SIAM Review of ``An Encyclopedia of Integer Sequences'' by N. J. A. Sloane & Simon Plouffe

J. M. Borwein and Robert M. Corless

In SIAM REVIEW, **38**, (1996), 333-337.

``The Encyclopedia of Integer Sequences'' by Sloane and Plouffe, published by Academic Press, is not a normal book. It contains a lexicographically ordered collection of integer sequences together with references to these sequences where they appear in the literature. The idea is that a researcher who encounters a sequence in her or his work, and wishes to quickly find out what is known about the sequence (does it have a name, for example, such as ``the Euler numbers'' or ``the Stirling numbers of the first kind''?), can look it up here.

On the face of it this seems a difficult task to accomplish, because surely there are very many sequences of interest. However, by Pareto's principle (80% of your work is done with 20% of your tools) we would expect that simple sequences would occur often, and thus such a book would be useful.

Indeed, this is the case, and even if the book were no more than the handsomely bound physical collection it is, it would have been worthwhile to create, publish, or buy, because it provides a very cheap and efficient route to answers that will work sometimes: if it doesn't work on a particular problem, no big effort has been expended, while if it *does* work you may save a lot of time.

But the physical book is *not* the whole story. Sloane and Plouffe have also created two ``avatars'' of the book, as freely available online computer programs (which we will call `sequences` and `superseeker`) for people to send their sequences to. Because the programs can be accessed by people who do not own the book, we think that Academic Press deserves considerable praise for its enlightened attitude towards the changing shape of publishing.

This is not the first, but is one of the first, of a growing list of sophisticated tools which are accessible to even relatively naive users, and which dramatically illustrate a positive use of the Internet. Our dream work environment would provide us with a whole palette of such tools and a simple key to what exists and how to use it. These tools should ideally be immediately integratable with your favourite working environment (MATLAB, Axiom, Maple, Mathematica, Whatever).

In our opinion the physical book is itself worthwhile not only because it is pleasant to browse in (electrons are so cold, in comparison) but also because of the discussion at the beginning on analysis of sequences. Some of the heuristics discussed in chapters 1, 2, and 3 (before the table of sequences proper begins) give useful hints as to what to do when the computer programs don't work; they also give a nice conceptual model of the inner workings of the programs.

One can turn the tables (so to speak) and use the sequences from the book as a test of each of the subprograms in `sequences` and `superseeker`. Simon Plouffe tells us that each subprogram was considered useful enough to be included if it could identify on the order of 10-100 of the sequences from the book. Further, about 25% of the sequences in the book are obtained from a rational generating function or elementary manipulation thereof (reversion, undoing a logarithmic differentiation, etc.). Addition of various other classes such as hypergeometric functions and pre-processing (adding `1' to each term or doubling the terms, etc) significantly increased the hit rate. It is to be emphasized that not every plausible transformation was included, and much expertise on the part of the authors was needed to choose useful transformations and to avoid `the curse of exponentiality'.

Finally, some `off-the-wall' sequences are also included, such as the numbers on the New York Subway stops, in Figure M5405.

Incidentally, due to a printer's error the table of Figures was not included in the book, and as the `silly' sequences are not actually indexed or numbered in the book, one must either use the programs or know that they are contained in Figure M5405 to find them.

We now give some examples of the uses of the book and the programs therein, to demonstrate their utility (and also some limitations).

---

jborwein@cecm.sfu.ca

How to use the programs

1. Prepare a message with a line of the form

   ```
   lookup 1 1 5 61 1385 50521 2702765
   ```

   in it (obviously change the sequence to the one you want to look up).
2. Send the message to `sequences@research.att.com` for the simple look-up service. In this case, omit the initial terms, and omit all minus signs. Separate the entries in the sequence with spaces, not commas. This simply looks up sequences in the Encyclopedia. The answer frequently comes back within minutes.
3. Send the message to `superseeker@research.att.com` for a more `enthusiastic' attempt to identify your sequence. This time, include the initial terms, *and the minus signs*. If possible, give from 10 to 20 terms. This program tries over 100 transformations in an attempt to match the given sequence with ones in the Encyclopedia.

