Lattice animals and heaps of dimers

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Abstract

The general quest of this paper is the search for new classes of square lattice animals that are both large and exactly enumerable.

The starting point is a bijection between a subclass of animals, called directed animals, and certain *heaps of dimers*, called pyramids, which was described by Viennot more than 10 years ago. The generating function for directed animals had been known since 1982, but Viennot's bijection suggested a new approach that greatly simplified its derivation.

We define here two natural classes of heaps that are supersets of pyramids and are in bijection with certain classes of animals, and we enumerate them exactly. The first class has an algebraic generating function and growth constant 3.5 (meaning that the number of n-celled animals grows like 3.5^n), while the other has a transcendental non-holonomic generating function and growth constant 3.58... The generating function for directed animals is algebraic, and has growth constant 3. Hence both these new classes are exponentially larger. We obtain similar results for triangular lattice animals.

1 Introduction

1.1 Counting animals: the state of the art in brief

The enumeration of *animals* (or *polyominoes*) is a longstanding "elementary" combinatorial problem that has some motivations in physics, for example in the study of branched polymers [21] and percolation [14]. A *polyomino* of *area* n is a finite connected union of n cells on a lattice. The polyominoes we consider in this paper have square or hexagonal cells. If we replace each cell of a polyomino by a vertex at its centre, we obtain the corresponding *animal*, which lives on the dual lattice (Figure 1).

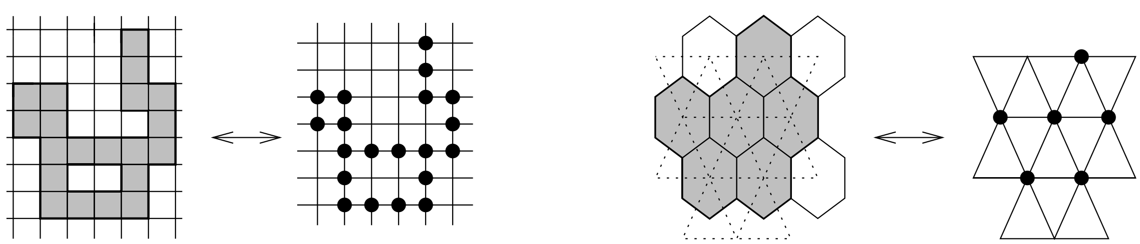

Figure 1: Polyominoes with square and hexagonal cells, and the corresponding animals on the square and triangular lattices.

Although these objects have been intensively studied for more than 40 years [22, 28, 39], exact results concerning general polyominoes have remained elusive. However, some asymptotic results are known: Let a_n denote the number of n-celled polyominoes on the square lattice. A concatenation argument [29] shows that there exists a constant μ, sometimes called *Klarner's constant*, or more generally *growth constant*, such that

$$\lim_{n\to\infty} (a_n)^{1/n} = \mu.$$

The exact value of μ is unknown, though numerical studies [14, 27] have shown that $\mu \simeq 4.06$. The best published bounds[1] for μ [27, 30] are

$$3.9 < \mu < 4.65.$$

Given the difficulty in solving this problem, what can we do to better understand polyominoes? A possible approach is the investigation and solution of large subclasses of polyominoes.

All the subclasses of polyominoes that have been solved to date have at least one of the following two properties: *convexity* or *directedness*. A polyomino is *column-convex* if its intersection with any vertical line is connected (Figure 2.a); it is *directed* if any cell can be reached from a fixed cell, called the *source*, by a north-east directed path that only visits cells of the animal (Figure 2.b). The most general polyominoes with these properties have been solved exactly (column-convex and directed polyominoes respectively): this shows that convexity limits the growth constant μ to 3.2..., while directedness limits it to 3. Table 1 gives more details, together with the nature of the associated *generating function* $\sum_n a_n q^n$.

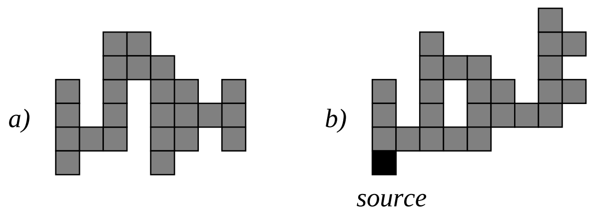

Figure 2: A column-convex polyomino and a directed polyomino.

