# Formula for the number of $m \times m$ nonnegative integer matrices with all row sums equal to $n$, up to row and column permutation 

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Let $Z\left(S_{m}^{\{n\}} ; x_{1}, x_{2}, \ldots\right)$ be the cycle index of (unrestricted disordered) $n$-ary symmetric group $S_{m}$ of degree $m$ (i.e. the group of permutations of all $n$-multisubsets of an $m$-set, induced by $S_{m}$ ), which can be calculated in the following way:
$Z\left(S_{m}^{\{n\}} ; x_{1}, x_{2}, \ldots\right)=\frac{1}{m!} \sum_{\pi(m)} \frac{m!}{k_{1}!1^{k_{1}}} k_{2}!2^{k_{2}} \ldots k_{m}!m^{k_{m}} \cdot \prod_{i \mid k} x_{i}^{e_{i}}$,
where $\pi(m)$ runs through all partitions of $m$ (i.e. nonnegative solutions of
$\left.k_{1}+2 k_{2}+\ldots+m k_{m}=m\right)$;
$k=\operatorname{lcm}\left\{i \mid k_{i} \neq 0\right\}$;
$e_{i}=e_{i}(\pi, n)=\frac{1}{i} \sum_{d \mid i} \mu\left(\frac{i}{d}\right) \cdot \sum \prod_{l=1}^{n}\binom{(l, d) k_{l}+t_{l}-1}{t_{l}}$,
where $\mu$ is Mobius function and the last sum is taken over all nonnegative solutions of $t_{1} \frac{1}{(1, d)}+t_{2} \frac{2}{(2, d)}+\ldots+t_{n} \frac{n}{(n, d)}=n$.
Let $Z\left(S_{m}^{\{n\}} ; 1 /(1-x)\right)=Z\left(S_{m}^{\{n\}} ; 1+x+x^{2}+\ldots, 1+x^{2}+x^{4}+\ldots, 1+x^{3}+x^{6}+\ldots, \ldots\right)$, i. e. $Z\left(S_{m}^{\{n\}} ; 1 /(1-x)\right)$ is obtained if we replace $x_{i}$ by $1+x^{i}+x^{2 i}+x^{3 i}+\ldots, i=1,2, \ldots$, in the cycle index $Z\left(S_{m}^{\{n\}} ; x_{1}, x_{2}, \ldots\right)$.
Then the number of of $m \mathrm{x} m$ nonnegative integer matrices with all row sums equal to $n$ is the coefficient of $x^{m}$ in $Z\left(S_{m}^{\{n\}} ; 1 /(1-x)\right)$.