---

jborwein@cecm.sfu.ca

Related Books and Programs

- A Handbook of Integer Sequences, by N. J. A. Sloane (1973). This might be considered the `first edition' of the book under review. It contained only 2372 sequences, compared to 5488 in the current volume. As of this writing, there are 6222 basic entries in the dynamic online version, and because of the transformations many more sequences can be identified.
- A Dictionary of Real Numbers, by Borwein & Borwein (1990).
- ISC--the Inverse Symbolic Calculator, which can be found easily from

  `http://www.cecm.sfu.ca`.

  When you give this program an approximation to a real number, it will do its best to decide what that number `really' is--in essence, this is a greatly expanded online version of the Dictionary of Real Numbers, mentioned above. For example, the ISC describes $\int_0^1 dx/\sqrt{1-x^4} = 1.311028777...$ as the `lemniscate number'. Simon Plouffe is currently working on this program here at the Centre.
- `gfun` the Generating Function Package by Salvy et al [4]. This Maple package from the `share` library contains functions for manipulating sequences, linear recurrences, differential equations, and generating functions of various types. Simon Plouffe and F. Bergeron had some input into this package, as well.
- `numapprox[pade]` (formerly `convert/ratpoly`) in Maple. This utility uses clever algorithms to convert power series into Padé approximants.
- Mathematica has facilities for conversion of series into Padé approximants and the like as well.

---

jborwein@cecm.sfu.ca

Examples for superseeker and sequences

---

jborwein@cecm.sfu.ca

Example 1

We begin with a classical analytic example, from ``Pi, Euler Numbers, and Asymptotic Expansions'' [1].

R. D. North asked for an explanation of the following fact:

$$4\sum_{k=1}^{500,000}\frac{(-1)^{k-1}}{2k-1} = 3.14159\underline{0}6535897932\underline{4}04626433832\underline{6}9502884197$$

The number on the right is not π to 40 places. As one would expect, the 6th digit after the decimal point is wrong. The surprise is that *only* the underlined digits are wrong. This is explained in detail in [1]. The discovery of the explanation is quite difficult from this result, but is somewhat easier from the following similar one:

$$\frac{\pi}{2} \approx 2\sum_{k=1}^{50,000}\frac{(-1)^{k-1}}{2k-1}$$
$$= 1.5707\underline{8}632679489\underline{7}619231321\underline{1}9163975\underline{205}20985\underline{833147388}$$

Here we note that if we add 1, -1, 5, and -61 to the incorrect digits, we get equality (to 40 places) with $\pi/2$. With the help of Sloane and Plouffe (or indeed with the help of Sloane's original Handbook, as was actually the case, or the computer programs) we can identify these as the first four nonzero Euler numbers. We conjecture, then, that the error is of the form

$$\frac{\pi}{2} = 2\sum_{k=1}^{50,000}\frac{(-1)^{k-1}}{2k-1} + \sum_{k\geq 1}\frac{E_{2k}}{100,000^{2k+1}},$$

where E_{2k} is the k^{th} nonzero Euler number. We can test this conjecture by computation, and find by adding the first 80 terms in the error formula above to the sum that we get $\pi/2$ to 500 digits. This does not tell us that our conjecture is true, but at least it encourages us that a proof might be possible. See [1] for the proof.

The point of this example was that recognition of the Euler numbers in this at first required ingenuity (to shift from π to $\pi/2$, because the original problem has *twice* the Euler numbers appearing in it). However, the case has changed: the new programs both recognize twice the Euler numbers.

