Formula for the number of *m* x *l* nonnegative integer matrices with all column sums equal to *n*, up to row and column permutation

Vladeta Jovović

Faculty of Technology and Metallurgy, University of Belgrade, Karnegijeva 4, P.O. Box 494, 11001 Belgrade, Yugoslavia. Email: vladeta@Eunet.yu
November 2000.

Let $Z(S_m^{\{n\}}; x_1, x_2, ...)$ be the cycle index of (unrestricted disordered) *n*-ary symmetric group S_m of degree *m* (i.e. the group of permutations of all *n*-multisubsets of an *m*-set, induced by S_m), which can be calculated in the following way:

$$Z(S_m^{\{n\}}; x_1, x_2, ...) = \frac{1}{m!} \sum_{\pi(m)} \frac{m!}{k_1! 1^{k_1} \, k_2! 2^{k_2} ... k_m! m^{k_m}} \cdot \prod_{i|k} x_i^{e_i} \, ,$$

where $\pi(m)$ runs through all partitions of *m* (i.e. nonnegative solutions of $k_1 + 2k_2 + ... + mk_m = m$) ;

$k = lcm\{i \mid k_i \neq 0\}$;

$$e_i = e_i(\pi, n) = \frac{1}{i} \sum_{d|i} \mu(\frac{i}{d}) \cdot \sum \prod_{l=1}^{n} \binom{(l,d) k_l + t_l - 1}{t_l} ,$$

where μ is Mobius function and the last sum is taken over all nonnegative solutions of

$$t_1 \frac{1}{(1,d)} + t_2 \frac{2}{(2,d)} + ... + t_n \frac{n}{(n,d)} = n .$$

Let $Z(S_m^{\{n\}}; 1/(1-x)) = Z(S_m^{\{n\}}; 1 + x + x^2 + ..., 1 + x^2 + x^4 + ..., 1 + x^3 + x^6 + ..., ...)$, i. e. $Z(S_m^{\{n\}}; 1/(1-x))$ is obtained if we replace x_i by $1 + x^i + x^{2i} + x^{3i} + ...$, *i*=1,2,…, in the cycle index $Z(S_m^{\{n\}}; x_1, x_2, ...)$.

Then the number of of *m* x *l* nonnegative integer matrices with all column sums equal to *n* is the coefficient of x^l in $Z(S_m^{\{n\}}; 1/(1-x))$.