A PASCAL DIAMOND

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Abstract. A variation of Pascal's triangle, which is called a Pascal diamond, is introduced and some resulting number sequences are analyzed. Numbers in the Fibonacci sequence arise naturally as alternating sums of row elements. Explicit formulas are presented and some problems and conjectures are discussed.

INTRODUCTION. A generalization of Pascal's triangle can be defined using the following recurrence scheme. Given two rows of values, we compute a new row by adding together the four numbers in the diamond above the value to be computed. A sample diamond is given in figure 1. The value 16 is the sum of the four numbers above it in the diamond configuration.

$$\begin{array}{ccc} & 3 & \\ 4 & 5 & 4 \\ & 16 & \end{array}$$

Figure 1: Sample Diamond

For our first application we start with one 1 in the first row and three 1's in the second row. The recurrence then determines the subsequent rows. The first few rows of the array are given in figure 2. We assume all blank positions are zero. So, for example, when calculating the second entry in the third row the 3 zeros are assumed to be up two places and up one and to the left. We call this array of

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numbers a Pascal's diamond. We consider some properties of the diamond array below. For now we note that each row contains two more entries than the previous row, and each row is symmetric around the center line.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & 1 & 1 & & & \\
 & & 1 & 2 & 4 & 2 & 1 & & \\
 & 1 & 3 & 8 & 9 & 8 & 3 & 1 & \\
1 & 4 & 13 & 22 & 29 & 22 & 13 & 4 & 1
\end{array}
$$

Figure 2: The First Five Rows of the Diamond

This pattern generation scheme arose while studying a switch setting problem [1]. Given an n by m arrangement of switches, some on and some off, the goal is to achieve an all on configuration of the switches. Many puzzles and computer games are built using this idea. The operation available involves activating a particular switch, causing it and its rectilinearly adjacent neighbors to change states. We found that we could solve the switch setting problem by determining the null-space of a particular matrix defined by the switch arrangement topology. To determine the null-space we begin with an initial (row) vector containing one 1 and a second vector containing the three 1's under the initial vector's 1. We then "grow" the null-space vector by applying the diamond rule recursively. Our work on the switches differed in two ways from the recursion algorithm presented above. First, the rows are bounded by a certain fixed length and are not allowed to grow outward without bound on either the left or the right, and second, since the switches (in the simplest case) have only two states, all of the arithmetic is done modulo 2.

SOME PROPERTIES OF THE DIAMOND. Let $[n, k]$ represent the kth value of the nth row. The row numbering begins at 0 and the elements in a row also are numbered beginning at 0. We have $[0, 0] = 1$, and $[n, 0] = [n, 2n] = 1$ for all n. The diamond then is the array beginning

$$\begin{array}{ccccccc} & & & [0,0] & & & \\ & & [1,0] & [1,1] & [1,2] & & \\ & [2,0] & [2,1] & [2,2] & [2,3] & [2,4] & \\ [3,0] & [3,1] & [3,2] & [3,3] & [3,4] & [3,5] & [3,6], \end{array}$$

and the diamond defining recurrence relation can be written as

$$[n+1,k] = [n,k] + [n,k-1] + [n,k-2] + [n-1,k-2]. \tag{1}$$

Letting $k = 1$ in (1) gives the following relationship for the second entry of each row:

$$[n+1,1] = [n,1] + [n,0] = [n,1] + 1 \tag{2}$$

Two of the terms are missing in (2) because $k-2$ is -1, and the array values for negative k are taken to be 0. It follows directly from (2) that $[n,1] = n$ for all n. Writing down the recurrences for subsequent terms and solving them gives rise to the following formulas:

$$[n,2] = (n^2 + 3n - 2)/2$$
$$[n,3] = (n^3 + 9n^2 - 22n + 12)/3!$$
$$[n,4] = (n^4 + 18n^3 - 49n^2 + 6n + 48)/4!$$
$$[n,5] = (n^5 + 30n^4 - 45n^3 - 570n^2 + 1904n - 1680)/5!$$
$$[n,6] = (n^6 + 45n^5 + 55n^4 - 2865n^3 + 12184n^2 - 18780n + 8640)/6!$$

We state the general result below.