If we do not give enough terms to `superseeker`, it fails to return anything (the heuristics of the program are not designed for short sequences, which, after all, can represent far too many things to be really useful). If we put in 7 terms, however, it returns

```
Report on [ 2,2,10,122,2770,101042,5405530]:
Many tests are carried out, but only potentially useful information
(if any) is reported here.


TEST: APPLY VARIOUS TRANSFORMATIONS TO SEQUENCE AND LOOK IT
UP IN THE ENCYCLOPEDIA AGAIN

        SUCCESS
        (limited to 10 matches):

Transformation T003 gave a match with sequence A0364
Transformation T004 gave a match with sequence A0364

List of sequences mentioned:

%I A0364 M4019 N1667
%S A0364
1,1,5,61,1385,50521,2702765,199360981,19391512145,2404879675441,
%T A0364 370371188237525,69348874393137901,15514534163557086905,
%U A0364 4087072509293123892361
%N A0364 Euler numbers: expansion of sec $x$ .
%R A0364 AS1 810. MOC 21 675 67.

<some cryptic material omitted>

References (if any):

[AS1] = M. Abramowitz and I. A. Stegun, { Handbook of Mathematical
Functions}, National Bureau of Standards, Washington DC, 1964.
[MOC] = { Mathematics of Computation} (formerly { Mathematical
Tables and
Other Aids to Computation}).
```

```
List of transformations used:
T003  sequence divided by the gcd of its elements
T004  sequence divided by the gcd of its elements, from the 2nd term

Abbreviations used in the above list of transformations:
u[j]    =       j-th term of the sequence
v[j]    =       u[j]/(j-1)!
Sn(z)   =       ordinary generating function
En(z)   =       exponential generating function
```

The Euler numbers appear as sequence M4019 in the book. [The code here is to the explicit tag in the book; A0364 is an internal absolute code while T003 tags the transformation used.]

---

jborwein@cecm.sfu.ca

Example 2

The following sequence arose in the analysis of the long-term dynamics of numerical methods. For details on the mathematics of this sequence, see [2], but for now note that this could (broadmindedly) be considered as `applied mathematics' because RMC was investigating the reliability of numerical methods for solving nonlinear differential equations over long time intervals (the classical theory gives results useful only on compact time intervals, and the presence of exponentially growing terms in the classical error bounds raises questions about the validity of numerical solutions over long time intervals).

Define the function $B(v) = 1 - v + 3v^2/2 - 8v^3/3 + 31v^4/6 - 157v^5/15 + \cdots$. Then multiplying each coefficient by *k*! we get the following sequence:

$$1, -1, 3, -16, 124, -1256, 15576, \ldots$$

This (modulo the obviously trivial minus signs) is sequence M3024 in the book, which gives the reference to [3].

The history of the example is perhaps more interesting than the mathematics. The first few terms of a series representing the `modified equation' solved by $u_{n+1} = u_n + h u_n^2$, which arises from forward Euler applied to $\dot{y} = y^2$, were laboriously computed using Maple. Bruno Salvy's `gfun` package was then used to identify the sequence; it succeeded, but on checking it was found that the wrong sequence had been generated in the first place (i.e. there was a bug in my Maple program--RMC). Once the bug was fixed, `gfun` could no longer identify the sequence; Bruno Salvy (who is at INRIA in France) was asked for help, and he remarked (immediately) that he *recognized the sequence*. It turned out that he had a pre-publication version of the book under review here, and as stated previously the sequence is listed in the book! Coincidentally, Gilbert Labelle (from Montréal, the author of the reference [3]) was visiting INRIA at this time, as well, so it is conceivable that even without the book the sequence would have been recognized, but the book did play a rôle.

It is worth remarking that the paper by Labelle that was uncovered by this recognition was extremely apt, and *would never have been discovered otherwise* because it is unlikely in the extreme that RMC would have looked in a combinatorics journal for a result on reliability of numerical methods for dynamical systems.

jborwein@cecm.sfu.ca

Example 3

Consider $\alpha = 1 + \sqrt{2}$, which is a Pisot number because the other root of $\alpha^2 - 2\alpha - 1$ is inside the unit circle. Then α^n is asymptotically an integer, and indeed $\alpha^n + (-1)^n / \alpha^n =$ $2, 6, 14, 34, 82, 198, \ldots$

The sequence as such is not in the book (we must divide by 2) but even without division by 2, `sequences` returns:

```
Matches (at most 7) found for  2 6 14 34 82 198  :

%I A2203 M0360 N0136
%S A2203
2,2,6,14,34,82,198,478,1154,2786,6726,16238,39202,94642,228486,
%T A2203 551614,1331714,3215042,7761798,18738638,45239074,109216786,
263672646
%N A2203 Companion Pell numbers: $a(n) = 2a(n-1) + a(n-2)$.
%R A2203 AJM 1 187 1878. FQ 4 373 66. BPNR 43.
%O A2203 0,1
%C A2203 njas
%K A2203


References (if any):

[AJM] = { American Journal of Mathematics}.
[BPNR] = P. Ribenboim, { The Book of Prime Number Records}, Springer-
Verlag, NY, 2nd ed., 1989.
[FQ] = { The Fibonacci Quarterly}.
```

Instead of mentioning Pisot numbers, the sequence is (correctly) identified as being related to Companion Pell numbers. This connexion also would have been unlikely without this compendium.

