SOME RESULTS ON PARTITIONS

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The word "partition" has numerous meanings in mathematics. Any time a division of some objects into subobjects is undertaken, the word partition is likely to pop up. For the purposes of this article a "partition of n" is a nonincreasing finite sequence of positive integers whose sum is n.

The theory of partitions has an interesting history. Certain special problems in partitions certainly date back to the Middle Ages; however, the first discoveries of any depth were made in the eighteenth century when L. Euler proved many beautiful and significant partition theorems. Euler indeed laid the foundations of the theory of partitions. Many of other great mathematicians – Cayley, Gauss, Hardy, Jacobi, Lagrange, Legendre, Littlewood, Rademacher, Ramanujan, Schur and Sylvester – have contributed to the development of the theory.

1 Introduction

Definition 1.1. *A* partition *of a positive integer n is a finite nonincreasing sequence of positive integers λ_1, λ_2,..., λ_r such that*

$$\sum_{i=1}^{r} \lambda_i = n.$$

The λ_i are called the parts *of the partition. Many times the partition is also denoted as $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$.*

Example 1.1. *The number 27 can be written as a sum:*

$$27 = 7 + 5 + 4 + 3 + 3 + 3 + 2$$

and we denote that partition: $\lambda = (7, 5, 4, 3, 3, 3, 2)$.

Also $\mu = (11, 4, 4, 3, 2, 1, 1, 1)$ is a partition of 27, because

$$27 = 11 + 4 + 4 + 3 + 2 + 1 + 1 + 1.$$

Example 1.2. *All the partitions of 8 are:*
(8), $(7, 1)$, $(6, 2)$, $(6, 1, 1)$, $(5, 3)$, $(5, 2, 1)$, $(5, 1, 1, 1)$, $(4, 4)$, $(4, 3, 1)$, $(4, 2, 2)$, $(4, 2, 1, 1)$, $(4, 1, 1, 1, 1)$, $(3, 3, 2)$, $(3, 3, 1, 1)$, $(3, 2, 2, 1)$, $(3, 2, 1, 1, 1)$, $(3, 1, 1, 1, 1, 1)$, $(2, 2, 2, 2)$, $(2, 2, 2, 1, 1)$, $(2, 2, 1, 1, 1, 1)$, $(2, 1, 1, 1, 1, 1, 1)$, $(1, 1, 1, 1, 1, 1, 1, 1)$.

Remark. We notice, that for every partition $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ of n:

a) $1 \leq \lambda_i \leq n \qquad \forall i \in \{1, 2, \ldots, n\}$,

b) $1 \leq r \leq n$.

2 The number of partitions

Definition 2.1. *The partition function $p(n)$ is the number of partitions of n.*

Example 2.1.

$$
\begin{array}{lllllllllllll}
p(1) = 1: & 1 & = & 1 \\
p(2) = 2: & 2 & = & 2 & = & 1+1 \\
p(3) = 3: & 3 & = & 3 & = & 2+1 & = & 1+1+1 \\
p(4) = 5: & 4 & = & 4 & = & 3+1 & = & 2+2 & = & 2+1+1 & = & 1+1+1+1 \\
p(5) = 7: & 5 & = & 5 & = & 4+1 & = & 3+2 & = & 3+1+1 & = & 2+2+1 \\
 & & = & 2+1+1+1 & = & 1+1+1+1+1
\end{array}
$$

Remark. It is useful to define $p(n)$ for all integers n. We shall set $p(0) = 1$ with the observation that the empty sequence forms the only partition of zero. For $n < 0$ it is obviously $p(n) = 0$.

The partition function increases quite rapidly with n. For example:

$p(10) =$	42		
$p(20) =$	627		
$p(30) =$	5604		
$p(40) =$	37338		
$p(50) =$	204226	$p(250) =$	230793554364681
$p(60) =$	966467	$p(300) =$	9253082936723602
$p(70) =$	4087968	$p(350) =$	279363328483702152
$p(80) =$	15796476	$p(400) =$	6727090051741041926
$p(90) =$	56634173	$p(450) =$	134508188001572923840
$p(100) =$	190569292	$p(500) =$	2300165032574323995027
$p(110) =$	607163746	$p(550) =$	34403115367205050943160
$p(120) =$	1844349560	$p(600) =$	458004788008144308553622
$p(130) =$	5371315400	$p(650) =$	5503637762499727151307095
$p(140) =$	15065878135	$p(700) =$	60378285202834474611028659
$p(150) =$	40853235313	$p(750) =$	610450747117966916191771809
$p(160) =$	107438159466	$p(800) =$	5733052172321422504456911979
$p(170) =$	274768617130	$p(850) =$	50349216918401212177548479675
$p(180) =$	684957390936	$p(900) =$	415873681190459054784114365430
$p(190) =$	1667727404093	$p(950) =$	3246724928206047105940972859506
$p(200) =$	3972999029388	$p(1000) =$	24061467864032622473692149727991

We can calculate the number of partitions of n in several ways. We can determine p(n) by listing all the partitions of n and counting them (as we did in the example 2.1) or we can use some recurrence furmulas (we'll talk about them later), but both ways soon become impractical. Therefore we usualy use the computer to calculate the $p(n)$. Mathematica[1] already has a built in function: PartitionsP[n], where n is a concrete positive integer.

But we are not always interested in calculating the number of *all* partitions of an integer n; instead of the set of all partitions we often observe just one of its subsets.

Let $\mathcal{L}$ denote any partition property. The number of partitions of n, having the property $\mathcal{L}$ is then denoted $p(n|\mathcal{L})$. In the case of observing all the partitions of n to exactly k parts we usually write $p_k(n)$.

Remark. It is obviously $p_k(n) = 0$ za $k < 0$ ali $k > n$.

Example 2.2. *In the example* 1.2 *we have listed all the* 22 *partitions of the integer* 8*. Now lets take a look how many are the partitions of* 8*, which correspond to some restrictions.*

$$\begin{aligned} p(8|\textit{every part is odd}) = 6: &\quad (7,1),\ (5,3),\ (5,1,1,1),\ (3,3,1,1),\\ &\quad (3,1,1,1,1,1),\ (1,1,1,1,1,1,1,1)\\ p(8|\textit{all the parts are distinct}) = 6: &\quad (8),\ (7,1),\ (6,2),\ (5,3),\ (5,2,1),\\ &\quad (4,3,1)\\ p_3(8) = 5: &\quad (6,1,1),\ (5,2,1),\ (4,3,1),\ (4,2,2),\\ &\quad (3,3,2) \end{aligned}$$

We notice, that $p(8|\textit{every part is odd}) = p(8|\textit{all the parts are distinct})$*. Later, in the theorem* 3.2*, we are going to show thathis relation is valid for every integer, not only* 8*.*

Theorem 2.1. *There is only one partition of any positive integer* n *to exactly* n *parts:* $p_n(n) = 1$*. When* $n > k$*, we can calculate the number of partitions of* n *to exactly* k *parts using the recurrence formula*

$$p_k(n) = p_k(n-k) + p_{k-1}(n-k) + \cdots + p_1(n-k). \tag{1}$$

Proof. It is obvious, that $p_n(n) = 1$: the only partition of a positive integer n to exactly n parts is represented by the sum of n ones: $1 + 1 + \cdots + 1$.

[1]Mathematica 4, Wolfram Research, Inc.

Now let be $n > k$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ any partition of a positive integer n to exactly k parts. Then

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k \tag{2}$$

is a denotion of n with exactly k positive parts. Therefore

$$n - k = (\lambda_1 - 1) + (\lambda_2 - 1) + \cdots + (\lambda_k - 1) \tag{3}$$

is a denotion of n with k positive parts, where some parts may equal zero since some λ_i may equal 1. In another words, (3) represents a sum of at most k positive parts, which corresponds to the partition of $n - k$ to k or less parts. It is easy to see that the mapping described above is bijective.
Therefore: $p_k(n) = p_k(n-k) + p_{k-1}(n-k) + \cdots + p_1(n-k)$. □

