CONTINUED FRACTION EXPANSIONS OF VALUES OF THE EXPONENTIAL FUNCTION AND RELATED FUN WITH CONTINUED FRACTIONS

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It is well known that one can obtain explicit continued fraction expansions of e^z for various interesting values of z; but the details of appropriate constructions are not widely known. We provide a reminder of those methods and do that in a way that allows us to mention a number of techniques generally useful in dealing with continued fractions. Moreover, we choose to consider some expansions in Gaussian integers, allowing us to display some new results and to indicate some generalisations of classical results.

1. Introduction

We have fun with continued fractions. That 'fun' is intended to demystify a variety of simple facts often disguised in the literature, or proved by turgid methods. In particular we provide some brief notes sketching an explanation for some of the well known continued fraction expansions of e^z for special values of z. Our remarks are motivated by suggestions [5] of Jerome Minkus on Gaussian integer continued fraction expansions of certain complex values of the exponential function. With his permission we mention and give our proof of his new results and conjectures. Once again we show the power and congeniality of the matrix methods inspired, for this author, by observations of Stark [10] and reintroduced in [7] and [8]. These methods had of course been used earlier, for example by Walters [11]; indeed, precisely in the context of the exponential function. We make liberal use of Walters' remarks below. Incidentally, the earliest mention of these matrix methods for continued fractions which I have looked at is Kolden [4]; scholarly colleagues also remind me to mention Frame [2]. I am occasionally asked who 'invented' the matrix approach to continued fractions. I feel that the correct answer is that the approach was invented by those who invented matrices. It is after all obvious to anyone who has learned matrix notation that the traditional recurrence formulas for the convergents insist on a matrix formulation.

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2. First principles

Suppose we are given a sequence of *arbitrary* nonsingular 2×2 matrices, say with complex elements, or even polynomials over $\mathbb{C}$,

$$\begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \qquad n=0,1,2,\ldots$$

with the property that the two limits $\lim_{n\to\infty} A_n/C_n$ and $\lim_{n\to\infty} B_n/D_n$ are equal to one another. Suppose the common limit is α. Then we will say that the sequence *yields an expansion* for α.

We claim that a sequence of 2×2 matrices yielding an expansion for α yields a formal continued fraction expansion of α. Indeed, there is no loss of generality in arranging that the matrices be unimodular, since multiplying each matrix of the sequence by some complex $k_n\neq 0$ does not change α, and then those matrices have a decomposition as a product of elementary unimodular matrices corresponding to a continued fraction expansion. We remind the reader that:

A continued fraction is an expression of the shape

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\ddots}}}$$

which one denotes in a space-saving flat notation by

$$[a_0\,,\,a_1\,,\,a_2\,,\,a_3\,,\,\ldots\ldots]\,.$$

Everything one needs now follows from the correspondence whereby

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \quad \text{for } n=0,\ 1,\ 2,\ \ldots\ldots$$

entails

$$\frac{p_n}{q_n}=[a_0\,,\,a_1\,,\,\ldots\ldots\,,\,a_n] \quad \text{for } n=0,\ 1,\ 2,\ \ldots\ldots\ ;$$

and conversely, up to the ambiguities in choosing the p_n and q_n given just their quotients p_n/q_n. If the a_n all are positive integers for $n\geq 1$ it suffices that p_n and q_n be chosen relatively prime.

Our remark is now no more than the observation that every unimodular matrix has decompositions corresponding, in the sense just indicated, to a continued fraction expansion; and this is not intended to be more than the remark that a unimodular matrix can be written as a product of elementary row transformations.

