PROGRESS ON RECTILINEAR CROSSING NUMBERS

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Abstract

Let $\overline{cr}(G)$ denote the rectilinear crossing number of a graph G. We show $\overline{cr}(K_{11}) = 102$ and $\overline{cr}(K_{12}) = 153$. Despite the remarkable hunt for the crossing number of the complete graph K_n, initiated by R. Guy in the 1960s, these quantities have been unknown for $n > 10$ to date. We also establish new upper and lower bounds on $\overline{cr}(K_n)$ for $13 \leq n \leq 20$, along with an improved general lower bound for $\overline{cr}(K_n)$. The results mainly rely on recent methods developed by the authors for exhaustively enumerating all combinatorially inequivalent sets of points (so-called order types).

1 Introduction and results

The *crossing number*, $cr(G)$, of a graph G is the least number of edge crossings attainable by a drawing[1] of G in the plane. Crossing number problems have quite a long history in the graph theory community (see, e.g., Tutte [15] and Erdös and Guy [5]) and have more recently been of interest to computer scientists (see Leighton [10] and Garey and Johnson [6]). We refrain from an attempt to survey the relevant literature in its generality, and point to a recent overview paper by Pach and Tóth [12] instead.

In the present note, we are concerned with the rectilinear version of the crossing number problem, where the edges of the underlying graph are required to be straight line segments. Following Harary and Hill [8], we will use the notion *rectilinear crossing number*, $\overline{cr}(G)$, of a graph G. The vertices of G are assumed to be in general position, meaning that no three vertices are allowed to be collinear in the drawing. Our specific interest is that of finding $\overline{cr}(K_n)$, the rectilinear crossing number of the complete graph on n vertices.

*Research supported by the Austrian Programme for Advanced Research and Technology

†Research supported by the FWF [Austrian Fonds zur Förderung der Wissenschaftlichen Forschung]

[1] *Drawing* may be interpreted in several ways leading to different concepts of crossing number; see Pach and Tóth [11].

Determining $\overline{cr}(K_n)$ is commonly agreed to be a difficult task. In fact, the asymptotic value, as n tends to infinity, is unknown for *any* interpretation of the crossing number $cr(K_n)$ considered in the literature; see Richter and Thomassen [13]. From an algorithmic point of view, deciding whether $\overline{cr}(G) \leq k$ for a given graph G and parameter k, is NP-hard; see Bienstock [3]. Only for very small n the exact values of $\overline{cr}(K_n)$ are known. Whereas the instances $n \leq 9$ have been settled quite a while ago [5], no progress has been made until in 2001 two groups of researchers (Brodsky et al. [4] and the authors [1], respectively) independently found $\overline{cr}(K_{10}) = 62$. In [4] the goal was reached by a purely combinatorial argument, while in [1] it arose as a byproduct in the exhaustive enumeration of all combinatorially inequivalent point sets (so-called order types) of size 10 or less.

Slightly more information about K_n can be extracted from the order type data base in [1]. For instance, beside the drawing constructed in [4] there is exactly one inequivalent drawing of K_{10} which also achieves $\overline{cr}(K_{10}) = 62$. The interested reader may consult [2] for a collection of new results on the crossing properties of small geometric graphs. For completeness, we include the following table from there.

n	4	5	6	7	8	9	10
$\overline{cr}(K_n)$	0	1	3	9	19	36	62
I_n	1	1	1	3	2	10	2

Table 1: $\overline{cr}(K_n)$ is attained by exactly I_n inequivalent drawings.

Encouraged by these achievements, we tailored the exhaustive search approach in [1] to compute crossing numbers; these modifications are reported in Section 2. The remainder of the present section states the obtained results.

Our first result completely resolves the situation for $n = 11$.