Remark. We calculate the number of all partitions of n as

$$p(n) = p_n(n) + p_{n-1}(n) + \cdots + p_1(n).$$

Corrolary 2.1. *When $n > k$, we can calculate the number of partitions of n to exactly k parts using the formula*

$$p_k(n) = p_k(n-k) + p_{k-1}(n-1). \tag{4}$$

Proof. From the theorem 2.1 we know, that $p_k(n) = p_k(n-k) + p_{k-1}(n-k) + \cdots + p_1(n-k)$. We can then write $p_{k-1}(n-1)$ as a sum in the same way:

$$\begin{aligned} p_{k-1}(n-1) &= p_{k-1}(n-1-(k-1)) + p_{k-2}(n-1-(k-1)) + \cdots + p_1(n-1-(k-1)) \\ &= p_{k-1}(n-k) + p_{k-2}(n-k) + \cdots + p_1(n-k). \end{aligned}$$

Instead of the right side of equivalence (4) we can now write

$$p_k(n-k) + p_{k-1}(n-k) + p_{k-2}(n-k) + \cdots + p_1(n-k),$$

and that is (theorem 2.1) exactly $p_k(n)$. □

The values of $p_k(n)$ can be grafically arranged into a triangle, similar to Pascal's or Stirling's triangles. The square table with dimension n, formed in Mathematica by the command

$$TableForm[Table[p[j, i], \{i, 1, n\}, \{j, 1, n\}]],$$

and preliminary defined

$$p[k\_, n\_] := Which[k > n, 0, k <= n, \sum_{i=1}^{k} p[i, n-k]] \qquad \text{in} \qquad p[1, 1] := 1,$$

approximates well the above described triangular table. Values, which the triangle to the square, are zero.
In the first row of the n times n table there are values $p_k(0)$, in the second $p_k(1)$, ..., in the n-th $p_k(n-1)$. There is $1 \leq k \leq n$ in each row, which generates n columns. The sum of the values $p_k(i-1)$ in i-th row equals $p(i-1)$.

Example 2.3. *For a choice $n = 20$ Mathematica with the command*

$$TableForm[Table[p[j,i],\{i,1,20\},\{j,1,20\}]]$$

forms the table:

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	3	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	3	4	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
1	4	5	5	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	4	7	6	5	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0
1	5	8	9	7	5	3	2	1	1	0	0	0	0	0	0	0	0	0	0
1	5	10	11	10	7	5	3	2	1	1	0	0	0	0	0	0	0	0	0
1	6	12	15	13	11	7	5	3	2	1	1	0	0	0	0	0	0	0	0
1	6	14	18	18	14	11	7	5	3	2	1	1	0	0	0	0	0	0	0
1	7	16	23	23	20	15	11	7	5	3	2	1	1	0	0	0	0	0	0
1	7	19	27	30	26	21	15	11	7	5	3	2	1	1	0	0	0	0	0
1	8	21	34	37	35	28	22	15	11	7	5	3	2	1	1	0	0	0	0
1	8	24	39	47	44	38	29	22	15	11	7	5	3	2	1	1	0	0	0
1	9	27	47	57	58	49	40	30	22	15	11	7	5	3	2	1	1	0	0
1	9	30	54	70	71	65	52	41	30	22	15	11	7	5	3	2	1	1	0

Another relation to calculate the number of partitions of n is the following:

$$n \cdot p(n) = S(n) + p(1) \cdot S(n-1) + p(2) \cdot S(n-2) + ... + p(n-1) \cdot S(1),$$

where S(n) denotes the sum of the divisors of n (for example S(4) = 1+2+4 = 7). We won't prove the relationship (see DEBONO, ALBERT N. The partitions of an integer, http://www.pcworldmalta.com/archive/iss37/num22.htm), we'll only do an example.

Example 2.4.

$$\begin{aligned} 6\cdot p(6) &= S(6)+p(1)\cdot S(5)+p(2)\cdot S(4)+p(3)\cdot S(3)+p(4)\cdot S(2)+p(5)\cdot S(1)\\ 6\cdot p(6) &= 12+1\cdot 6+2\cdot 7+3\cdot 4+5\cdot 3+7\cdot 1\\ p(6) &= 11 \end{aligned}$$

3 Some interesting results on the partitions

To describe the number of partitions we often use generating functions: the generating function for the sequence $(a_n)_{n=0}^{\infty}$ is the (formal) power series $f(q)=\sum_{n=0}^{\infty} a_n q^n$.

