The number $a_{n,k}$ of minimal covers of an unlabeled n-set that cover k points of that set uniquely

Define a unit column of a binary matrix to be a column with only one 1.
Let $b_{m,n,k}$ be the number of m x n binary matrices with k=0, 1, ..., n unit columns, up to row and column permutations and $b_{m,n}(x)=\sum_{k=0}^{n} b_{m,n,k}x^k$. Also let $Z(S_n;x_1,x_2,...,x_n)$ denote the cycle index of symmetric group S_n of degree n. Then

$$b_{m,n}(x)=\frac{1}{m!}\sum_{\pi(m)}\frac{m!}{k_1!1^{k_1}\,k_2!2^{k_2}...k_m!m^{k_m}}\;Z(S_n;c_1(\pi,x),c_2(\pi,x),...,c_n(\pi,x)),$$

where $\pi(m)$ runs through all partitions of m (i.e. nonnegative solutions of $k_1+2k_2+...+mk_m=m$) ;

$$c_j(\pi,x)=(2^{\sum_{i=1}^{m}(j,i)k_i}-\sum_{i|j}ik_i)+\sum_{i|j}ik_i\cdot x^j\ ,\ j=1,2,\ldots,n,$$

where (j,i) is the greatest common divisor of j and i.

Specially for small values of m and n we have:

$$b_{0,n}(x)=1$$

$$b_{1,n}(x)=Z(S_n;1+x,1+x^2,1+x^3,...)$$

$$b_{2,n}(x)=\frac{1}{2!}(Z(S_n;2+2x,2+2x^2,2+2x^3+...)+Z(S_n;2,2+2x^2,2,2+2x^4,...))$$

$$b_{3,n}(x)=\frac{1}{3!}(Z(S_n;5+3x,5+3x^2+...)+3Z(S_n;3+x,5+3x^2,3+x^3,5+3x^4,...)+$$
$$+2Z(S_n;2,2,5+3x^3,2,2,5+3x^6,...))$$

$$b_{4,n}(x)=\frac{1}{4!}(Z(S_n;12+4x,12+4x^2,...)+8Z(S_n;3+x,3+x^2,12+4x^3,3+x^4,3+x^5,12+4x^6,...)+$$
$$+6Z(S_n;6+2x,12+4x^2,6+2x^3,12+4x^4,...)+3Z(S_n;4,12+4x^2,4,12+4x^4,...)+$$
$$+6Z(S_n;2,4,2,12+4x^4,2,4,2,12+4x^8,...))$$

$$b_{5,n}(x)=\frac{1}{5!}(Z(S_n;27+5x,27+5x^2,...)+10Z(S_n;13+3x,27+5x^2,13+3x^3,27+5x^4,...)+$$
$$+15Z(S_n;7+x,27+5x^2,7+x^3,27+5x^4,...)+20Z(S_n;6+2x,6+2x^2,27+5x^3,6+2x^4,6+2x^5,27+5x^6,...)+$$
$$+20Z(S_n;4,6+2x^2,13+3x^3,6+2x^4,4,27+5x^6,4,6+2x^8,13+3x^9,6+2x^{10},4,27+5x^{12},...)+$$
$$+30Z(S_n;3+x,7+x^2,3+x^3,27+5x^4,3+x^5,7+x^6,3+x^7,27+5x^8,...)+$$
$$+24Z(S_n;2,2,2,2,27+5x^5,2,2,2,2,27+5x^{10},...)).$$

$b_{0,n}(x) = 1$

$b_{1,n}(x) = 1 + x + x^2 + x^3 + ... + x^n$

$b_{2,0}(x) = 1$
$b_{2,1}(x) = 2 + x$
$b_{2,2}(x) = 3 + 2x + 2x^2$
$b_{2,3}(x) = 4 + 3x + 4x^2 + 2x^3$
$b_{2,4}(x) = 5 + 4x + 6x^2 + 4x^3 + 3x^4$
$b_{2,5}(x) = 6 + 5x + 8x^2 + 6x^3 + 6x^4 + 3x^5$, *etc.*

$b_{3,0}(x) = 1$
$b_{3,1}(x) = 3 + x$
$b_{3,2}(x) = 7 + 4x + 2x^2$
$b_{3,3}(x) = 14 + 11x + 8x^2 + 3x^3$
$b_{3,4}(x) = 25 + 24x + 22x^2 + 12x^3 + 4x^4$
$b_{3,5}(x) = 41 + 46x + 48x^2 + 33x^3 + 17x^4 + 5x^5$, *etc.*

$b_{4,0}(x) = 1$
$b_{4,1}(x) = 4 + x$
$b_{4,2}(x) = 14 + 6x + 2x^2$
$b_{4,3}(x) = 44 + 27x + 13x^2 + 3x^3$
$b_{4,4}(x) = 127 + 102x + 62x^2 + 21x^3 + 5x^4$
$b_{4,5}(x) = 335 + 333x + 239x^2 + 105x^3 + 35x^4 + 6x^5$, *etc.*

Let $\overline{b}_{m,n}(x) = b_{m,n}(x) - b_{m,n-1}(x)$. If $a_{n,k}$ is the number of minimal covers of an unlabeled n-set that cover k points of that set uniquely and $a_n(x) = \sum_{k=0}^{n} a_{n,k} x^k$ then

$$a_n(x) = \overline{b}_{0,n}(x) + x\overline{b}_{1,n-1}(x) + x^2\overline{b}_{2,n-2}(x) + x^3\overline{b}_{3,n-3}(x) + x^4\overline{b}_{4,n-4}(x) + x^5\overline{b}_{5,n-5}(x) + ... \; .$$

Example:

$$a_5(x) = x \cdot x^4 + x^2(1 + x + 2x^2 + 2x^3) + x^3(4 + 3x + 2x^2) + x^4(3 + x) + x^5 =$$
$$= x^2 + 5x^3 + 8x^4 + 7x^5.$$