SOME PROBLEMS CONCERNING RECURRENCE SEQUENCES

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There are questions about recurrence sequences that seem to crop up again and again. Plainly, though their answers are well known they are not known well. We endeavour to explain these answers in context so that they may become more widely known.

The sequence $0, 1, -1, 2, -2, \dots$, in which each integer occurs exactly once, is a *recurrence sequence*; that is, it satisfies a linear, homogeneous recurrence relation with constant coefficients, namely,

$$a_n = -a_{n-1} + a_{n-2} + a_{n-3}.$$

It is not hard to produce a recurrence sequence in which each integer occurs exactly twice, or for that matter exactly n times, for any given n—we will show how to do this later. Can there be a recurrence sequence in which each integer occurs infinitely often? In which every rational number occurs? Every Gaussian integer? We will present the theory that enables us to answer these and many other questions about the range of a recurrence. At the pinnacle of this theory is the beautiful Skolem-Mahler-Lech Theorem, which deserves to be more widely known.

Let us first make some very general remarks about recurrence sequences. Suppose that the sequence $a_0, a_1, \dots$ satisfies the relation

$$a_{h+n} = s_1 a_{h+n-1} + \cdots + s_n a_h$$

for some complex numbers $s_1, \dots, s_n$ and for $h = 0, 1, \dots$. Taking $h = 0$, we see that a_n is in the ring $\mathbf{Z}[a_0, \dots, a_{n-1}, s_1, \dots, s_n]$. An easy induction argument shows that, in fact, all the terms in the sequence belong to this ring. Thus, the entire sequence belongs to a ring finitely generated over $\mathbf{Z}$, the integers.

It follows immediately that it is impossible for every rational number to occur in a recurrence sequence, as the rationals are not contained in any finitely generated extension of the integers.

A little more is true. If we are dealing with rational (or even algebraic) numbers then it makes sense to speak of a common denominator d_0 for the numbers $a_0, \dots, a_{n-1}$ and a common denominator d for $s_1, \dots, s_n$. It is clear by induction (or immediate by what we say below) that then the numbers $d_0 d^h a_h$ all are integers.

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1. The Skolem-Mahler-Lech Theorem

To settle the other questions raised in our opening paragraph, we must invoke the theorem of Skolem, Mahler, and Lech;

Theorem A. *If $a_0, a_1, \dots$ is a recurrence sequence, then the set of all k such that $a_k = 0$ is the union of a finite (possibly empty) set and a finite number (possibly zero) of full arithmetic progressions.*

Here, a *full arithmetic progression* means a set of the form $\{r, r+d, r+2d, \dots\}$ with $0 \le r < d$. To illustrate, consider the sequence given by the recurrence $a_{n+6} = 6a_{n+4} - 12a_{n+2} + 8a_n$, with initial conditions $(a_0, \dots, a_5) = (8, 0, 9, 0, 8, 0)$;

$$8, 0, 9, 0, 8, 0, 4, 0, 0, 0, 16, 0, 128, 0, \dots .$$

The set of k such that $a_k = 0$ is the union of the finite set $\{8\}$ and the full arithmetic progression $\{1, 3, 5, \dots\}$; in fact, the sequence is given by $a_n = 0$ if n is odd, $a_n = (n-8)^2 2^{(n-6)/2}$ if n is even.

As so often happens, the proof of the theorem involves notions rather more sophisticated than its statement; so much so, that we can give only the barest sketch here. We will first tell the story of *generalized power sums* and make some introductory remarks about *p-adic analysis*, two of the important notions underlying the proof of the Skolem-Mahler-Lech Theorem, and of interest in their own right. The reader who is willing to accept the theorem on faith and eager to see the solutions of the problems posed above can read enough of the next section to understand the notation and then skip to Section 6 for the applications. The ambitious reader may then go on to the more advanced exposition written by the second author [vdP], or the detailed proof of Theorem A given by Cassels [Cas].

We note in passing that, for our purposes, $7, 0, 0, 0, \dots$ is not a recurrence sequence; the recurrence must hold from the start. The reader will experience no difficulty in extending the results given here to recurrences that only kick in after one or more terms of a sequence.