Definition 3.1. *Let be $H\subseteq\mathbb{N}$. By $P(H)$ we denote the set of all partitions whose parts lie in H. By $p(n|\lambda\in P(H))$ we denote the number of all partitions of n that have all their parts in H.*

Example 3.1. *For any n is $p(n|\lambda\in P(L))=p(n|$every part is odd$)$, where L is the set of all odd positive integers.*

Definition 3.2. *Let be $H\subseteq\mathbb{N}$. By $P(H_d)$ we denote the set of all partitions whose parts lie in H and where no part occures no more than d times.*

Example 3.2. *For any n is $p(n|\lambda\in P(\mathbb{N}_1))=p(n|$all the parts are distinct$)$.*

Theorem 3.1. *Let be $H\subseteq\mathbb{N}$ and let be*

$$f(q)=\sum_{n=0}^{\infty} p(n|\lambda\in P(H))q^n, \tag{5}$$

$$f_d(q)=\sum_{n=0}^{\infty} p(n|\lambda\in P(H_d))q^n. \tag{6}$$

the generating functions. Then for $|q|<1$:

$$f(q)=\prod_{n\in H}\frac{1}{(1-q^n)}, \tag{7}$$

$$f_d(q)=\prod_{n\in H}\left(1+q^n+\cdots+q^{dn}\right)=\prod_{n\in H}\frac{\left(1-q^{(d+1)n}\right)}{(1-q^n)}. \tag{8}$$

Proof. Let be $H = \{h_1, h_2, h_3, h_4, \ldots\} \subseteq \mathbb{N}$. Then we can write

$$\begin{aligned}\prod_{n\in H} \frac{1}{(1-q^n)} &= \prod_{n\in H} \left(1+q^n+q^{2n}+q^{3n}\cdots\right) \\ &= \left(1+q^{h_1}+q^{2h_1}+q^{3h_1}+\cdots\right) \\ &\quad \cdot \left(1+q^{h_2}+q^{2h_2}+q^{3h_2}+\cdots\right) \\ &\quad \cdot \left(1+q^{h_3}+q^{2h_3}+q^{3h_3}+\cdots\right) \\ &\quad \vdots\end{aligned}$$

We observe that the exponent of q is just the partition: $N = a_1h_1+a_2h_2+a_3h_3+\cdots$. Hence q^N will occur in the foregoing summation once for each partition of n into parts taken from H. Therefore

$$\prod_{n\in H} \frac{1}{(1-q^n)} = \sum_{n=0}^{\infty} p(n|\lambda \in P(H))q^n.$$

If we are to view the foregoing procedures as operations with convergent infinite products, then the multiplication of infinitely many series together requires some justification. The simplest procedure is to truncate the infinite product to

$$\prod_{i=1}^{k} \frac{1}{1-q^{h_i}}.$$

This truncated product will generate those partitions whose parts are in $H' = \{h_1, h_2, \ldots, h_k\}$ and we can therefore write it as

$$\begin{aligned}\prod_{n\in H'} \frac{1}{(1-q^n)} &= \prod_{n\in H'} \left(1+q^n+q^{2n}+q^{3n}\cdots\right) \\ &= \left(1+q^{h_1}+q^{2h_1}+q^{3h_1}+\cdots\right) \\ &\quad \cdot \left(1+q^{h_2}+q^{2h_2}+q^{3h_2}+\cdots\right) \\ &\quad \vdots \\ &\quad \cdot \left(1+q^{h_k}+q^{2h_k}+q^{3h_k}+\cdots\right)\end{aligned}$$

$$= \sum_{a_1=0}^{\infty} \sum_{a_2=0}^{\infty} \cdots \sum_{a_k=0}^{\infty} q^{a_1h_1+a_2h_2+\cdots+a_kh_k}\ .$$

In this product we multiply only finite number of absolutely convergent series and we know the result is then a convergent. For $q \in \mathbb{R}$, $0 < q < 1$, $h_1, h_2, \ldots, h_k \in H'$, $h_1 < h_2 < \ \ldots\ < h_k$ it is:

$$\sum_{n=0}^{h_k} p(n|\lambda \in P(H))q^n \ \le\ \prod_{i=1}^{k} \frac{1}{1-q^{h_i}} \ \le\ \prod_{i=1}^{\infty} \frac{1}{1-q^{h_i}} \ <\ \infty.$$