3. The exponential function

It is not too difficult to verify directly that the continued fraction expansion of e^z corresponds, in the sense just described, to the sequence of matrices

$$\begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} = \prod_{h=0}^{n} \begin{pmatrix} (2h+1)+z & (2h+1) \\ (2h+1) & (2h+1)-z \end{pmatrix}. \tag{1}$$

Indeed, $A_nD_n - B_nC_n = (-1)^{n+1}z^{2(n+1)}$ shows that the formal power series $\lim A_n/C_n$ and $\lim B_n/D_n$ coincide. One sees readily that $A_n(z) = D_n(-z)$ and $B_n(z) = C_n(-z)$ and one confirms more laboriously that

$$n!A_n(z)/(2n+1)! = e^{\frac{1}{2}z}\big(1 + \frac{z}{2(n+1)} - \frac{z^2}{8(2n+1)} + \dots \qquad \big)$$

and

$$n!B_n(z)/(2n+1)! = e^{\frac{1}{2}z}\big(1 - \frac{z}{2(n+1)} - \frac{z^2}{8(2n+1)} + \dots \qquad \big).$$

For further details see Walters [11].

Since

$$\begin{pmatrix} 2h+2 & 2h+1 \\ 2h+1 & 2h \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2h & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

we immediately obtain the expansion

$$e = [1\,,0\,,1\,,1\,,2\,,1\,,1\,,4\,,1\,,1\,,6\,,1\,,\dots].$$

Noting that

$$\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x+y & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

we have the familiar expansion

$$e - 1 = [1\,,1\,,2\,,1\,,1\,,4\,,1\,,1\,,6\,,\dots] = [\overline{1\,,1\,,2h}\,]_{h=1}^{\infty}.$$

Here the overline indicates quasi-periodicity with the variable h sequentially taking the values $1,\ 2,\ \dots$ with each repetition of the quasi-period.

4. Digression: Fun with 2×2 matrices

Everyone knows about the transpose

$$A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \text{ of a matrix } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

but, correctly, less well known[1] is the *false transpose*

$$A' = \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

Writing $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, we see that $JAJ = A'^t = A^{t\prime}$ is appropriately called the *double transpose* of A and then $A' = (JAJ)^t$ readily yields such properties of the false transpose as $(AB)' = B'A'$.

We already know that continued fraction expansions correspond to matrix products

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

In the sequel it will be convenient to define

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \text{ and to note that } JL^x = R^x J = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, observing that $J^2 = I$, we see that continued fraction expansions also correspond to so-called RL–sequences in the sense that

$$\begin{aligned} &\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\cdots\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}\cdots = \\ &\qquad = R^{a_0}J\cdot JL^{a_1}R^{a_2}J\cdot\cdots = R^{a_0}L^{a_1}R^{a_2}\cdots\ . \end{aligned}$$

5. Some expansions in Gaussian integers

We notice the decomposition

$$\begin{aligned} &\begin{pmatrix} (2a+1)+2i & (2a+1) \\ (2a+1) & (2a+1)-2i \end{pmatrix} = \\ &\qquad = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -ia & 1 \end{pmatrix}\begin{pmatrix} 1 & -1+i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1+i & -1-i \\ 1 & 1-2i \end{pmatrix}, \end{aligned}$$

whence alternatively it also equals

$$\begin{pmatrix} 1+2i & -1+i \\ 1 & 1-i \end{pmatrix}\begin{pmatrix} 1 & -1-i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ ia & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Of course one need do no more than check a claim of this kind, yet it warrants explanation. The first decomposition is a more or less natural row decomposition;

[1]Cynical remark to be omitted by publications that believe that mathematics cannot be fun: However, it seems that 32% of undergraduate students cannot distinguish the false transpose and the transpose; if students are explicitly warned to avoid this blunder this number increases to 45%. One also must not tell students that the double transpose has the congenial property that the double transpose of a product is the product of the double transposes without any bothersome reversal of order.

the fun way to obtain the second from it is to note that the conjugate of the given matrix is its false transpose.

