SOLUTIONS TO PROBLEMS IN CHAPTER FOUR

1. **Prove that the number of partitions of an integer n is equal to the number of partitions of $n+1$ whose smallest part is 1.**

   Solution: Given a partition of n, we can produce a partition of $n+1$ by adding a single one; this will produce a partition of $n+1$ whose smallest part is 1, and which possesses one more part than our original partition. Similarly, given a partition of $n+1$ whose smallest part is 1, we remove this one to obtain a partition of n. This is a one-one correspondence between partitions of n and partitions of $n+1$ whose smallest part is one, and the proof is complete.

2. **Let $p^*(n)$ denote the number of partitions of n whose summands all exceed one. Prove that $p^*(n) = p(n) - p(n-1)$**

   Solution: We know from the previous problem that $p(n-1)$ is equal to the number of partitions of n having one as its smallest part. It follows that $p(n) - p(n-1)$ counts the number of partitions of n that do not have one as their smallest part. Since that is equivalent to our definition of p^* the proof is complete.

3. **Recall that we defined the quantity $p_k(n)$ to be the number of partitions of n with summands no larger than k. Let k and n be integers with $1 < k < n$. Prove that**

   $$p_k(n) = p_{k-1}(n) + p_k(n-k).$$

   Solution: Let us divide the partitions of n into two sets: those possessing k as a summand and those not possessing k as a summand. If we do not allow k as a summand, then we must partition n into summands not exceeding $k-1$. The number of such partitions is $p_{k-1}(n)$. If we do have k as a summand, then the remaining summands constitute a partition of $n-k$ into parts not exceeding k. There are $p_k(n-k)$ such partitions. Since these correspondences are clearly one-one, we have $p_k(n) = p_{k-1}(n) + p_k(n-k)$ as desired.

4. **Prove that for any integer** n**, we have** $p(n+2)+p(n) \geq 2p(n+1)$.

Solution: The inequality

$$p(n+2) + p(n) \geq 2p(n+1)$$

is equivalent to

$$[p(n+2) - p(n+1)] - [p(n+1) - p(n)] \geq 0.$$

But we know from problem two that the quantity in the first set of brackets is equal to the number of partitions of $n+2$ whose summands all exceed 1, while the quantity in the second set of brackets denotes the number of partitions of $n+1$ whose summands all exceed 1.

Given any partition of $n+1$ into parts greater than one, we can add one to its largest part to obtain a partition of $n+2$ into parts greater than one. Distinct partitions of $n+1$ map to distinct partitions of $n+2$ in this way. It follows that the number of partitions of $n+1$ into parts not exceeding one can not be larger than the number of partitions of $n+2$ into parts not exceeding one, and the desired inequality follows.

5. **In how many ways can** 100 **idenitical apples be placed into** 20 **different bags if we require that each bag has at least three apples? A formula is fine; there is no need to come up with the exact number.**

Solution: We begin by placing three apples into each of the bags. That accounts for 60 of the apples, leaving 40 left over. Any partition of 40 into no more than 20 summands will correspond to an acceptable placement of the remaining apples. By the result of problem eight, that number is precisely $\binom{39}{19}$.

6. **A given partition of an integer k is said to be self-conjugate if it is equal to its own conjugate. Prove that the number of self-conjugate partitions of k is equal to the number of partitions of k into distinct, odd summands.**

   Solution: Let P be a partition of k into distinct, odd summands. Then every summand in P has the form $2r-1$ for some choice of r. We will construct a Ferrers diagram out of this partition as follows: Arrange the $2r-1$ squares corresponding to each summand in P into one row and one column, each of size r, so that the first square in the row and the first square in the column overlap. For example, given the partition $9+3+1$ of 13 into distinct, odd summands, construct the Ferrers diagram:

   ⊠ ⊠ ⊠ ⊠ ⊠
   ⊠ ⊞ ⊞
   ⊠ ⊞ □
   ⊠
   ⊠

   Here, the squares marked ⊠ correspond to the summand 9, the squares marked ⊞ correspond to the summand 3, while the □ corresponds to the summand 1. The result is the self-conjugate partition 5+3+3+1+1 of 9.

   Given a self-conjugate partition, we could reverse the construction given above to produce a new partition consisting of distinct odd summands. It follows that the number of self-conjugate partitions is equal to the number of partitions into distinct odd summands, and the proof is complete.

7. **Prove that the number of partitions of m is equal to the number of partitions of $2m$ into m parts.**

   Solution: Given a partition of $2m$ into m parts, we can subtract one from each part to obtain a partition of m. Conversely, given a partition of m into k parts, with $k \leq m$, we add one to each of the k parts, and add $m-k$ new parts each of size one. The result is a partition of $2m$ into precisely m parts. Since these two operations are inverses of each other, we have established a one-one correspondence between partitions of m and partitions of $2m$ into precisely m parts.

8. **Prove that the number of partitions of n into exactly r parts where partitions differing in the order of their summands are to be considered different is $\binom{n-1}{r-1}$.**

Solution: A partition of n into exactly r parts can be viewed in the following manner: Imagine a row consisting of n ones. We insert $r-1$ dividers between some of the ones, thereby producing a partition of n. For example, the partitions 11/111/1/1111 and 1/1111/111/11 are two different ways of inserting three slashes to produce a partition into four parts of the number 10 (in this case, the two partitions are $2+3+1+4$ and $1+4+3+2$. These partitions are different according to the requirements of the problem.)