Model	μ	Nature of the GF	Who solved it (first)
Rectangles	1	q-series	obvious
Ferrers diagrams (Partitions)	1	q-series	Euler [17]
Stacks	1	q-series	Auluck [1], Wright [44]
Staircase (Parallelogram)	2.30...	q-series	Klarner & Rivest [31]
Directed convex	2.30...	q-series	Bousquet & Viennot [12]
Convex	2.30...	q-series	Bousquet & Fédou [11]
Bargraph (Compositions)	2	rational	obvious
Directed column convex	2.62...	rational	Moser, Klarner [28]
Column convex	**3.20...**	**rational**	Temperley [39]
Diagonally convex directed	2.66...	q-series	Privman & Svrakić [36] Bousquet [8], Feretić [18]
Directed	**3**	**algebraic**	Dhar [15]

Table 1: Some of the solved subclasses of square lattice polyominoes and their growth constants.

1.2 Two new solvable classes of animals

In order to advance this line of polyomino enumeration one must now *invent* new exactly solvable classes of animals. Since the largest directed and convex classes have been solved, if these new classes are to be larger then they cannot be restricted by directedness or convexity. In this paper we define two new classes of square lattice animals that are neither directed nor convex. Each of them is in one-to-one correspondence with a natural class of *heaps of dimers*. These objects, defined in Section 2.1, have already proved useful in the enumeration of directed animals, by suggesting a canonical way to recursively factor them into two smaller directed animals (see Section 2.2 for details). The same kind of factorisation allows us to enumerate

[1] This topic is evolving rapidly; see Steve Finch's web page on mathematical constants for an up-to-date information `http://www.mathsoft.com/asolve/constant/constant.html`

our first new class of animals, called *stacked directed animals*: their generating function is algebraic (like for directed animals), and their growth constant is 3.5 (Section 2.3). The second and more general new class (*multi-directed animals*) is also related to heaps, but resists the factorisation method. To solve this second model we use a different approach based on a column-by-column construction of animals (Section 3). This construction gives a functional equation for their generating function which we solve explicitly in two steps. The first step yields an expression of the generating function in terms of the Fibonacci (or Tchebytcheff) polynomials (Section 3.2), which we re-interpret in Section 4 using Viennot's inversion lemma for heaps. A second step leads us to an expression that is easier to analyse (Section 5): we prove that it is transcendental (and even non-holonomic, see below for definitions), and that its growth constant is approximately 3.58. The column-by-column construction also works for directed animals, and allows us to take into account their width, which we prove to be a transcendental parameter (Section 6).

It is interesting to note that the enumeration of polyominoes having a convexity property can also be attacked via two methods: a *wasp-waist factorisation* that consists in splitting the polyomino into two smaller ones, at a place where it is especially thin (see for instance [9, 35]), and a column-by-column construction, sometimes called Temperley's method (see, e.g., [7, 39]). This is somehow paralleled by the existence of two constructions for directed animals.

All our square lattice results have analogues for triangular lattice animals (polyominoes with hexagonal cells). The growth constants of the two new classes we define are respectively 4.5 and 4.58..., which surpass the constants 3.86... and 4 obtained by counting column-convex animals and directed animals on the triangular lattice [29, 15]. The largest class we enumerate (multi-directed animals) is equinumerous with a class of animals introduced by Klarner [29], for which he simply gives a lower bound (Section 3.1). Also, the column-by-column construction suggests a third new class of animals (on the triangular lattice only). This class shares many of the features of multi-directed animals, and in particular, is not holonomic. However, its growth constant is smaller, being $2+\sqrt{5}=4.23...$ The number of n-celled animals in each family, for small values of n, is given at the end of the paper in Tables 4 (square lattice) and 5 (triangular lattice).

We are tempted to write that the classes of animals we define and count in this paper are "the largest classes of animals ever counted exactly". This sentence has to be written with a few caveats: the lower bound on Klarner's constant $\mu \geq 3.9$ mentioned above suggests that one has been able to enumerate classes of animals whose growth constant is significantly larger than our "record" of 3.58... The classes of animals that provide such lower bounds have usually a rational generating function that *could* be evaluated exactly; but the details of this rational function are less interesting than the value of its smallest singularity (the inverse of which is the growth constant). The description of these classes usually depends on a parameter k: a simple example is provided by polyominoes of width at most k, whose generating function can be evaluated by transfer matrix techniques [45]; another example is based on a factorisation of polyominoes into *prime* polyominoes: the generating function for polyominoes whose prime factors have area less than k is $(1-P_k(x))^{-1}$, where $P_k(x)$ enumerates prime polyominoes of area less than k. We refer to [42] for variations on this factorisation. While certainly interesting as approximants of the real problem, the dependence on k, and the obvious linear structure of these classes of polyominoes, make them not very interesting in themselves.