Next we observe that

$$\begin{pmatrix} 1+i & -1-i \\ 1 & 1-2i \end{pmatrix}\begin{pmatrix} b+2i & b \\ b & b-2i \end{pmatrix}\begin{pmatrix} 1+2i & -1+i \\ 1 & 1-i \end{pmatrix} = -8\begin{pmatrix} 1 & 0 \\ -b+1 & 1 \end{pmatrix}.$$

Finally let $z = 2i$ in (1) and take its matrices in threes to obtain

$$e^{2i} \longleftrightarrow \prod_{k=0}^{\infty} RL^{-3ki}R^{-1+i}L^{-6k-2}R^{-1-i}L^{(3k+2)i}R\,,$$

which corresponds to

Theorem 1. (Minkus [5])

(3) $$e^{2i} = [1\,,\,\overline{-3hi\,,\,-1+i\,,\,-6h-2\,,\,-1-i\,,\,(3h+2)i\,,\,2}\,]_{h=0}^{\infty}\,.$$

The experienced reader will recognise, however, that hindsight assisted the quoted calculation; it did: it is a formula demonstrated by less friendly methods in [5].

Might one have discovered (3) spontaneously? To that end consider a continued fraction expansion of

$$\coth 2z = (e^{2z}+1)/(e^{2z}-1).$$

Our remarks show that $\coth 2z$ corresponds to

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\prod_{h=0}^{n}\begin{pmatrix} (2h+1)+2z & (2h+1) \\ (2h+1) & (2h+1)-2z \end{pmatrix},$$

and because

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} (2h+1)+2z & (2h+1) \\ (2h+1) & (2h+1)-2z \end{pmatrix}$$
$$= 2\begin{pmatrix} (2h+1)+z & (2h+1)-z \\ z & z \end{pmatrix} = 2z\begin{pmatrix} (2h+1)/z & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

this immediately yields a formula known to Lambert (see [4]), that

(4) $$\frac{e^{2z}+1}{e^{2z}-1} = [\,\overline{(2h+1)/z}\,]_{h=0}^{\infty}\,.$$

Taking this as our starting point, we seek to recover an expansion for e^{2i} by multiplying the continued fraction expansion (4) by the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

We do that by following suggestions of Raney [9], and to that end, recalling (4), interpret our task as being to transform the matrix product

$$R^{-1}\begin{pmatrix}2&0\\0&-1\end{pmatrix}L^{-1}R^{-i}L^{-3i}R^{-5i}L^{-7i}R^{-9i}L^{-11i}\cdots$$

so that it once again corresponds to a continued fraction expansion, to wit to an RL–sequence consisting just of (Gaussian) integer powers of R and of L. We do that by applying appropriate *transition formulae* to move the offending matrix through the RL-sequence until it disappears in the $\ldots$ on the right. This will be less complex once we see that we may write

$$\begin{pmatrix}1&0\\0&-1\end{pmatrix}L^{-1}R^{-i}L^{-3i}R^{-5i}L^{-7i}R^{-9i}L^{-11i}\cdots = \\ = \begin{pmatrix}i&0\\0&1\end{pmatrix}L^{i}RL^{-3}R^{5}L^{-7}R^{9}L^{-11}\cdots .$$

The trick is to notice that multiplying a continued fraction $[a\,,\,b\,,\,c\,,\,d\,,\,e\,,\ldots]$ by x yields $[ax\,,\,b/x\,,\,cx\,,\,d/x\,,\,ex\,,\ldots]$. We have multiplied by i and have neglected a $-I$, taking advantage of the fact that the $=$ sign can only mean that the two sequences of matrices yield the same continued fraction expansion.

We set

$$A_+=\begin{pmatrix}1+i&0\\0&1\end{pmatrix};\ \text{so its conjugate is } A_-=\begin{pmatrix}1-i&0\\0&1\end{pmatrix}.$$

We apply the transition formulæ[2]

$$\begin{aligned} A_+R &=R^{1+i}A_+ & \qquad A'_+L &=L^{1+i}A'_+\\ A_+L^{1+i} &=LA_+ & A'_+R^{1+i} &=RA'_+\\ A_+L^iR &=RL^iA'_+ & A'_+R^iL &=LR^iA_+\\ A_+LR^i &=R^iLA'_+ & A'_+RL^i &=L^iRA_+ \end{aligned}$$