Thus the sequence of partial sums $\sum_{n=0}^{h_k} p(n|\lambda \in P(H))q^n$ is a bounded increasing sequence and must therefore converge. Therefore $\sum_{n=0}^{\infty} p(n|\lambda \in P(H))q^n$ converges. On the other hand

$$\sum_{n=0}^{\infty} p(n|\lambda \in P(H))q^n \geq \prod_{i=1}^{k} \frac{1}{1-q^{h_i}} \qquad \text{in} \qquad \prod_{i=1}^{k} \frac{1}{1-q^{h_i}} \overset{k\to\infty}{\to} \prod_{i=1}^{\infty} \frac{1}{1-q^{h_i}}.$$

Therefore

$$\sum_{n=0}^{\infty} p(n|\lambda \in P(H))q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^{h_i}} = \prod_{n\in H} \frac{1}{1-q^{h_i}}.$$

Before proving the equivalence (8), let's show that

$$\prod_{n\in H} \left(1+q^n+\cdots+q^{dn}\right) = \prod_{n\in H} \frac{\left(1-q^{(d+1)n}\right)}{(1-q^n)}$$

is true. All we have to do is to use the formula for calculating the first $r+1$ parts of geometric series:

$$1+x+x^2+\cdots+x^r = \frac{1-x^{r+1}}{1-x}.$$

Proof of the equivalence

$$f_d(q) = \prod_{n\in H} \frac{\left(1-q^{(d+1)n}\right)}{(1-q^n)}$$

is similar to proof of the equivalence (7), only the partial sums of the geometric series are taken:

$$\begin{aligned} \prod_{n\in H} \left(1+q^n+\cdots+q^{dn}\right) &= \sum_{a_1=0}^{d} \sum_{a_2=0}^{d} \sum_{a_3=0}^{d} \cdots q^{a_1h_1+a_2h_2+a_3h_3+\cdots} \\ &= \sum_{n=0}^{\infty} p(n|\lambda \in P(H_d))q^n. \end{aligned}$$

Similar justification can be given to the proof of (8). □

Theorem 3.2 (Euler).

$$p(n|\textit{every part is odd}) = p(n|\textit{all the parts are distinct}) \tag{9}$$

for all n.

Proof. Using the theorem 3.1 we can write:

$$\sum_{n=0}^{\infty} p(n|\lambda \in P(L))q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}} \qquad \text{in}$$

$$\sum_{n=0}^{\infty} p(n|\lambda \in P(\mathbb{N}_1))q^n = \prod_{n=1}^{\infty} \frac{1-q^{2n}}{1-q^n} = \prod_{n=1}^{\infty} (1+q^n)\,.$$

Since

$$\prod_{n=1}^{\infty} (1+q^n) = \prod_{n=1}^{\infty} \frac{1-q^{2n}}{1-q^n} = \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}},$$

it is

$$\sum_{n=0}^{\infty} p(n|\lambda \in P(\mathbb{N}_1))q^n = \sum_{n=0}^{\infty} p(n|\lambda \in P(L))q^n.$$

or, in another form, $p(n|\text{all the parts are distinct}) = p(n|\text{every part is odd})$. □

Definition 3.3. *For $a, q \in \mathbb{C}$ we define:*

$$\begin{aligned}
(a)_n &= (a;q)_n = (1-a)(1-aq)\cdots\left(1-aq^{n-1}\right), \\
(a)_\infty &= (a;q)_\infty = \lim_{n\to\infty} (a;q)_n = (1-a)(1-aq)\cdots\left(1-aq^{n-1}\right)\cdots \;\; za\ |q| < 1, \\
(a)_0 &= 1.
\end{aligned}$$

Theorem 3.3. *For $a, q \in \mathbb{C}$, $|q| < 1$ and any positive integer n it is*

$$(a)_n = \frac{(a)_\infty}{(aq^n)_\infty}\ .$$

Proof. Let be $a, q \in \mathbb{C}$, $|q| < 1$ in $n \in \mathbb{N}$. Then (definition 3.3) it is

$$\frac{(a)_\infty}{(aq^n)_\infty} = \frac{(a;q)_\infty}{(aq^n;q)_\infty} = \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})\cdot(1-aq^n)\cdot(1-aq^{n+1})\cdots}{(1-aq^n)(1-aq^{n+1})\cdot(1-aq^{n+2})\cdot(1-aq^{n+3})\cdots},$$

and therefore

$$(1-a)(1-aq)\cdots\left(1-aq^{n-1}\right),$$

which (definition 3.3) equals $(a)_n$. □

Without proving them (here) let me introduce you to the following identities.