Result 1 *The rectilinear crossing number of K_{11} is* 102*, and this value is attained by exactly* 374 *inequivalent drawings. Each drawing shows exactly* 3 *extreme vertices.*

For larger n our results are less complete. Still, by carefully constructing drawings with low numbers of crossings, cf. Section 2, we obtained upper bounds superior (except for $n = 13$) to all previously known experimental results. These bounds are listed in line 2 of Table 2. Line 3 comments on the number of inequivalent drawings that exist for the respective number of crossings.

n	11	12	13	14	15	16	17	18	19	20
upper bound	102	153	229	324	447	603	798	1030	1318	1658
drawings	374	≥ 1	≥ 4272	≥ 22	≥ 2360	≥ 17	≥ 17532	≥ 62	≥ 3069	≥ 44
lower bound	102	153	221	310	423	564	738	949	1204	1505

Table 2: New bounds on $\overline{cr}(K_n)$.

Earlier experiments, by Thorpe and Harris [14], were based on randomized search and achieved drawings of K_{12} and K_{13} with 155 and 229 crossings, respectively. Their latter result thus competes with our best examples.

All drawings counted in Table 2 possess exactly three extreme vertices. This adds evidence to the commonly believed conjecture that the convex hull of every minimal rectilinear drawing of K_n is a triangle.

Concerning lower bounds, Result 1 immediately implies an improvement in the following way. A well-known argument for counting crossings in subgraphs [13] leads to the recurrence relation

$$\overline{cr}(K_n) \geq \lceil \overline{cr}(K_{n-1}) \frac{n}{n-4} \rceil. \tag{1}$$

Moreover, $\overline{cr}(K_n)$ and $\binom{n}{4}$, the maximum number of crossings in a K_n, are known to have the same parity for n odd; see [5]. Plugging in $\overline{cr}(K_{11}) = 102$ yields the specific lower bounds listed in line 4 of Table 2 and, when driven to larger n, the general lower bound in Result 3. Incidentally, lower and upper bounds happen to match for $n = 12$.

Result 2 *The rectilinear crossing number of K_{12} is 153. The only known drawing that attains this value is depicted in Figure 1.*

Inequality (1) implies that the ratio $\overline{cr}(K_n)/\binom{n}{4} < 1$ is a strictly increasing function and thus exists in the limit. By performing calculations through large n we obtained the value below. Actually, for $n > 200$ the improvement became marginal and could be disregarded.

Result 3 $\lim_{n\to\infty} \overline{cr}(K_n)/\binom{n}{4} > 0.311507$.

Lower bounds for the limit in Result 3 have been improved several times. The previously best value has been 0.3001, in [4].

2 Graph enumeration

This section describes our approach for generating inequivalent drawings of K_n.

The *order type* of a set of n points (in general position in the plane) is a mapping that assigns to each ordered triple of points an orientation (either clockwise or counter-clockwise); see Goodman and Pollack [7]. Two n-point sets S_1 and S_2 are called *(combinatorially) equivalent* if they have the same order types (up to global reversal of orientation), that is, if there exists a bijection $\varphi : S_1 \to S_2$ such that every point triple a, b, c in S_1 agrees in orientation with the triple $\varphi(a), \varphi(b), \varphi(c)$ in S_2.

Equivalent point sets exhibit identical crossing properties in the following sense. Any two line segments ab and cd spanned by S_1 cross if and only if the corresponding