2. Generalized power sums

A *generalized power sum* $a(h)$, $h = 0, 1, 2, \dots$ is an expression of the shape

$$a(h) = \sum_{i=1}^{m} A_i(h)\alpha_i^h \,, \ h = 0, 1, 2, \dots \tag{1}$$

with *roots* α_i, $1 \le i \le m$, which are distinct non-zero quantities, and *coefficients* $A_i(h)$ which are polynomials respectively of degree $n_i - 1$, for positive integers n_i, $1 \le i \le m$. The generalized power sum $a(h)$ is said to have *order* $n = \sum_{i=1}^m n_i$.

Set

$$s(X) = \prod_{i=1}^{m} (1 - \alpha_i X)^{n_i} = 1 - s_1 X - \cdots - s_n X^n \,. \tag{2}$$

Then the sequence (a_h) with $a_h = a(h)$, $h = 0, 1, 2, \ldots$ satisfies the recurrence relation

$$a_{h+n} = s_1 a_{h+n-1} + \cdots + s_n a_h\,, \quad h = 0, 1, 2, \ldots\,. \tag{3}$$

To see this let $E : f(h) \mapsto f(h+1)$ be the shift operator. Its properties include:
(i) $E^n\big(f(h)\big) = f(n+h)$,
(ii) $E(f+g) = E(f) + E(g)$, and
(iii) for all complex α and β,

$$(E-\alpha)(E-\beta) = (E-\beta)(E-\alpha) = E^2 - (\alpha+\beta)E + \alpha\beta\,.$$

We have

$$(E-\alpha_i)\big(A_i(h)\alpha_i^h\big) = A_i(h+1)\alpha_i^{h+1} - A_i(h)\alpha_i^{h+1} = \big(\Delta A_i(h)\big)\alpha_i^{h+1},$$

where $\Delta A_i(h) = A_i(h+1) - A_i(h)$ is a polynomial of lower degree than that of A_i. By induction, $(E-\alpha_i)^{n_i}\big(A_i(h)\alpha_i^h\big)$ is identically zero. Let P be the operator given by $P = \prod_{i=1}^m (E-\alpha_i)^{n_i}$. It follows that

$$P\big(a(h)\big) = P\sum_{j=1}^m A_j(h)\alpha_j^h = \sum_{j=1}^m P\big(A_j(h)\alpha_j^h\big) = 0.$$

But

$$P\big(a(h)\big) = (E^n - s_1E^{n-1} - \cdots - s_n)a(h) = a(h+n) - s_1a(h+n-1) - \cdots - s_na(h).$$

Thus generalized power sums correspond to the sequences satisfying the recurrence relations (3). They also correspond to the Taylor coefficients of power series expansions of rational functions. Indeed, it follows from the above that there is a polynomial $r(x)$, of degree less than n, so that the power series

$$\sum_{h=0}^\infty a_h X^h = \frac{r(X)}{s(X)} \tag{4}$$

is a rational function; to see this multiply by $s(X)$ and note the recurrence relation (3).

Conversely, suppose we are given a rational function (4) as above, and suppose $\deg r < \deg s$. A partial fraction expansion, together with the well-known identity

$$(1-Y)^{-j} = \sum_{h=0}^\infty \binom{h+j-1}{j-1} Y^h,$$

yields

$$\frac{r(X)}{s(X)} = \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{r_{ij}}{(1-\alpha_iX)^j} = \sum_{h=0}^\infty \left(\sum_{i=1}^m \sum_{j=1}^{n_i} r_{ij} \binom{h+j-1}{j-1} \alpha_i^h \right) X^h.$$

The combinatorial symbols displayed are polynomials of degree $j-1$ in h, so the coefficients of X^h, $h=0,1,2,\dots$ are indeed the values of a generalized power sum as described.

Accordingly, results on generalized power sums are equivalent to corresponding results on the Taylor coefficients of power series expansions of rational functions.

Later, we will need to deal with exponential polynomials

$$a(z)=\sum_{i=1}^{m}A_i(z)\exp(z\log\alpha_i)\,, \tag{5}$$

the continuations to $\mathbf{C}$ of generalized power sums. These are the solutions of linear differential equations with constant coefficients. To be precise, with $D=d/dz$, (5) is annihilated by the differential operator $\prod_{i=1}^{m}(D-\log\alpha_i)^{n_i}$. The order of the exponential polynomial (5) is n, as for the corresponding generalized power sum.