Lemma 3.1. *For all $q \in \mathbb{R}$, $q \neq 1$, $n \in \mathbb{N}$ it is*

$$\frac{1-q^n}{(q)_n} = \frac{1}{(q)_{n-1}}.$$

Theorem 3.4 (Cauchy). *For all $a, q, t \in \mathbb{C}$, $|q| < 1$ in $|t| < 1$ it is:*

$$1+\sum_{n=1}^{\infty} \frac{(1-a)(1-aq)\cdots(1-aq^{n-1})\,t^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1-atq^n}{1-tq^n} \tag{10}$$

Corrolary 3.1 (Euler). *For $q, t \in \mathbb{C}$, $|t| < 1$, $|q| < 1$ are true the identities*

$$1+\sum_{n=1}^{\infty} \frac{t^n}{(q)_n} = \prod_{n=0}^{\infty} \frac{1}{1-tq^n} \qquad \text{and} \tag{11}$$

$$1+\sum_{n=1}^{\infty} \frac{t^n q^{\frac{1}{2}n(n-1)}}{(q)_n} = \prod_{n=0}^{\infty} (1+tq^n)\,. \tag{12}$$

Theorem 3.5 (Jacobi triple product identity). *For $z, q \in \mathbb{C}$, $z \neq 0$ and $|q| < 1$ it is*

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n=0}^{\infty} \left[\left(1-q^{2n+2}\right)\left(1+zq^{2n+1}\right)\left(1+\frac{q^{2n+1}}{z}\right)\right]. \tag{13}$$

Corrolary 3.2. *For $q \in \mathbb{C}$, $|q| < 1$ is*

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{(2k+1)n(n+1)}{2}-in} = \sum_{n=0}^{\infty} (-1)^n q^{\frac{(2k+1)n(n+1)}{2}-in}\left(1-q^{(2n+1)i}\right) \qquad \text{and} \tag{14}$$

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{(2k+1)n(n+1)}{2}-in} = \prod_{n=0}^{\infty}\left(1-q^{(2k+1)(n+1)}\right)\left(1-q^{(2k+1)n+i}\right)\left(1-q^{(2k+1)(n+1)-i}\right). \tag{15}$$

Theorem 3.6 (Gordon). *Let be $1 \leq i \leq k$ and let $B_{k,i}(n)$ denote the number of partitions $(b_1, b_2, \ldots, b_s)$ of n, where $b_j - b_{j+k-1} \geq 2$ for $j = 1, 2, \ldots, s-k+1$ and at most $i-1$ of the parts b_u, $1 \leq u \leq s$ equal 1.*
Let $A_{k,i}(n)$ denote the number of partitions $(a_1, a_2, \ldots, a_r)$ of n, into parts a_v, $1 \leq v \leq r$ where $a_v \not\equiv 0 (mod\ 2k+1)$ and $a_v \not\equiv \pm i (mod\ 2k+1)$.
Then $A_{k,i}(n) = B_{k,i}(n)$ for all n.

Corrolary 3.3 (The first Rogers – Ramanujan identity). *The partitions of an integer* n*, in which the difference between any two parts is at least* 2*, are equinumerous with the partitions of* n *into parts* a_j*,* $a_j \not\equiv 1 (mod\ 5)$ *and* $a_j \not\equiv 4 (mod\ 5)$*.*

Corrolary 3.4 (The second Rogers – Ramanujan identity). *The partitions of an integer* n*, in which each part exceeds* 1 *and the difference between any two parts is at least* 2*, are equinumerous with the partitions of* n *into parts* a_j*,* $a_j \not\equiv 2 (mod\ 5)$ *and* $a_j \not\equiv 3 (mod\ 5)$*.*

The both Rogers – Ramanujan identities are better known in another form:

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})} \qquad (16)$$

and

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}. \qquad (17)$$

Further reading:

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