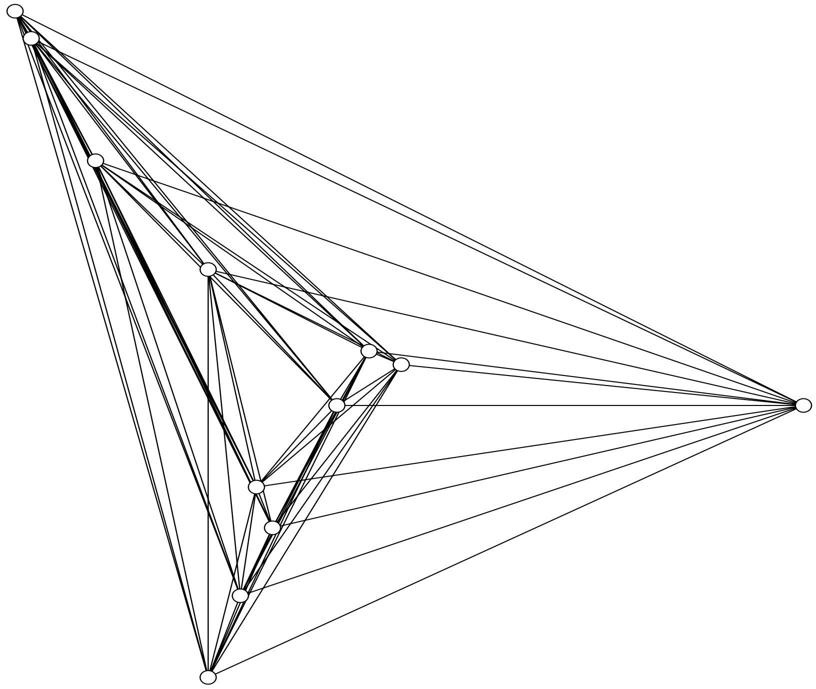

Figure 1: Minimal drawing of K_{12}.

two line segments $\varphi(a)\varphi(b)$ and $\varphi(c)\varphi(d)$ do. In particular, whether or not two line segments ab and cd do cross can be read off the orientations of the four point triples in $\{a, b, c, d\}$. We will call two drawings of K_n *equivalent* if and only if they live on equivalent n-point sets.

We are left with the problem of enumerating all different order types of given size n. This task has been recently accomplished for values of n through 10 in [1]. In a nutshell, their approach proceeds in two steps (for fixed size n).

Combinatorial step: Generate a candidate list of so-called *pseudo order types* which is guaranteed to contain all order types.

Geometrical step: Find a *realizing point set* for each of the pseudo order types in the list which *are* realizable, i.e., which are in fact order types.

To our advantage, the combinatorial step is comparatively easy, whereas the geometrical step is an intricate problem. The general problem of deciding the existence of a realizing point set is NP-hard. An exponential algorithm is known which, however, turned out to be too slow already for $n = 10$ when the number of sets is large. The interested reader is referred to [1] for a detailed discussion of this material. As we shall see, the geometrical step needs only partially be carried out for our present purpose.

Following are the details on the combinatorial step. A pseudo order type is a mapping that assigns to each ordered triple i, j, k in $\{1, \ldots, n\}$ a clockwise or counter-clockwise orientation. A pseudo order type does not necessarily have a realizing point set, however, because the geometry of a point set imposes restrictions on the choice of orientations. Still, whether two (abstract) line segments ij and kl cross can be decided by checking orientations. As has been shown in [7], each pseudo order type can be encoded by a λ-*matrix*. In this matrix, each entry $\lambda(i, j)$ gives the number of ordered triples i, j, k which are oriented counter-clockwise. Moreover, a list of pseudo order types which contains all the (realizable) order types can be obtained by considering combinatorially different *wiring diagrams*. The latter are combinatorial abstractions of straight line arrangements and correspond to certain pseudo order types by duality; see [7].

These observations led us to the following implementation of the combinatorial step for $n = 11$. Utilizing wiring diagrams, we generated a list of pseudo order types in λ-matrix representation. For the sake of time efficiency, we generated all matrices in *natural labelling*: index 1 is extreme, and indices 2 through n are sorted clockwise around 1. (Note that these geometric notions still make sense in the abstract setting; for example, a line segment ij is extreme if and only if $\lambda(i, j) \in \{0, n-2\}$.) Natural labelling suggests itself from wiring diagrams; see [1] for more details. As there are several naturally labelled λ-matrices for a fixed pseudo order type (namely the number of extreme indices times two, for re-orientation), only the (lexicographically) minimal matrix was taken into account for each pseudo order type.