It is plain that an exponential polynomial vanishes identically if and only if all its coefficients vanish. We see this readily by induction on the order. Indeed, a one term exponential polynomial $A(z)\exp(z\log\alpha)$ obviously vanishes identically if and only if $A(z)$ vanishes identically. If (5) vanishes identically, then so does $(D-\log\alpha_1)a(z)$, which has order $n-1$. By the induction hypothesis all its polynomial coefficients vanish; that is for all i the polynomials $(D-\log\alpha_1+\log\alpha_i)A_i(z)$ vanish identically. Then, with the exception of the constant coefficient of A_1, all the polynomials A_i must vanish identically. Our remark about a one term exponential polynomial guarantees that also that coefficient vanishes, and we are done.

3. An application to recurrence sequences

Let us use the equivalence of recurrence relations and rational functions to produce a recurrence sequence in which each integer occurs exactly k times. We write $c^{(k)}$ for the block $c,c,\dots,c$ of length k. The sequence $0^{(k)},1^{(k)},-1^{(k)},2^{(k)},-2^{(k)},\dots$ clearly contains each integer exactly k times. The corresponding power series is

$$f(x)=x^k+\cdots+x^{2k-1}-x^{2k}-\cdots-x^{3k-1}+2x^{3k}+\dots,$$

which factors as

$$x^k(1+x+\cdots+x^{k-1}-x^k-\cdots-x^{2k-1})(1+2x^{2k}+3x^{4k}+\dots).$$

This is a rational function, since $1+2x^{2k}+3x^{4k}+\cdots=(1-x^{2k})^{-2}$. Thus, the original sequence is a recurrence sequence. With a bit more algebra, we see

$$f(x)=\frac{x^k}{(1-x)(1+x^k)^2}=\frac{x^k}{1-x+2x^k-2x^{k+1}+x^{2k}-x^{2k+1}},$$

so the sequence satisfies the relation

$$a_{h+2k+1}=a_{h+2k}-2a_{h+k+1}+2a_{h+k}-a_{h+1}+a_h,$$

together with the initial conditions $a_0=\cdots=a_{k-1}=0$, $a_k=\cdots=a_{2k-1}=1$, $a_{2k}=-1$.

4. An introduction to p-Adic analysis

The absolute value function defined on the integers has the following properties;

(i) $|x| \geq 0$ for all x,
(ii) $|x| = 0$ if and only if $x = 0$,
(iii) $|xy| = |x| \cdot |y|$ for all x and y, and
(iv) $|x + y| \leq |x| + |y|$ for all x and y.

There are other functions that have the same properties. Given any non-zero integer n, and any prime number p, we can write $n = p^a m$ with a and m integers, $a \geq 0$, and p and m relatively prime. Moreover, this expression is unique. Define the function $|\ |_p$ by $|n|_p = p^{-a}$. Thus, for example, $|35|_7 = \frac{1}{7}$, $|36|_7 = 1$, and $|36|_3 = \frac{1}{9}$. If by convention we take $|0|_p = 0$ for all p, then it is not hard to see that all the properties of $|\ |$ listed above hold for $|\ |_p$, for each p. In fact, the last property holds in a stronger form, namely,

(iv') $|x + y|_p \leq \max(|x|_p, |y|_p)$.

We call $|\ |_p$ the *p-adic absolute value.* Thinking about convergence with respect to this absolute value leads to some peculiar-looking formulas. For example, for the geometric series with first term 6 and common ratio 7, the equation

$$6 + 42 + 294 + 2058 + \cdots = -1$$

is a blunder in the usual run of things, but quite correct in the 7-adics.

The p-adic absolute value is easily continued to a function on the rational numbers, enjoying properties (i) through (iv'); any rational x can be written as $x = p^a \frac{r}{s}$ with a, r, and s integers, and r and s both relatively prime to p. Thus, $|\frac{35}{36}|_7 = \frac{1}{7}$, and $|\frac{35}{36}|_3 = 9$.

