TRIANGULAR NUMBERS ARE EVERYWHERE !

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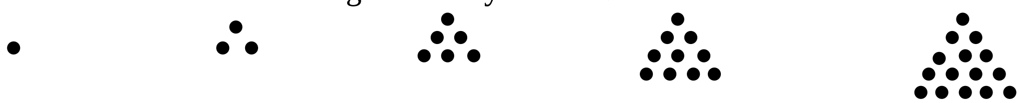
The sequence 1, 3, 6, 10, 15, . . . , $\frac{n(n+1)}{2}$, . . . shows up in many places of mathematics. To the Greeks these numbers were known as the triangular numbers due to the association with the triangular array of dots.



1 = 1   3 = 1 + 2   6 = 1 + 2 + 3   10 = 1 + 2 + 3 + 4   15 = 1 + 2 + 3 + 4 + 5

We observe that the triangular numbers can also be associated with the sums of consecutive natural numbers beginning with 1.

If we let $T_n = 1 + 2 + 3 + \ldots + n$ we can find a closed form for T_n.
For convenience T_0 is defined to be 0.

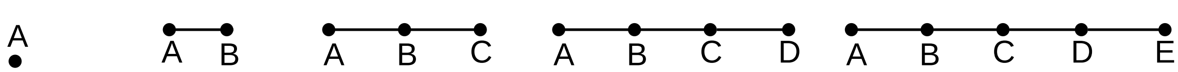
$$T_n = 1 + 2 + 3 + \ldots + (n - 1) + n$$
$$T_n = n + (n - 1) + \ldots + 2 + 1$$
$$2\,T_n = \underbrace{(n + 1) + (n + 1) + \ldots + (n + 1) + (n + 1)}_{n \text{ - groupings of } n + 1.} = n(n + 1)$$

$$T_n = \frac{n(n+1)}{2}$$

We shall now look at several examples of how triangular numbers appear in mathematical settings.

Example 1:

Complete the table :



number of points	1	2	3	4	5	6	7	n
number of line segments	0	1	3	6	10			

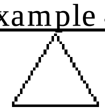
Example 2: Complete the table:

number of rays	1	2	3	4	5	6	n
number of angles	0	1	3	6			

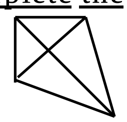
Example 3:

	number of terms	number of non-square terms
$(a)^2 = a^2$	1	0
$(a + b)^2 = a^2 + b^2 + 2ab$	3	1
$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$	6	3
$(a + b + c + d)^2 =$ _______________	____	____
$(a + b + c + d + e)^2 =$ _______________	____	____
$\left(a_1 + a_2 + \ldots + a_n\right)^2 =$ _______________	____	____

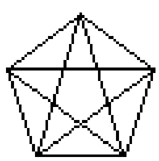
Example 4: Complete the table:



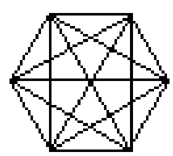
0 diagonals



2 diagonals



5 diagonals

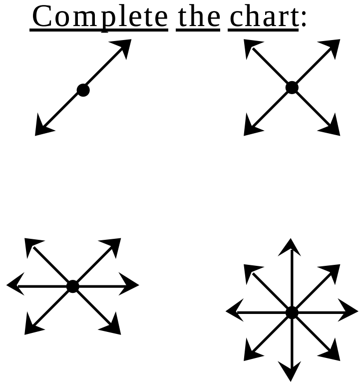


9 diagonals

number of sides of polygon	number of diagonals
3	0 = 1 - 1 = 0
4	2 = 3 - 1 = 1 + 1
5	5 = 6 - 1 = 2 + 3
6	9 = 10 - 1 = 3 + 6
7	___ = ______ = _____
8	
⋮	
n	___ = ______ = _____

Example 5:

Complete the chart:



number of straight lines	number of straight angles
1	0 = 4.0
2	4 = 4.1
3	12 = 4.3
4	24 = 4.6
5	____ = ____
6	____ = ____
n	____ = ____

Exploration 1:

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(a) Find the sum of 1 + 2 + 3 + . . . + 500

(b) Find the sum of 100 + 101 + 102 + . . . + 500

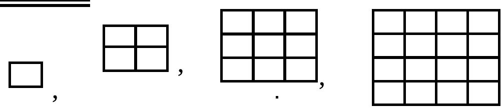
(c) Find the sum of 2 + 4 + 6 + 8 + . . . + 2n

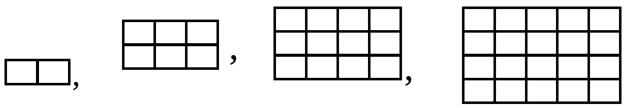
(d) Find the sum of 1 + 3 + 5 + . . . + (2n - 1)

(e) Find the sum of 1 + 3 + 5 + . . . + 99

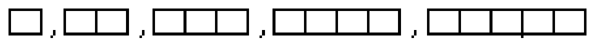
(f) Find the sum of 47 + 49 + 51 + . . . + 99

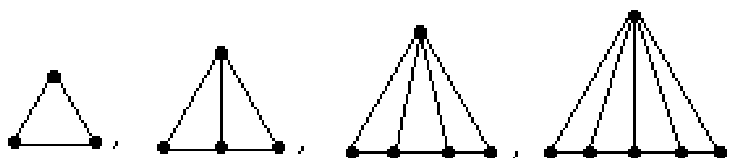
Exploration 2: Determine the number of rectangles in each checkerboard.

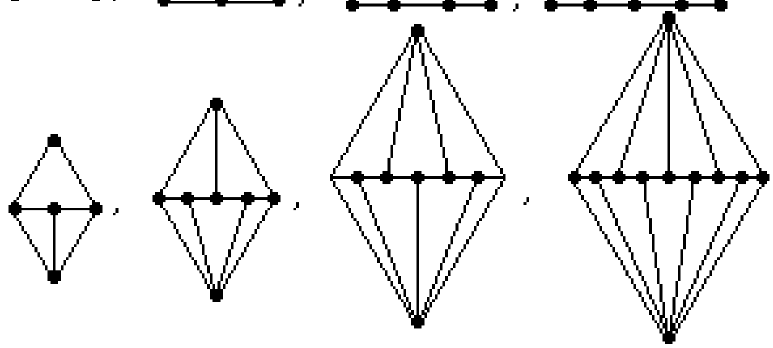
a) 

b) 

Exploration 3:

(a) Count the rectangles in each: 

(b) Count the triangles in each: 

(c) Count the triangles in each: 

Exploration 4:

Find the sum of all the numbers in the triangular array of numbers:

1
2 3
4 5 6
• • •
•
37 38 • • • 44 45