Lacunary Prime Numbers

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One of the fundamental aspects of Number Theory is the distinction between prime and composite numbers. The Fundamental Theorem of Arithmetic shows that prime numbers are the building blocks of the integers. It is clear that prime numbers are an important aspect of Number Theory, and they are the focus of attention from many researchers. A popular area of study related to prime numbers is the search for prime numbers of particular form. Mersenne or Fermat prime numbers [2, 3] are the best-known example of this research. For example, the five largest known primes of the time are Mersenne primes. While no large Fermat primes have been found to date, the quest to factor these numbers is still a common area of research.

On investigation of On Line Encyclopaedia of Integer Sequences [6], it was found that no reference to Lacunary numbers or Lacunary primes were present. This repository is intended to act as a reference guide to integer sequences and includes lists of Mersenne and Fermat numbers.

1. Lacunary Numbers

A particular form of primes that have come to our attention is lacunary prime numbers. The simplest way to explain these numbers is to say that there is a 'gap' of zeros between the first and last digit.

Definition 1. *A Lacunary number has the following form*

$$L(a,b,c) = a \times 10^b + c = \overline{a\underbrace{00...0}_{b-1}c} \qquad (1.)$$

where $0 < a,c < 10$ *are digits and* $b > 0$*. We write* $L_{a,c}$ *to signify the family of Lacunary numbers* $L(a,b,c), \forall b > 0$.

For example, the number $L(1,5,3) = 1 \times 10^5 + 3 = 100003$ is a Lacunary Number. Notice that the number has 4 zeros despite $b = 5$. In the family $L_{1,3}$ we find several primes as 13, 103, 1000003, etc (see Table 1).

Definition 2. *A Lacunary prime number is a prime number of the form* $L(a,b,c) = a \times 10^b + c$ *where* $0 < a,c < 10$ *and* $b > 0$. *We write* $L_{a,c}^p$ *to signify the family of Lacunary prime numbers.*

It is clear that some pairs a,c do not yield any prime numbers, the simplest example of this occurring when $c = 2$. Using some simple divisibility rules it is possible to eliminate many possible pairs. The divisibility with 2, 3 and 5 generates the following Lemma.

Lemma 1.

1. *If* $2 | c$ *then* $2 | L(a,b,c), \forall a,b$.
2. *if* $5 | c$ *then* $5 | L(a,b,c), \forall a,b$.
3. *if* $3 | a + c$ *then* $3 | L(a,b,c), \forall b$.

♦

Lemma 2.

a) If $a = c$ *and b is odd then* $11 | L(a,b,c)$.

b) If $a + c = 11$ *and b is even then* $11 | L(a,b,c)$.

Proof. We start from

$$L(a,b,c) = a \cdot 10^b + c = a \cdot (11-1)^b + c = a \cdot (-1)^b + c \pmod{11},$$

which further gives the cases of Lemma 2.

♦

Lemma 3.

a) If $b = 6 \cdot k$ *and* $7 | a + c$ *then* $7 | L(a,b,c)$.

b) If $b = 6 \cdot k + 1$ *and* $7 | 3 \cdot a + c$ *then* $7 | L(a,b,c)$.

c) If $b = 6 \cdot k + 2$ *and* $7 | 2 \cdot a + c$ *then* $7 | L(a,b,c)$.

d) If $b = 6 \cdot k + 3$ *and* $7 | 6 \cdot a + c$ *then* $7 | L(a,b,c)$.

e) If $b = 6 \cdot k + 4$ *and* $7 | 4 \cdot a + c$ *then* $7 | L(a,b,c)$.

f) *If* $b = 6 \cdot k + 5$ *and* $7 | 5 \cdot a + c$ *then* $7 | L(a,b,c)$.