Any rational x has a unique decimal expansion, $x = \sum_{j=m}^{\infty} a_j 10^{-j}$ with a_j in $\{0, 1, \ldots, 9\}$, the series converging in the usual absolute value. So, too, for each p, any rational x has a unique p-adic expansion $x = \sum_{j=m}^{\infty} a_j p^j$ with a_j in $\{0, 1, \ldots, p-1\}$, converging in the p-adic absolute value. For example, in the 7-adics we have

$$\begin{aligned}\frac{17}{98} = 7^{-2} \cdot \frac{17}{2} = 7^{-2}\left(9 + \frac{3}{1-7}\right) &= 7^{-2}(2 + 1 \cdot 7 + 3 + 3 \cdot 7 + 3 \cdot 7^2 + \ldots) \\ &= 5 \cdot 7^{-2} + 4 \cdot 7^{-1} + 3 + 3 \cdot 7 + 3 \cdot 7^2 + \ldots,\end{aligned}$$

where we have used the geometric series expansion $\frac{1}{1-7} = 1 + 7 + 7^2 + \ldots$.

Now consider the sequence 1, 1.4, 1.41, 1.414, 1.4142, ... of decimal approximations to the square root of two. If m is less than n, then the mth and nth terms of this sequence differ by less than 10^{-m}, a quantity which goes to zero as m increases. Such a sequence is called a *Cauchy sequence* (with respect to the usual absolute value). You can't help feeling such a sequence ought to have a limit, but this one doesn't—*if* you confine yourself to the rationals [Euc]. In analysis, it is useful for Cauchy sequences to have limits, so we embed the rationals in the larger set called the *reals*. Every real number has a decimal expansion, and every Cauchy sequence converges—we say the reals are *complete*. The details of the completion process can be found in many introductory analysis texts, for example [Gle].

Now consider the sequence $7, 7+7^2, 7+7^2+7^4, 7+7^2+7^4+7^8, \dots$. In 7-adic absolute value, the difference between the mth and nth terms in this sequence is $|7^{2^m}+\cdots+7^{2^{n-1}}|_7 = 7^{-2^m}$, which goes to zero as m increases. That is to say, this is a Cauchy sequence—*if* you view it 7-adically. It ought, then, to have a limit. It is not a geometric series, so it cannot have a rational limit. By a process formally identical to the construction of the reals, we embed the rationals in a larger set we denote $\mathbf{Q}_p$, and call *the p-adic rationals*. Every p-adic rational has a p-adic expansion, and the p-adic rationals are complete.

Back to the reals. There are non-constant polynomials which have real coefficients but no real roots, for example, x^2+1. If we extend the reals to a field containing a root of x^2+1, we obtain the complex numbers. *Mirabile dictu*, every non-constant polynomial with complex coefficients has a complex root. We say that the complex numbers are *algebraically closed*. The absolute value function is continued to the complex numbers by $|a+bi| = (a^2+b^2)^{1/2}$. *Mirabile* squared, the complex numbers are complete (with respect to this absolute value). The important functions of calculus (rational, exponential, trigonometric, ...) can be continued to functions of a complex variable, and many problems about real functions become easier to handle in this larger domain.

Back to the p-adic rationals. They are not algebraically closed. For example, if α in $\mathbf{Q}_7$ were a root of $x^2-7=0$, we would have $|\alpha|_7 = 7^{-1/2}$, but if α had the 7-adic expansion $\alpha = \sum_{j=m}^{\infty} a_j 7^j$, we would have $|\alpha|_7 = 7^{-m}$, with m an integer. We can embed $\mathbf{Q}_p$ in an algebraically closed field $\overline{\mathbf{Q}}_p$, although the miracle of "add one number, get the rest free" does not occur here. We can extend $|\ |_p$ to $\overline{\mathbf{Q}}_p$, but $\overline{\mathbf{Q}}_p$ is not complete. We can complete $\overline{\mathbf{Q}}_p$ to a field $\mathbf{C}_p$, and this field is the p-adic analogue of the complex numbers; it is complete and algebraically closed. There is a rich theory of analytic functions on $\mathbf{C}_p$, mirroring that on the complex numbers.

This material can be found in less telegraphic form in [Kob].

What is really going on is this: The set of all Cauchy sequences forms a ring once we define the operations termwise; that the set is closed under the operations is a consequence of the rules (i)–(iv). One defines the field of reals (respectively p-adic rationals, according to the particular valuation defining 'Cauchy') to be this ring with sequences 'with the same limit' identified. What that means is that we take the subset of null sequences, those converging to 0, and notice again by the rules (i)–(iv) that this set is a maximal ideal in the ring of Cauchy sequences. Then the quotient ring is a field.