and their conjugates to obtain sequentially the *transductions*

$$\begin{array}{ccccccccc} L^iR & L^{-3} & R^4 & RL^i & L^{-7-i} & R^9 & L^{-11-i} & L^iR & \ldots\\ A_+ & A'_+ & A'_+ & A'_+ & A_+ & A_+ & A_+ & A_+ & A'_+\\ RL^i & L^{-3-3i} & R^{2-2i} & L^iR & L^{-4+3i} & R^{9+9i} & L^{-6+5i} & RL^i & \ldots \end{array}$$

and then

$$\begin{array}{cccccccccccccc} R & L^{-3-i} & L^{-i}R & R^{1-i} & R^{-i}L & L^{-1+i} & R & L^{-4+4i} & L^{-i}R & R^{8+10i} & R^{-i}L & L^{-7+5i} & \ldots\\ A_- & A_- & A_- & A'_- & A'_- & A_- & A_- & A_- & A_- & A'_- & A'_- & A_- & A_-\\ R^{1-i} & L^{-1-2i} & RL^{-i} & R & LR^{-i} & L^{-1} & R^{1-i} & L^{-4} & RL^{-i} & R^{-1+9i} & LR^{-i} & L^{-6-i} & \ldots \end{array}$$

[2]By the way, these require only a few calculations, since they are mostly transposes and or false transposes one of the other.

telling us that

$$\begin{aligned} e^{2i} &\longleftrightarrow R^{-1}\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} A_- A_+ L^i R L^{-3} R^5 L^{-7} R^9 L^{-11}\cdots \\ &= R^{-1}\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} R^{1-i} L^{-1-2i} R L^{-i} R L R^{-i} L^{-1} R^{1-i} L^{-4} R L^{-i} R^{-1+9i} L R^{-i} L^{-6-i}\cdots \\ &= R^i L^{-2+i} R^i L^{-1} R^i L^{-i} R L^i R^{1+i} L^{4i} R^i L^{-1} R^{-9-i} L^{-i} R L^{-1+6i}\cdots \end{aligned}$$

which corresponds to the continued fraction expansion

$$[i\,,\,-2+i\,,\,i\,,\,-1\,,\,i\,,\,-i\,,\,1\,,\,i\,,\,1+i\,,\,4i\,,\,i\,,\,-1\,,\,-9-i\,,\,-i\,,\,1\,,\,-1+6i\,,\,\ldots]\,.$$

The trouble is that this is not obviously the same result as (3).

6. Digression: Fun with continued fractions

The curious calculation

$$\begin{aligned} -\alpha &= 0 + -\alpha \\ -1/\alpha &= -1 + (\alpha-1)/\alpha \\ \alpha/(\alpha-1) &= 1 + 1/(\alpha-1) \\ \alpha - 1 &= -1 + \alpha \\ 1/\alpha &= 0 + 1/\alpha \\ \alpha &= \alpha \end{aligned}$$

says that

$$\begin{aligned} &[\ldots\,,\,a\,,\,-b\,,\,\gamma] = \\ &\qquad = [\ldots\,,\,a\,,\,0\,,\,-1\,,\,1\,,\,-1\,,\,0\,,\,b\,,\,-\gamma] = [\ldots\,,\,a-1\,,\,1\,,\,b-1\,,\,-\gamma]. \end{aligned}$$

Its converse,

$$[A\,,\,1\,,\,B\,,\,\Gamma] = [A+1\,,\,-B-1\,,\,-\Gamma]$$

is just a special case. Almost identical trickery shows that also

$$\begin{aligned} &[\ldots\,,\,a\,,\,-b\,,\,\gamma] = \\ &\qquad = [\ldots\,,\,a\,,\,0\,,\,1\,,\,-1\,,\,1\,,\,0\,,\,b\,,\,-\gamma] = [\ldots\,,\,a+1\,,\,-1\,,\,b+1\,,\,-\gamma], \end{aligned}$$

