.)

Problem A. Construct explicitly generators and relations for the $\boldsymbol{C}$-algebras $Cov(F_{n_1}, \dots F_{n_s})$ and $Inv(F_{n_1}, \dots F_{n_s})$.

We next discuss some of what is known for the sub-problem:

Problem B. Find a minimal generating set for the $\boldsymbol{C}$-algebra $Inv(n)$ of invariants of binary $n - ic$ forms.

Let $\mu(n)$ denote the minimal number of generators for $Inv(n)$; then we have the table

n	1	2	3	4	5	6	7	8
$\mu(n)$	0	1	1	2	4	5	30	9

The values of $\mu(n)$ for $n \leq 4$ were proved by Cayley and Sylvester, using precisely the techniques (though not with their modern names) one would use today to study these questions, i.e., roots and weights for $SL_2(\boldsymbol{C})$, Hilbert-Samuel polynomials, etc.; for a 20th century account of their techniques, cf. [25].

These techniques of Cayley and Sylvester consisted of utilizing correct and beautiful explicit formulas for the Hilbert-Samuel polynomials of the rings $Inv(n)$ and $Cov(n)$, to analyze the structures of these rings. However, a subtle error in the use of these techniques led Cayley to assert incorrectly in [5] that the rings $Cov(5)$, $Inv(7)$ were not finitely generated. As Elliott ([9], p. 178) explains: "The error arose from considering all syzygies independent, whereas there are syzygies of the second order connecting syzygies."

The German school of invariant theorists, comprising at this point especially Aronhold, Gordan and Clebsch, had been constructing (it is perhaps fair to say, in rivalry with the English school) a quite different (though also ingenious and beautiful) machinery for the study of Problems A and B, based on the Aronhold symbolic method and the theory of "Ueberschiebung" (the latter closely related to the Racah-Wigner algebra of $3j$- and $6j$-symbols used by quantum physics (cf. [1])). When Sylvester incorrectly denied that $Cov(5)$ and $Inv(7)$ were finitely generated, Gordan pounced. Not only did he correct these statements, but Gordan was able to show that all the rings in Problem A and Problem B were finitely generated. When once the emphasis was thus shifted from the *combinatorial* problem of computing explicit generators and relations for the rings in Problems A and B to the problem of *proving* the rings involved are finitely generated over $\boldsymbol{C}$, the stage was set for Hilbert, who showed the latter easier question could be settled by bold and simple new techniques. Hilbert utilized *neither* the techniques developed by Cayley and Sylvester (which nevertheless evolved into some important techniques of modern representation theory) *nor* the techniques of Clebsch and Gordan (which essentially dropped out of sight for many years, with the important exception of their utilization in Alfred Young's important series of papers 'On Quantitative Substitutional Analysis', [32] I (1901) - IX (1952); these techniques have only recently entered mathematics again in the work of Dixmier and others).

Speaking roughly, the techniques developed by the English school of invariant theory give a *lower* bound for $\mu(n)$; those developed by the German school (pre-Hilbert) give an *upper* bound (so are better adapted to proving finite generation). In the cases of $\mu(5)$, $\mu(6)$ and $\mu(8)$, this lower bound and upper bound coincide, giving the precise values $\mu(5) = 4$, $\mu(6) = 5$, $\mu(8) = 9$. The 19th century result $\mu(8) = 9$ was reconfirmed in 1967 by work of T. Shioda [26] studying $Inv(8)$.

The case $Inv(7)$ was studied by F. von Gall in 1888 [12]; here, the lower bound for $\mu(7)$ given by the Cayley-Sylvester techniques is $28 \leq \mu(7)$, while the upper bound given by the Clebsch-Gordan techniques is $\mu(7) \leq 33$. Thus this troublesome case remained open (contrary to the incorrect assertion in [15], footnote to §177) until fairly recently, when the Clebsch-Gordan Ueberschiebung formalism was once again studied (after a gap of many years) by Dixmier and others; in particular, Dixmier and Lazard in 1985 proved [8] that $\mu(7) = 30$. (Further material concerning these matters can be found in [7].)