The 'miracle' of $\mathbf{R}$ and $\mathbf{C}$ actually is rather special. It turns out that if a field $\mathbf{F}$ is algebraically closed and if $\mathbf{L}$ is a subfield of finite codimension in $\mathbf{F}$ (in English: if $\mathbf{F}$ is a finite-dimensional vector space over some field $\mathbf{L}$) then necessarily $[\mathbf{F}:\mathbf{L}] = 2$ (compare $[\mathbf{C}:\mathbf{R}] = 2$) and $\mathbf{L}$ is an *ordered field*. That means that $\mathbf{L}$ is the disjoint union of three sets N, $\{0\}$ and P with P closed under addition and multiplication and $N = -P$; P is of course the set of *positive* elements of $\mathbf{L}$. It turns out that $\mathbf{L}$ can be ordered if and only if -1 is not a sum of squares. A complete orderable field is known a *real field* and always is a subfield of codimension 2 in an algebraically closed field; for all this see for example [L], Chapter XI. By contrast, it is not hard to see that $\mathbf{Q}_p$ is not an ordered field.

5. On proving the Skolem-Mahler-Lech Theorem

Recall that the terms of a recurrence are given by a generalized power sum,

$$a(h) = \sum_{i=1}^{m} A_i(h)\alpha_i^h \ , \ h = 0, 1, 2, \dots \tag{1}$$

Given a positive number p, every h can be written uniquely as

$$h = r + (p-1)t, \text{ with } r = 0, 1, \dots, p-2 \text{ and } t = 0, 1, 2, \dots$$

If we write $a_{p,r}(t)$ for $a(h)$, we get

$$a_{p,r}(t) = \sum A_i\bigl(r + (p-1)t\bigr)\alpha_i^r \exp(t \log \alpha_i^{p-1}). \tag{6}$$

Now it can be shown that there exist primes p such that the logarithmic and exponential functions can be continued to analytic functions on $\mathbf{C}_p$—more accurately, on regions of $\mathbf{C}_p$ large enough for the formula above to make p-adic sense for t in a closed set D containing the integers. Then $a_{p,r}(t)$ is a p-adic analytic function on D for $r = 0, 1, \dots, p-2$.

Suppose that there are infinitely many h such that $a(h) = 0$. Then there must be at least one r for which the analytic function $a_{p,r}(t)$ is zero for infinitely many integers t. Of course, there are *complex* analytic functions which are zero for infinitely many integer values of their argument; for example, $\sin \pi z$. This can occur because the integers are an unbounded set in $\mathbf{C}$. Things are different in $\mathbf{C}_p$, since the integers form a *bounded* set there; after all, $|n|_p \le 1$ for all integers n. It turns out that a function (whether complex or p-adic) analytic on a closed, bounded region and with infinitely many zeroes in that region must be identically zero. Thus, $a\bigl(r + (p-1)t\bigr)$ vanishes identically for all integer t, and in particular $a(h)$ is zero for all h in an arithmetic progression. This concludes our sketch of the proof.

It is a little strange that Theorem A should force us to enter the realm of p-adic analysis. Actually that can sort of (but not really) be avoided. It turns out that p must be selected so that $\alpha_i^p \equiv \alpha_i \bmod p$ for each i. Then (6) has no more than $n-1$ integer zeroes (so certainly not infinitely many); otherwise it vanishes identically [RvdP]. The trouble is that there seems only to be a p-adic proof for the bound.

6. Applying the Skolem-Mahler-Lech Theorem

So if there are infinitely many h such that $a(h) = 0$ then there must be at least one r for which the analytic function

$$a_{p,r}(t) = \sum A_i\bigl(r + (p-1)t\bigr)\alpha_i^r \exp(t \log \alpha_i^{p-1})$$

vanishes identically. So, by our discussion at the end of §2, since the A_i are not identically zero, the $\log\alpha_i^{p-1}$ cannot all be distinct.

Indeed, the numbers α_i^{p-1} must coincide at least in pairs. Plainly $p-1$ is not arbitrary and depends only on the roots α_i.