Theorem 1. *$[n,k]$ is a polynomial in n of degree k, such that $k![n,k]$ is monic with integer coefficients.*

Proof. First rewrite (1) as

$$[n,k] - [n-1,k] = [n-1,k-1] + [n-1,k-2] + [n-2,k-2].$$

Treat this as an identity in the variable n and constant k, and sum over n. The least value of n to use is the last non-zero entry in the appropriate diagonal. It can be written as $\lfloor (k+1)/2 \rfloor$ to account for parity of k. Then

$$\begin{aligned}[n,k] &= \sum_{i=\lfloor \frac{k+1}{2} \rfloor}^{n} ([i,k-1]+[i,k-2]+[i-1,k-2]) \\ &+ [\lfloor \frac{k+1}{2} \rfloor, k] \\ &= \sum_{i=\lfloor \frac{k+1}{2} \rfloor}^{n} ([i,k-1]+2[i,k-2]) \\ &+ [\lfloor \frac{k+1}{2} \rfloor - 1, k-2] - [n-1,k-2] + [\lfloor \frac{k+1}{2} \rfloor, k]\end{aligned}$$

The sequence of polynomials thus continues, with the general recurrence establishing by induction that $[n,k]$ is a polynomial in n of degree k, such that $k![n,k]$ is monic with integer coefficients.

Theorem 2. *Let T_n be the sum of the elements in row n of the Pascal diamond. Then $lim_{n\to\infty} T_{n+1}/T_n = (3+\sqrt{13})/2$.*

Proof. Using the recurrence in (1) we have that

$T_{n+1} = 3T_n + T_{n-1}$. Now let $x = T_{n+1}/T_n$ and we see that x satisfies the quadratic equation $x = 3 + 1/x$, equivalently $x^2 - 3x - 1 = 0$. So we have that the ratio of sums of consecutive rows of the diamond approaches $(3+\sqrt{13})/2 = 3.3027756\ldots$

The first few values of the row sums are 1, 3, 10, 33, 109, 360, 1189, 3927, 12970, 42837, 141481, and 467280. This sequence has arisen several times in the literature, with several references available in the Encylopedia of Integer Sequences [2]. Another famous sequence also arises in connection with row sums, as the next theorem notes.

Theorem 3. $\sum_{i=0}^{n}(-1)^i[n,i] = F_{n+1}$, *the $n+1st$ Fibonacci number.*

Proof. This follows immediately by induction.

A combinatorial problem to which the numbers in the Pascal diamond provide an insight is the focus of the next theorem.

Theorem 4. *Define an infinite directed graph $G(V, E)$ by using as the vertex set V points corresponding to the non-zero entries $[n, k]$ of the Pascal diamond array, and creating directed edges in E from the vertex $[n, k]$ to the vertices $[n+1, k]$, $[n+1, k+1]$, $[n+1, k+2]$, and $[n+2, k+2]$. Then the number of distinct paths from $[0, 0]$ to $[n, k]$ is given by the value of $[n, k]$.*

Proof. Again, an easy proof is available by induction.

A SECOND RECURRENCE. In the switch setting problem, null-space vectors were built using the diamond rule modified to use a leftmost column entries that remained zero. In this case an array arises that is left justified: the only new non-zero values in successive rows appear on the right. This results in an array as shown in figure 3.