with converse

$$[A\,,\,-1\,,\,B\,,\,\Gamma] = [A-1\,,\,-B+1\,,\,-\Gamma]$$

The imaginary analogue of the first formula is just

$$\begin{aligned} (5)\quad &[a\,,\,b\,,\,\gamma] = i[-ia\,,\,ib\,,\,-i\gamma] = \\ &\qquad = i[-ia-1\,,\,1\,,\,-ib-1\,,\,i\gamma] = [a-i\,,\,-i\,,\,b-i\,,\,\gamma] \end{aligned}$$

and its converse conjugate is,

$$[A\,,\,i\,,\,B\,,\,\Gamma] = [A-i\,,\,B-i\,,\,\Gamma]\,.$$

7. Related results

With some care the reader will succeed in showing that such transductions do provide a systematic fairly uncontrived method for obtaining the expansion (3). Specifically,

$$[i\,,\,-2+i\,,\,i\,,\,-1\,,\,i\,,\,-i\,,\,1\,,\,i\,,\,1+i\,,\,4i\,,\,i\,,\,-1\,,\,-9-i\,,\,-i\,,\,1\,,\,-1+6i\,,\,\ldots]$$

plainly commences $[1\,,\,0\,,\,-1+i\,,\,\ldots]$. That's good because we're claiming to be rediscovering the expansion

$$[1\,,\,\overline{-3hi\,,\,-1+i\,,\,-6h-2\,,\,-1-i\,,\,(3h+2)i\,,\,2}\,]_{h=0}^{\infty}\,.$$

Next

$$[-2+i\,,\,i\,,\,-1\,,\,i\,,\,\ldots]=[-2\,,\,-1-i\,,\,i\,,\,\ldots]$$

and

$$[i\,,\,-i\,,\,1\,,\,1+i\,,\,4i\,,\,\ldots]=[2i\,,\,1+i\,,\,i\,,\,1+i\,,\,4i\,,\,\ldots]=[2i\,,\,1\,,\,1\,,\,4i\,,\,\ldots]\,.$$

Fortunately

$$\begin{aligned}[1\,,\,1\,,\,4i\,,\,i\,,\,-1\,,\,-9-i\,,\,\delta]&=[2\,,\,-1-4i\,,\,-i\,,\,1\,,\,9+i\,,\,-\delta]\\ &=[2\,,\,-1-3i\,,\,1\,,\,0\,,\,i\,,\,9+i\,,\,-\delta]=[2\,,\,-3i\,,\,-1\,,\,-i\,,\,-9-i\,,\,\delta]\\ &=[2\,,\,-3i\,,\,-1+i\,,\,-9\,,\,\delta]\,.\end{aligned}$$

Moreover

$$\begin{aligned}[-9\,,\,-i\,,\,1\,,\,-1+6i\,,\,\varepsilon]&=[-8\,,\,-1\,,\,1+i\,,\,-1\,,\,1-6i\,,\,-\varepsilon]\\ &=[-8\,,\,-1\,,\,i\,,\,6i\,,\,\varepsilon]=[-8\,,\,-1-i\,,\,5i\,,\,\varepsilon]\end{aligned}$$

and we've pretty well completed the second quasi-period.

Of course, if one were set on obtaining (3) one would avoid tedious applications of this 'fun with continued fractions' by contriving more appropriate transition formulæ, but that would miss the point.

Our instancing Raney's remark that linear fractional transformations of a continued fraction may be effected by one or more finite state transductions of a corresponding RL–sequence should readily yield a proof of a generalisation, as suggested by Minkus, of a theorem of Hurwitz [3] according to which if α has a continued fraction expansion of the shape

$$[a_0\,,\,a_1\,,\,\ldots\,,\,a_p\,,\,\overline{f_1(h)\,,\,f_2(h)\,,\,\ldots\,,\,f_r(h)}\,]_{h=0}^{\infty}\,,$$

with polynomials f_1, …, f_r taking integer values at nonnegative integers h, then each linear fractional transformation $(a\alpha+b)/(c\alpha+d)$ of α with $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in$ $\mathrm{GL}_2(\mathbb{Z})$ has a continued fraction expansion of the same form. The generalisation replaces $\mathbb{Z}$ by the ring of integers of (or indeed, by any order of) an algebraic number field. Our hint of the argument relies upon observing that a unimodular transformation of α changes only the entries preceding its quasi-period, and our example illustration that multiplication by matrices $\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)$ corresponds to finite state transductions of an RL–sequence should easily allow a verification that it inter alia preserves the property of present interest.