1.3 Formal power series: some definitions and notations

The (area) *generating function* for a class $\mathcal{A}$ of animals is $\sum_n a_n x^n$, where a_n denotes the number of animals of $\mathcal{A}$ having area n. We shall often enumerate animals according to several parameters, like the area and width, which will give rise to multivariate generating functions, like

$$\sum_{n,k} a_{n,k} x^n u^k,$$

where $a_{n,k}$ is the number of animals of $\mathcal{A}$ having area n and width k.

Given a ring $\mathbb{L}$ and n indeterminates $x_1,\ldots,x_n$, we denote by $\mathbb{L}[x_1,\ldots,x_n]$ the ring of polynomials in $x_1,\ldots,x_n$ with coefficients in $\mathbb{L}$, and by $\mathbb{L}[[x_1,\ldots,x_n]]$ the ring of formal power series in $x_1,\ldots,x_n$ with coefficients in $\mathbb{L}$. If $\mathbb{L}$ is a field, we denote by $\mathbb{L}(x_1,\ldots,x_n)$ the field of rational functions in $x_1,\ldots,x_n$ with coefficients in $\mathbb{L}$.

A formal power series F in $\mathbb{L}[[x_1,\ldots,x_n]]$ is said to be *algebraic* if there exists a non-trivial polynomial P in $n+1$ variables, with coefficients in $\mathbb{L}$, such that $P(x_1,\ldots,x_n,F)=0$. It is said to be *holonomic* (or *D-finite*) if the partial derivatives of F with respect to the variables x_i span a finite dimensional vector space over $\mathbb{L}(x_1,\ldots,x_n)$ (see [33]). In other words, for $1\le i\le n$, the series F satisfies a non-trivial partial differential equation of the form

$$\sum_{k=0}^{d_i} P_{k,i}(x_1,\ldots,x_n)\frac{\partial^k F}{\partial x_i^k}=0,$$

where $P_{k,i}(x_1,\ldots,x_n)$ is a polynomial in the x_j.

Any algebraic series is holonomic. Any derivative of a holonomic series F is holonomic, as is any (well-defined) specialisation of F. The classical theory of linear differential equations implies that a holonomic series $F(x)$ with complex coefficients has only finitely many singularities.

2 Factorisation of heaps of dimers

2.1 Heaps of dimers and animals

Heaps of pieces are simple combinatorial structures that were first introduced by Viennot [41]. They provide a convenient geometric representation of the elements of a partially commutative monoid (see Section 4 for more details). Intuitively, a heap of *dimers* is obtained by dropping a finite number of dimers towards a horizontal axis. Each dimer falls until it touches the horizontal axis or another dimer; see Figure 3, in which the following elementary definitions are also demonstrated. A heap is *strict* if no dimer has another dimer directly above it. It is *connected* if its orthogonal projection on the horizontal axis is connected. The *width* of a connected heap is the number of non-empty columns. The dimers that touch the axis are *minimal*. A heap having only one minimal dimer is called a *pyramid*. If, moreover, this minimal dimer lies in the rightmost non-empty column, the heap is a *half-pyramid*. The *right width* of a pyramid is the number of nonempty columns to the right of the minimal dimer (so that a single column of dimers has zero right width). The *left width* is defined similarly. A pyramid with zero right width is hence a half-pyramid.

For instance, the heap of Figure 3.a is neither strict nor connected, while the heap of Figure 3.b is both strict and connected, and has width 8. Two examples of pyramids can be found on Figure 5; both have right width and left width 2. A (schematic) half-pyramid is shown on Figure 7.

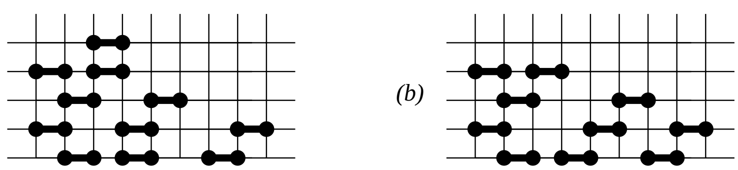

Figure 3: Two heaps of dimers; each has three minimal dimers.

Take an animal A on the square or triangular lattice, and rotate it by 45 degrees counterclockwise; replace each cell by a dimer, and let the dimers fall. We call $V(A)$ the resulting heap of dimers. Note that $V(A)$ is connected (Figure 4). It was observed by Viennot [40] that this mapping induces a bijection between directed animals on the square (resp. triangular) lattice and strict (resp. general) pyramids of dimers (Figure 5). The correspondence V will be the leading thread of this paper: all the classes of animals we are going to enumerate will be in bijection with natural families of heaps of dimers. We define the *width* of an animal to be the width of the corresponding heap.