---

jborwein@cecm.sfu.ca

Example 4

A problem that recently arose on `sci.math` was the (well-known) problem of finding when triangular numbers are square numbers; the first few quickly lead to the sequence $1, 8, 49, 288, 1681, \ldots$, as can be determined with a few minutes computation. This is sequence M4536 in the book, and references are provided to Dickson's History, to Beiler and other recreational mathematics books. In some sense this is what the book is for: to give people an *index* into what is `well-known' and perhaps to avoid ingenious but ultimately wasted rediscovery.

---

jborwein@cecm.sfu.ca

Failed examples

The book and programs are not oracles, and cannot perform miracles. For example, if we submit the following sequence, which simply counts the number of terms in a particular arrangement of a perturbation solution of a heat transfer problem (we would like to know how quickly the size of the solution is growing),

$$2, 12, 44, 100, 203, 344, 558, 824, 1189, 1620, 2176, 2812, \ldots$$

we get no answer.

Other failures can of course occur. The following example shows what *might* happen, and the potential for mis-identification of a sequence. If we submit the sequence [0, 1, 2, 3, 5, 7, 9, 12, 15, 18] to `superseeker`, it returns matches for both M0638 and for M0639, which agree to the first 10 entries. The 11th entry for M0638 is 22, whilst the 11th entry for M0639 is 23. One of them must be wrong, and this brings home the fact that even if the programs or book say that the sequence you give it is X *only*, that might just be a numerical coincidence.

The user of the book and programs must remember that a match does not prove that the sequence found is the one you are looking for, and it is up to the user to demonstrate that any matches found by the programs or in the book are really appropriate for the problem at hand.

---

jborwein@cecm.sfu.ca

About the reviewers

One of us is more pure than applied, while the other is more vice than versa. However, neither of us believes in drawing artificial boundaries between branches of mathematics, and to us the most exciting thing about the book under review is that it helps to erase such boundaries (and indeed narrow the real gaps that exist). RMC is currently visiting the Centre for Experimental and Constructive Mathematics, where JMB is Director. It should be noted that this review is not entirely at `arm's length', because (as previously mentioned) Simon Plouffe has recently joined the CECM.

---

jborwein@cecm.sfu.ca

References

1
J. M. Borwein, P. B. Borwein, and K. Dilcher, ``Pi, Euler Numbers, and Asymptotic Expansions", American Mathematical Monthly, **96**, No. 8, (October 1989), *pp* 681-687.

2
Robert M. Corless, ``Error Backward", in *Chaotic Numerics*, AMS Contemporary Mathematics Series vol 172, Peter Kloeden and K. J. Palmer, eds, (1994) *pp* 31-62.

3
G. Labelle, ``Sur l'Inversion et l'Itération Continue des Séries Formelles", Europ. J. Combinatorics (1980) **1**, *pp* 113-138.

4
Bruno Salvy and Paul Zimmermann, ``GFUN: A Maple Package for the Maniuplation of Generating and Holonomic Functions in One Variable" *preprint*.

---

jborwein@cecm.sfu.ca

About this document ...

SIAM Review of ``An Encyclopedia of Integer Sequences'' by N. J. A. Sloane & Simon Plouffe

This document was generated using the **LaTeX**2HTML translator Version 96.1 (Feb 5, 1996) Copyright © 1993, 1994, 1995, 1996, Nikos Drakos, Computer Based Learning Unit, University of Leeds.

The command line arguments were:
latex2html `-address jborwein@cecm.sfu.ca sloane.tex`.

The translation was initiated by Jonathan Borwein on Sat Jul 13 11:30:21 PDT 1996

---

jborwein@cecm.sfu.ca