Proof. In order to prove this lemma we remark that

$$10^{6\cdot k} = 100^{3\cdot k} = 2^{3\cdot k} \cdot 50^{3\cdot k} = (7+1)^{k} \cdot (7^2+1)^{3\cdot k} = 1 \pmod 7.$$

Therefore, we have

- $b = 6\cdot k \Rightarrow L(a,b,c) = a\cdot 10^{6\cdot k} + c = a + c \pmod 7$.
- $b = 6\cdot k + 1 \Rightarrow L(a,b,c) = a\cdot 10^{6\cdot k+1} + c = 10\cdot a\cdot 10^{6\cdot k} + c = 3\cdot a + c \pmod 7$.
- $b = 6\cdot k + 2 \Rightarrow L(a,b,c) = a\cdot 10^{6\cdot k+2} + c = 100\cdot a\cdot 10^{6\cdot k} + c = 2\cdot a + c \pmod 7$.
- $b = 6\cdot k + 3 \Rightarrow L(a,b,c) = a\cdot 10^{6\cdot k+3} + c = 1000\cdot a\cdot 10^{6\cdot k} + c = 6\cdot a + c \pmod 7$
- $b = 6\cdot k + 4 \Rightarrow L(a,b,c) = a\cdot 10^{6\cdot k+4} + c = 10000\cdot a\cdot 10^{6\cdot k} + c = 4\cdot a + c \pmod 7$
- $b = 6\cdot k + 5 \Rightarrow L(a,b,c) = a\cdot 10^{6\cdot k+5} + c = 100000\cdot a\cdot 10^{6\cdot k} + c = 5\cdot a + c \pmod 7$

which solves Lemma 3.

♦

These lemmas eliminate quite a few possibilities immediately and leave only 23 pairs of (*a*,*c*) that might contain Lacunary primes. Moreover, they also give some restrictions on *b*. Based on these results we can develop efficient methods to search for lacunary primes.

Some special lacunary numbers are obtained for *a*=*c*=1, which are

$$L(1,b,1) = 10^b + 1 = \overline{1\underbrace{00...0}_{b-1}1}. \quad (2.)$$

The following proposition gives some properties of them.

Proposition 4. *The lacunary numbers* $L(1,b,1)$, $b > 1$ *satisfy:*

a) $L(1,b,1) \mid L(1, b\cdot(2k+1), 1)$.

b) If $L(1,b,1)$ *is prime then b is power of 2.*

Proof.

For the first part of the proof we use

$$L(1,b\cdot(2k+1),1) = 10^{b\cdot(2k+1)} + 1 = (10^b)^{2k+1} + 1 = (L(1,b,1)-1)^{2k+1} + 1 \vdots L(1,b,1).$$

Let suppose the *b* is not a power of 2 thus it has an odd factor $b = (2\cdot b_1 + 1)\cdot b_2$. Since $L(1,b,1) = L(1, b_1\cdot(2b_2+1), 1) \vdots L(1,b_1,1)$ we find that $L(1,b,1)$ is composite. Therefore, *b* is a power of 2.

♦

Our computation has proven that even for $b = 2^{b_1}$ the numbers $L(1,b,1)$ are still composite. The prime number decomposition of the first four numbers $L(1,2^{b_1},1)$ is shown below.

$10^4+1 = 73 * 137$

$10^8+1 = 17 * 5882353$

$10^{16}+1 = 353 * 449 * 641 * 1409 * 69857$

$10^{32}+1 = 19841 * 976193 * 6187457 * 834427406578561$.

2. Lacunary Prime Numbers

Searching for Lacunary numbers by hand would quickly become tedious and would severely limit the search. Therefore, computers are needed. A simple Java program has been used to search for lacunary primes. This program utilises Java's BigInteger class as well as the incorporated isProbablePrime() method [1], which is an implementation of the Miller Rabin test [4, 5]. The prototype of this method is

```
public boolean isProbablePrime(int certainty);
```