Our computations resulted in the tremendous number of 18 410 581 880 matrices, thereof 2 243 203 071 being minimal and thus corresponding to pairwise inequivalent pseudo order types. Clearly, matrix generation and minimality check had to be done on-line to avoid storage problems. For each minimal matrix, we calculated the number of crossings in the (abstract) K_{11} for the respective pseudo order type. The minimum number, 102, was obtained by 374 pseudo order types with exactly 3 extreme indices each. These calculations took about one month on a 1GHz PIII processor under Linux. Reliability of the calculations is given by their completely combinatorial nature.

At this late point, geometry came into play. We tried to find realizing point sets for the 374 pseudo order types above, utilizing methods from [1]. Either strategy (point insertion using the existing order type data base for $n = 10$, and simulated annealing starting from scratch, respectively) was successful in realizing all pseudo order types in question. In particular, 16-bit integer coordinates have been obtained for all the corresponding 11-point sets. The point sets have been re-checked by re-calculating their λ-matrices: the sets are pairwise inequivalent and yield drawings of K_{11} with exactly 102 crossings. This proves the correctness of our geometric computations. The calculations required a few minutes only.

Once having available all existing minimal drawings of K_{11}, constructing drawings of K_n for $n \geq 12$ with few crossings suggests itself. Following suit, for each of the corresponding 11-point sets we considered the arrangement of straight lines through each pair of points and placed a 12^{th} point in an arrangement cell in all possible ways. This provided us with a large collection of 12-point sets, the 'best' of which were used in turn to generate a collection of 13-point sets in the same manner, and so on. We continued through $n = 20$ with this strategy, examining hundred-thousands of sets for each value of n. The obtained results are displayed in Table 2. Particularly pleasing is the discovery of a 12-point set with 153 crossings, see Figure 1, which we could prove minimal by examining lower bounds. It should be noted, however, that the above insertion strategy in general fails to produce *all* inequivalent n-point sets when applied to all inequivalent $(n-1)$-point sets.

3 Remarks

The experimental approach reported in this note led to various new results concerning the rectilinear crossing number of K_n. This constitutes one more example where computational experiments, when carried out carefully and combined with existing theoretical knowledge, shed new light into notoriously difficult combinatorial problems. The interested reader will find detailed output data of the computations (point sets in coordinate representation) at our web page[2].

We plan to build up a data base for $n = 11$ of all *projective* pseudo order types (rank 3 oriented matroids). This will enable us to compute the (Euclidean) pseudo order types of size 11 much faster while using reasonable sized disk space. An examination of questions similar to those in the present paper thus will be easier in the future.

A tantalizing question is that of the value of $\overline{cr}(K_{13})$. According to the new bounds in Table 2, the latest rage is $\overline{cr}(K_{13}) \in \{221, 223, 225, 227, 229\}$. Unfortunately, with current methods it seems out of reach to even perform a computation of all different order types of size 12. Still there is hope that an inspection of the currently 'best' drawings of K_n for small n might reveal regularities leading to new theoretical insights. A small step in this direction is the observation that, as of yet, all these drawings showed triangular convex hulls. Also, from Tables 1 and 2 we see that, for n odd, the number of inequivalent drawings with small numbers of crossing is quite large. This indicates that drawings attaining $\overline{cr}(K_n)$ might be easier to find for n odd, and leads us to believe that $\overline{cr}(K_{13}) = 229$.

Finally, let us remark on a related problem. The *(rectilinear) Hamiltonian cycle problem* asks for the largest number $h(n)$ of crossing-free Hamiltonian cycles a rectilinear drawing of K_n can realize. A conjecture originally stated in Hayward [9] and adopted in Brodsky et al. [4] reads: does there always exist a drawing of K_n which simultaneously attains $\overline{cr}(K_n)$ and $h(n)$? The conjecture has been proved for $n \leq 8$ and disproved for $n = 9$ and 10 in [2]. The present computations for $n = 11$

[2] `http://www.igi.TUgraz.at/oaich/triangulations/crossing.html`

add to the negative side: the largest number of cycles achieved among the 374 sets attaining $\overline{cr}(K_{11})$ is 20328, and we constructed an 11-point set having more cycles.

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