The preceding is an example of what Weyl's classical groups cite as "in the tradition of classical invariant theory" (2nd Edition, p. 239 and reference 1, to Chapter 8, p. 312), and the phrase "classical invariant theory" is used in essentially the same way in [22], pp. 95-104. As noted above, the Gordan-Wallach text seems rather to use this phrase to refer to the portion of invariant theory treated in Weyl's text. Thus the reader should be aware that "classical invariant theory" is used in these two rather differing ways in the literature.

The ring $Inv(1, 1, \ldots, 1)$ of invariants of s binary linear forms $a^{(1)}x + b^{(1)}y$, $a^{(2)}x+b^{(2)}y, \ldots, a^{(s)}x+b^{(s)}y$ may be proved to be generated by the $\frac{s}{2}$ determinants

$$\begin{vmatrix} a^{(i)} & b^{(i)} \\ a^{(j)} & b^{(j)} \end{vmatrix} \quad (1 \le i < j \le s).$$

This easily proved fact underlies the Aronhold symbolism, which was utilized by the German school to study the much deeper facts about Problem B discussed above. The generalization to $SL(n)$ and $GL(n)$ of this easy fact is what Weyl called the 'First Fundamental Theorem of Classical Invariant Theory'. The treatment of invariant theory in Weyl's Classical Groups, and in the Goodman-Wallach text, enters the long story at this point.

Finally, we conclude our discussion of the Goodman-Wallach text. The preceding review has only touched on the major highlights of the material treated therein. An enormous amount of care and intelligent work has obviously gone into the preparation of this text. The result is an incredibly rich (though only partial) selection of beautiful topics from the current frontiers of representation theory and of invariant theory, whose study will be rewarding both to beginners and to experts in these fields.

References

[1] L.C. Biedenharn and J.D. Louck, *Angular Momentum in Quantum Physics, Theory and Application*, (1981) Encyclopedia of Mathematics and Its Applications, G.-C. Rota, editor, vol. 8, Addison-Wesley Publishing Company, Reading, Mass. MR **83a:**81001

[2] H. Boerner, *Representations of Groups with Special Consideration for the Needs of Modern Physics,* (1970) North Holland, New York. MR **42:**7792

[3] A. Borel, *Linear Algebraic Groups,* (1969) W.A. Benjamin, New York. MR **40:**4273

[4] T. Bröcker and T. tomDieck, *Representations of Compact Lie Groups,* (1985) Graduate Texts in Mathematics 98, Springer-Verlag (New York, Berlin, Heidelberg, Tokyo). MR **86i:**22023

[5] A. Cayley, *A Second Memoir upon Quantics,* CLXVI (1856) Philosophical Transactions of the Royal Society of London [Collected Works, Vol. II (1889), Cambridge at the University Press (141), pp. 250-281].

[6] A. Clebsch and F. Lindemann, *Vorlesungen über Geometrie*, (1876-1891) Leipzig.

[7] J. Dixmier, *Quelques aspects de la théorie des invariants* **43** (1990), Gazette des Math., 39-64. MR **90m:**15047

[8] J. Dixmier and D. Lazard, *Le Nombre Minimum d'Invariants Fondamentaux Pour Les Formes Binaires de Degré 7,* 43 (1985-1986) Portugaliae Mathematica. MR **88f:**15045

[9] E.B. Elliott, *An Introduction to the Algebra of Quantics,* 1895, Oxford at the Clarendon Press.

[10] W. Fulton and J. Harris, *Representation Theory: A First Course,* (1991) Graduate Texts in Mathematics 129, Springer-Verlag, (New York, Berlin, Heidelberg). MR **93a:**20069

[11] H. Freudenthal and H. de Vries, *Linear Lie Groups,* (1969) Academic Press, New York. MR **41:**5546

[12] F. von Gall, *Das vollständige Formensystem der binären Formen 7ten Ordnung,* **31** (1888) *Math. Ann.,* 318-336.