Moreover, we see that the original function

$$a(z)=\sum_{i=1}^{m}A_i(z)\exp(z\log\alpha_i)$$

vanishes at all $z=r+t(p-1)$ with $t\in\mathbf{Z}$. As an aside we mention that then it follows that $a(z)$ must be the product of

$$\sin\tfrac{\pi}{p-1}(z-r)=\tfrac{1}{2i}\bigl(e^{\frac{\pi i}{p-1}(z-r)}-e^{-\frac{\pi i}{p-1}(z-r)}\bigr)$$

with some other exponential polynomial. In that sense a recurrence sequence has infinitely many zeroes if and only if it is 'sinful'.

So, in particular (taking $l=p-1$, say) we have:

Proposition 1. *If a recurrence sequence vanishes infinitely often, then it vanishes on an arithmetic progression with a common difference l that depends only on the roots.*

Now suppose there is a number k such that $a(h)=k$ for infinitely many h. Let $b(h)=a(h)-k$. Then $b(h)=\sum_1^m A_i(h)\alpha_i^h-k\cdot 1^h$ is a generalized power sum with the same roots as $a(h)$ (and, possibly, the root 1 if it was not already a root of $a(h)$), hence the same l-value as $a(h)$, and $b(h)$ is zero whenever $a(h)=k$. Thus, $a(h)$ takes on the value k on an arithmetic progression with common difference l.

Now there are only l different complete arithmetic progressions of integers with common difference l. So we have established a principal remark of this note, namely,

Proposition 2. *The number of values that a recurrence sequence can take on infinitely often is bounded by some integer l that depends only on the roots.*

It follows immediately that there is no recurrence sequence in which each integer occurs infinitely often.

Nor is there a recurrence sequence in which every Gaussian integer occurs. For suppose a_h were such a sequence, and let $\sum_{h=0}^{\infty}a_hX^h=\dfrac{r(X)}{s(X)}$. Then

$$\sum_{h=0}^{\infty}\mathcal{R}\mathrm{e}(a_h)X^h=\mathcal{R}\mathrm{e}\,\frac{r(X)}{s(X)}.$$

Now it is easy to see that the real part of a rational function is again a rational function, so $\mathcal{R}\mathrm{e}(a_h)$ is a recurrence sequence, and it takes on every integer infinitely often. As we have seen, this cannot happen.

7. Multiplicity: A good question.

We restrict ourselves to recurrence sequences of integers. By the results just explained an integer recurrence sequence either takes the value 0 infinitely many times, in which case it has special properties that allow us to say it is *degenerate*, or only finitely many times. Is there a bound $\mu(n)$ so that a nondegenerate integer recurrence sequence of order n has at most $\mu(n)$ zeroes? Of course any given nondegenerate integer recurrence sequence has a bound on its number of zeroes. Our question is whether there is a *uniform* bound for the *multiplicity*, depending only on the order of the sequence.

It is obvious that $\mu(2) = 1$. (Truly. Give this a few minutes thought.) The bound $\mu(3) = 6$ is very much more difficult and has only been confirmed recently [Beu]. The extreme case is

$$a_{h+3} = 2a_{h+2} - 4a_{h+1} + 4a_h\,, \quad a_0 = a_1 = 0\,,\ a_2 = 1\,.$$

Its six zeroes are $a_0 = a_1 = a_4 = a_6 = a_{13} = a_{52} = 0$.

For larger n there are not even any worthwhile conjectures. The problem deserves some computer time, say at least so as to guess $\mu(4)$ (which is ≥ 9).

8. Recurrence

The question, whether there is a recurrence sequence in which each rational occurs, was raised in *Crux Mathematicorum* in October, 1989.

Proposition 2 was published in 1959 by Shapiro [Sha], and again some years later by Berstel and Mignotte [Ber]. The question, whether there is a recurrence sequence in which each Gaussian integer occurs infinitely often, was posed in *Crux Mathematicorum* in June, 1988, and repeated in October 1989. These sequences are recurrent in more ways than one! Indeed, Theorem A for recurrence sequences of algebraic numbers was first proved by Mahler in the 30's, based upon an idea of Skolem. Then, Lech published the result for general recurrence sequences in 1953. In 1956 Mahler published the same result, apparently independently (but later realised to his chagrin that he had actually reviewed Lech's paper some years earlier, but had forgotten it).

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The challenge that Gerry presented,
Was one that Alf rather resented;
Alf wanted to write,
In prosaical light,
In the hope it'd mean that he meant 't.

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