As an example, we mention the continued fraction expansion

Theorem 2.

$$\frac{e^{2i}+i}{e^{2i}-i} = i[-5\,,\,2\,,\,\overline{2\,,\,3k+2\,,\,1\,,\,12k+16\,,\,1\,,\,3k+4}\,]_{k=0}^{\infty}\,. \tag{6}$$

We might obtain this from (3) by multiplying the corresponding matrix products by $M=\left(\begin{smallmatrix}1 & i\\ 1 & -i\end{smallmatrix}\right)$. However, it happens that first principles are more convenient and we note that

$$\begin{aligned} M\begin{pmatrix} a+2i & a\\ a & a-2i\end{pmatrix}M^{-1} &= \begin{pmatrix} a & ia+2i\\ -ia+2i & a\end{pmatrix}\\ &= \begin{pmatrix}1 & i\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -ih & 1\end{pmatrix}\begin{pmatrix}1 & 2i\\ 0 & 1\end{pmatrix}\begin{pmatrix}4 & 0\\ i & 1\end{pmatrix},\\ \text{with } 2h+1=a\text{; and also}\qquad &= \begin{pmatrix}1 & 0\\ i & 4\end{pmatrix}\begin{pmatrix}1 & 2i\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -ih & 1\end{pmatrix}\begin{pmatrix}1 & i\\ 0 & 1\end{pmatrix}\end{aligned}$$

by false transposition.

Moreover

$$\begin{pmatrix}4 & 0\\ i & 1\end{pmatrix}\begin{pmatrix} b & ib+2i\\ -ib+2i & b\end{pmatrix}\begin{pmatrix}1 & 0\\ i & 4\end{pmatrix} = -8\begin{pmatrix}1 & -2i(b+2)\\ 0 & 1\end{pmatrix}.$$

So, taking the matrices in threes yields

$$\begin{aligned}\frac{e^{2i}+i}{e^{2i}-i} &\longleftrightarrow \begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}\prod_{h=0}^{\infty}\begin{pmatrix}(2h+1)+2i & 2h+1\\ 2h+1 & (2h+1)-2i\end{pmatrix}\\ &\longleftrightarrow \prod_{k=0}^{\infty} R^{i}L^{-3ki}R^{2i}R^{-2((6k+3)+2)i}R^{2i}L^{-(3k+2)i}\\ &\longleftrightarrow i[1\,,\,\overline{3k\,,\,-2(6k+3)\,,\,3k+2}\,]_{k=0}^{\infty}\,.\end{aligned}$$

We now employ ‘fun with continued fractions’ to obtain (6). An expansion equivalent to (6) is conjectured in [5].

8. Concluding remarks

It is useful to recall the remarks of Walters [11] and to apply those methods to retrieving complex analogues suggested by Minkus [5] of classical results of Euler and Stieltjes; see Perron [6]. Our methods also provide an opportunity to mention ideas of Raney [9] and to demonstrate the utility of a number of simple transformations of continued fractions that we have labeled ‘fun with ... ’.

It may not be evident from our remarks — which appear to cite just isolated wonders — just why $e^{1/q}$ and $e^{2/q}$, and $e^{i/q}$ and $e^{2i/q}$, should have continued fraction expansions of Hurwitz type whilst other powers of e seemingly do not (see, for example, [1]). A review of our examples reveals, however, that taking the matrices

$$\begin{pmatrix}(2h+1)q+x & (2h+1)q\\ (2h+1)q & (2h+1)q-x\end{pmatrix}$$

three at a time if $|x|=2$ (or just one at a time if $|x|=1$) gives 8 times a unimodular matrix (respectively, a unimodular matrix), whilst, as [1] shows for the case $x=3$, apparently no collecting the matrices for other values of $|x|$ yields products of unimodular matrices multiplied just by a constant.

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