[13] O.E. Glenn, *A Treatise on the Theory of Invariants,* (1915) Ginn and Company, Boston.

[14] P. Gordan, *Vorlesungen über Invariantentheorie,* (ed. G. Kerschensteiner) Vol. 1 (1885), Vol. 2 (1887) Leipzig, 2nd Edition (1987) Chelsea Publishing Company, New York. MR **89g:**01034

[15] J.H. Grace and A. Young, *The Algebra of Invariants,* (1903) Cambridge University Press, (Reprinted Chelsea Publishing Company, Bronx, New York).

[16] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces,* (1978) Pure and Applied Math. 80, Academic Press, NY. MR **80k:**53081

[17] R. Howe, *Perspectives on Invariant Theory,* The Schur Lectures (1992) (ed., I. Piatetski-Shapiro and S. Gelbart), Israel Mathematical Conference Proceedings 8. MR **96e:**13006

[18] R. Howe, *The first fundamental theorem of invariant theory and spherical subgroups,* (333-346), Vol. 56 Part One, Proc. of Summer Research Institute on Algebraic Groups and Their Generalizations, (1991) (Eds. Haboush and Parshall), Proceedings of Symposia in Pure Mathematics, AMS, Providence, Rhode Island. MR **95f:**13008

[19] J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory* (1972) Springer-Verlag. MR **48:**2197

[20] N. Jacobson, *Lie Algebras,* (1962) Wiley-Interscience, New York (Reprinted Dover 1979). MR **80k:**17001

[21] D. Mumford and J. Fogarty, *Geometric Invariant Theory,* 2nd edition (1982) Ergebnisse der Math. und Ihrer Grenzgebiete 34, Springer-Verlag (Berlin, Heidelberg, New York). MR **86a:**14006

[22] P.J. Olver, *Equivalence, Invariants and Symmetry,* (1995) Cambridge University Press. MR **96i:**58005

[23] G. Salmon, *A Treatise on Higher Plane Curves,* 3rd Edition (1889), Hodges, Foster and Figgis, Dublin.

[24] G. Salmon, *Lessons Introductory to the Modern Higher Algebra,* 3rd Edition, (1885), Hodges, Foster and Figgis, Dublin.

[25] I. Schur (H. Grunsky), *Vorlesungen über Invariantentheorie,* (1968) Springer-Verlag. MR **37:**5248

[26] T. Shioda, *On the graded ring of invariants of binary octavics,* **89** (1967) Amer. J. Math., 1022-1046. MR **36:**3790

[27] J. Tits, *Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen,* (1967) Lecture Notes in Math. **40**, Springer-Verlag, Berlin. MR **36:**1575

[28] H.W. Turnbull, *The Theory of Determinants, Matrices and Invariants,* (1929) Blackie (London, Glasgow).

[29] V.S. Varadarajan, *Lie Groups, Lie algebras and Their Representations,* (1974; 1984) Graduate Texts in Math. 102, Springer-Verlag, New York. MR **51:**13113; MR **85e:**22001

[30] N.R. Wallach, *Harmonic Analysis on Homogeneous Spaces,* (1973) Marcel Dekker, Inc., New York. MR **58:**16978

[31] H. Weyl, *The Classical Groups, Their Invariants and Representations* (First Edition, 1939, Second Edition with supplements, 1946) Princeton University Press. MR **1:**42c; MR **98k:**01049

[32] A. Young, *Collected Papers*, Univ. of Toronto Press Math. Expositions, no. 21 (1977), Univ. of Toronto Press. MR **55:**12438

[33] D.P. Želobenko, *Compact Lie Groups and Their Representations*, (1973) Translations of Mathematical Monographs 40, Amer. Math. Soc., Providence, RI. MR **57:**12776b

Jacob Towber
De Paul University
E-mail address: jtowber@math.uchicago.edu