Since we do not allow zero as a summand in a partition, we have $n-1$ choices for where to put a divider. Out of these $n-1$ choices, we must select $r-1$ of them. It follows that the total number of ordered partitions of n into exactly r parts is $\binom{n-1}{r-1}$ as desired.

9. **Find the number of ordered quadruples (x_1, x_2, x_3, x_4) of positive odd integers such that $x_1 + x_2 + x_3 + x_4 = 98$.**

Solution: We can represent each of the (x_i)'s in the form $x_i = 2y_i - 1$ for some positive integer y_i. We then have

$$98 = \sum_{i=1}^{4} 2y_i - 1 = 2\left(\sum_{i=1}^{4} y_i\right) - 4.$$

It follows that

$$51 = \sum_{i=1}^{4} y_i.$$

We have thus reduced the problem to that of finding the number of ordered partitions of 51 into exactly 4 parts. By the result of the previous problem, we know the number of such partitions is prescisely $\binom{50}{3}$ and the solution is complete.

Prove that the number of partitions of $(a-c)$ into exactly $(b-1)$ parts, none of which is larger than c, is equal to the number of partitions of $(a-b)$ into $(c-1)$ parts none of which is larger than b.

Solution: Let P be a partition of $(a-c)$ into exactly $(b-1)$ parts, none of which is greater than c. If we add one additional part of length c we obtain a partition of a into exactly b parts, having c as its largest part. Similarly, given a partition of a into b parts the largest of which is c, we may remove the part of size c to return to a partition of $(a-c)$ into $(b-1)$ parts none of which is larger than c. We have thus established a one-one correspondence between these two sorts of partitions.

Now, given a partition of a into b parts the largest of which is c, we take the conjugate to obtain a partition of a into c parts the largest of which is b. Since every partition is the conjugate of precisely one other, we have established a one-one correspondence between these two sorts of partitions. It follows that the number of partitions of $(a-c)$ into $(b-1)$ parts none of which is larger than c is equal to the number of partitions of a into c parts the largest of which is b.

The final step is to take a given partition of a into c parts the largest of which is b and remove its largest part (namely b). By doing this we obtain a partition of $(a-b)$ into $(c-1)$ parts none of which is larger than b. Once more, we can reverse this operation by taking a partition of $(a-b)$ into $(c-1)$ parts with no part larger than b and adding a single part of size b. This results in a partition of a into c parts the largest of which is b. We thus have a one-one correspondence between these two sorts of partitions.

Putting everything together establishes a one-one correspondence between partitions of $(a-c)$ into $(b-1)$ parts none of which is larger than c and the number of partitions of $(a-b)$ into $(c-1)$ parts none of which is larger than b, as desired.

Prove that the number of partitions of n with no summand greater than k is equal to the number of partitions of $n+k$ with exactly k parts.

Solution: We know from the chapter that the number of partitions of n with no summand greater than k is equal to the number of partitions of n with no more than k parts.

Any partition of $n+k$ with exactly k parts can be viewed as combining a partition of k into k ones with any partition of n into no more than k parts. Conversely, given a partition of n into no more than k parts, we obtain a partition of $n+k$ with k parts by adding one to every part of the partition, and appending as many aditional parts of size one as we need to give us a partition of $n+k$ into exactly k parts. This establishes a one-one correspondence between partitions of $n+k$ into exactly k parts with partitions of n having no more than k parts.

And since the number of partitions of n into no more than k parts is equal to the number of partitions of n with no summand greater than k, the proof is complete.

Notice that problem seven is the special case of this problem where $n = k$.

Given a partition P of a positive integer n, we say P is perfect if it contains precisely one partition of every number less than n. Prove that the number of perfect partitions of an integer n is equal to the number of ordered factorizations of $n+1$.

Solution: First, notice that any perfect partition P must contain at least one 1, for otherwise it would be impossible to partition 1 using elements of P. So let $q_1 - 1$ denote the number of 1's in P. It follows that any number less that q_1 now has one partition in P. Therefore, the next smallest element of our partition must be q_1.

Continuing in this vain, we suppose there are $q_2 - 1$ parts of P equal to q_1. We can now partition any number between 1 and $q_1q_2 - 1$ in exactly one way using only the numbers 1 and q_1. Similarly, we would then have $q_3 - 1$ copies of q_1q_2, and this would allow us to partition any number between one and $q_1q_2q_3 - 1$ in precisely one way using only the numbers 1, q_1 and q_2.

Let us suppose that P contains k elements. Then we have that

$$n = (q_1 - 1) + q_1(q_2 - 1) + q_1q_2(q_3 - 1) + \cdots (q_1q_2q_3 \cdots q_{k-1})(q_k - 1).$$

This implies that $n + 1 = q_1 q_2 q_3 \cdots q_k$. We have thus obtained an ordered factorization of $n+1$, and this factorization uniquely determines the partition P. It follows that the number of perfect partitions is equal to the number of such factorizations, and the proof is complete.