1					
1	1				
3	2	1			
6	7	3	1		
16	18	12	4	1	
40	53	37	18	5	1

Figure 3: Another Recurrence

In this left bounded array, each row contains one more element than the previous row. Clearly the last element of each row is 1 and the next to last element is n. We choose a notation that makes it easiest to describe a unimodal property of the rows of this array, rather than highlighting the correspondence between this array and the Pascal diamond. Thus we index from the center to keep track of these elements as

$\langle 1,1\rangle$				
$\langle 2,1\rangle$	$\langle 2,2\rangle$			
$\langle 3,1\rangle$	$\langle 3,2\rangle$	$\langle 3,3\rangle$		
$\langle 4,1\rangle$	$\langle 4,2\rangle$	$\langle 4,3\rangle$	$\langle 4,4\rangle$	
$\langle 5,1\rangle$	$\langle 5,2\rangle$	$\langle 5,3\rangle$	$\langle 5,4\rangle$	$\langle 5,5\rangle$

It is natural to look for connections between this array and the Pascal diamond. Then the second array arises to the right of a central column of zeros, with the same array of negative numbers arising to the right. An engine for generating connections is the identity (which can be easily verified inductively)

$$[n,k]-[n,k-2]=\langle n+1,k-n+1\rangle. \tag{3}$$

Identity (3) can be used to extend the left bounded array leftward beyond the (implicit) column of zeros. Since the Pascal diamond is symmetrical, what is generated is a mirror image of the left bounded array, except that all the entries are negative. In fact, one way of obtaining the left bounded array is to start the Pascal diamond using the original recurrence, but with the two initial rows 0 and -1 0 1. Identity (3) also applies to provide an analogous result to Theorem 1, giving a second family of monic polynomials in n with integer coefficients. The first few values are listed below.

$$\langle n,n-2\rangle=(n^2+n-6)/2!$$
$$\langle n,n-3\rangle=(n^3+6n^2-43n+48)/3!$$
$$\langle n,n-4\rangle=(n^4+14n^3-109n^2+142n+24)/4!$$
$$\langle n,n-5\rangle=(n^5+25n^4-175n^3-385n^2+3534n-4920)/5!$$
$$\langle n,n-6\rangle=(n^6+39n^5-185n^4-3075n^3+23584n^2-56364n+43200)/6!$$

The row sums for the left bounded array,

$$\sum_{i=1}^{n}\langle n,i\rangle,$$

generating $\{U_n\}=\{1,2,6,17,51,154,473,1464,\ldots\}$, are more difficult to analyze than the row sums in the Pascal diamond. Nevertheless, the same limiting value of ratios of successive rows exists.

Theorem 5. *Let* U_n *be the sum of the elements in row* n *of the left bounded array. Then* $lim_{n\to\infty} U_{n+1}/U_n = (3 + \sqrt{13})/2$.

Proof. The row sum recurrence is $U_{n+1} = 3U_n + U_{n-1} - \langle n, n\rangle$, so it is enough to show that $\langle n, n\rangle = o(U_n)$ as $n \to \infty$ in order to make the argument of Theorem 2 apply. We show more, that the rows of the left bounded array are unimodal with a maximum value that moves ever rightward. First, note that path counting result of Theorem 4 applies to the left bounded array as well, with the change that the infinite directed graph has a restricted set of edges down the left column (thus the vertices labeled $\langle n, 1\rangle$ have only three outgoing edges, and the other vertices have four).

Now we need to establish the "more".

We believe the maximum value in row n occurs in position $O(\log n)$ or greater, perhaps as much as $O(n)$. The latter conjecture is based on an analogy with Pascal's triangle explored in the next section.

A CONNECTION WITH PASCAL'S TRIANGLE. A left bounded array can be constructed using the recurrence for Pascal's triangle as well:

	1						
0		1					
	1		1				
0		2		1			
	2		3		1		
0		5		4		1	
	5		9		5		1

FIGURE 3: LEFT BOUNDED PASCAL'S TRIANGLE

This time, the analog of (3) holds to give these table entries as differences of binomial coefficients. Hence the maximum value in row n of this array occurs at the k value that gives the maximum value of the difference in binomial coefficients in row n of Pascal's triangle. But as n grows, by the classical limit theorem of De Moivre and Laplace the binomial distribution approaches a normal distribution, and the maximum (absolute) derivative of this function occurs at a fixed value of x. Scaling n to x gives the value of k where the maximum difference occurs, a value that increases linearly with n.

References

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2. N. J. A. Sloane and S. Plouffe, *The Encyclopedia of Integer Sequences*, Academic Press, 1995.

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