and returns false if the number is composite and true if the number is probable prime. Since the method uses a probabilistic test, the probability that a BigInteger is prime is bigger that $1-\frac{1}{2^{certaincy}}$. In our case the accuracy of computation is over $1-\frac{1}{2^5}=\frac{31}{32}=0.96$.

```
public boolean searchLacunaryABC(int a, int b, int c){
      // Construct L(a,b,c)
      String s = new String("”+a);
      for(int i=0;i<b-1;i++)s+=”0”;
      s+=c;
      // Construct the BigInteger associated with L(a,b,c)
      BigInteger bi = new BigInteger(s);
      return bi.isProbablePrime(5);
}

public void searchLacunaryAC(int a, int c){
      // check for first 1000 numbers
      for(int b=1;b<1001;b++) if(searchLacunaryABC(a,b,c))
            System.out.println("    ”+b);
}
```

Figure 1. Java program to test primality of L(a,b,c).

With these methods and the above lemmas we have found 260 lacunary prime numbers for values of *b* less than 1000. These are listed in the following table.

a	c	b	No. primes
1	1	1,2	2
1	3	1,2,5,6,11,17,18,39,56,101,105,107,123,413,426	15
1	7	1,2,4,8,9,24,60,110,134,222,412,700,999	13
1	9	1,2,3,4,9,18,22,45,49,56,69,146,202,272	14
2	3	1,3,5,6,7,12,16,17,22,24,35,115,120,358	14
2	9	1,5,25,455,761	5
3	1	1,3,5,7,10,28,36,67,81,147,483,643	12
3	7	1,2,5,8,24,29,84,110,129,176,593	11
4	1	1,2,3,13,229,242,309,957	8
4	3	1,3,7,10,40,419,449	7
4	7	1,3,9,39	4
4	9	2,4,5,8,9,28,191,196	8
5	3	1,2,3,8,18,20,31,42,103,175,181,531,706	13
5	9	1,2,3,5,8,20,29,86,283,757	10
6	1	1,2,8,9,15,20,26,38,45,65,112,244,303,393,560,839	16
6	7	1,2,3,8,9,19,58,121,187,806,855	11
7	1	1,2,3,4,5,8,9,45,136,142,158,243,923	13
7	3	1,4,6,16,22,39,103,163,240	9
7	9	1,2,4,6,11,12,13,35,46,57,128,156,263,353,396,429,783,982	18
8	3	1,31,105,113,369	5
8	9	1,2,3,6,12,20,21,37,42,55,60,98,100,104,223,237,260,501,570,600,698	21
9	1	3,4,5,9,22,27,36,57,62,78,201,537,696,790,905	15
9	7	1,2,3,4,5,15,19,20,46,52,53,192,380,588,776,906	16

Table 1. Table of Lacunary Prime Numbers.

Conclusions

The largest probable prime found in this search is the huge 1000 digit $L(1,999,7)$ which has an extraordinary 998 zero gap. This is in itself a fascinating find, and is one of 6 numbers that exceed the 900-digit mark. The table clearly shows how the frequency of primes decreases as the size of the numbers increase. Indeed for some pairs, primes are very dense at low values but are extremely rare for the larger values of *b*. For example $L_{4,9}^{p}$ has 6 primes with $b < 30$ and only two more occur for the remainder of the search, both of which occur with $b < 200$. There is no prime in this family that is greater in length than 200 digits.

It is also interesting to compare certain pairs with each other. There are many distinguishing features that can be used for comparisons; number of primes, regularity of occurrence, maximum size attained or average size. For example, on analysis of the table, it is interesting to ask what is special about the pair 8,9 that provides so many primes or does the pair 4,7 provide any more primes other than the four found in this search.

Extending the search could perhaps provide further insight into these questions but ultimately there is a flaw in this approach. For the pair 1,1 it has been shown that only powers of 2 are suitable values for *b,* if indeed there are any more primes of that form. This was achieved by a simple examination of the factors that occurred. Clearly, if similar work was done with the factors of other pairs it is possible that similar results could be found. In this way it may be possible to discover if there are infinitely many primes in a given Lacunars family or if in fact each family has a finite number of primes. Perhaps also, this would be the only way to truly understand the difference between pairs.

References.

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