We can associate with any strict heap H an infinite set $\mathcal{E}(H)$ of general heaps by replacing each dimer of H by a column of dimers of any positive height (Figure 6). Conversely, given an element H' of $\mathcal{E}(H)$, we can recover H by compressing all columns of H'. Consequently, if $F(x)$ is the generating function of a class $\mathcal{F}$ of strict heaps (where x counts the number of dimers), then the generating function for heaps of $\mathcal{E}(\mathcal{F})$ is $F(x/(1-x))$. Almost all the classes $\mathcal{A}$ of animals we are going to count will have a square lattice version $\mathcal{A}_s$ and a triangular lattice version $\mathcal{A}_t$ (with one exception). The animals of $\mathcal{A}_s$ will be in bijection with a set $\mathcal{F}_s$ of strict heaps, and the animals of $\mathcal{A}_t$ will be in bijection with heaps of the set $\mathcal{E}(\mathcal{F}_s)$. Hence, the generating functions of the corresponding animals will always be related by $A_t(x) = A_s(x/(1-x))$, and their growth constants by $\mu_t = 1 + \mu_s$.

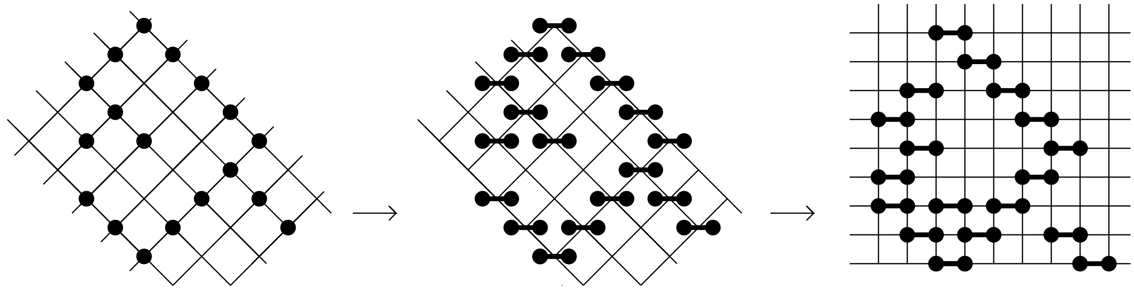

Figure 4: From animals to connected heaps: the transformation V.

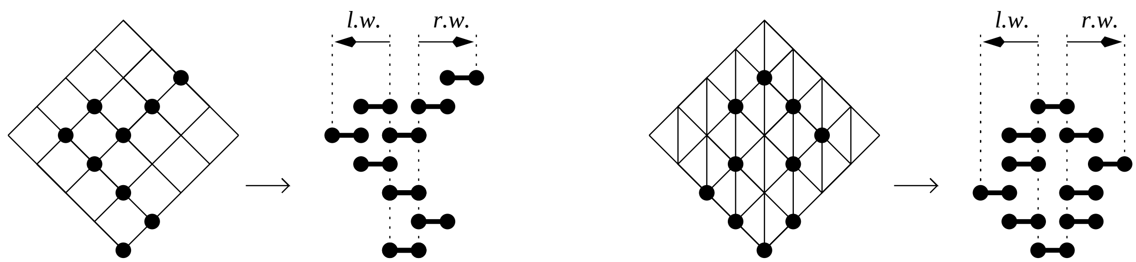

Figure 5: From directed animals to pyramids: the square lattice and the triangular lattice.

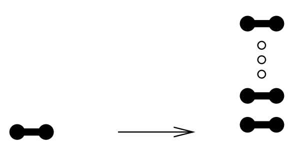

Figure 6: From strict heaps to general heaps.

2.2 Directed animals

Directed animals on the square and triangular lattices were first enumerated by Dhar [15, 16], Hakim and Nadal [25], followed by Gouyou-Beauchamps and Viennot [23]. However, all their proofs are complicated when compared to the simplicity of the generating functions (see Proposition 1 below). Simpler proofs were given later [4, 5, 6, 34, 37]: each of them either considers explicitly directed animals as heaps of dimers, or describes a construction that is clearly compatible with the heap structure. We present below the very direct proof of [6], which uses the *monoid structure* of the set of heaps (the product of two heaps is obtained by putting one heap above the other and dropping its pieces). From unambiguous factorisations of pyramids, one derives algebraic equations for their generating function.

Proposition 1 (Directed animals) *The generating functions $Q_s(x)$ and $Q_t(x)$ for strict (resp. general) half-pyramids are given by*

$$\begin{aligned} Q_s(x) &= \frac{1-x-\sqrt{(1+x)(1-3x)}}{2x} \\ Q_t(x) &= Q_s\left(\frac{x}{1-x}\right) = \frac{1-2x-\sqrt{1-4x}}{2x}. \end{aligned}$$

Denoting the generating function for strict (resp. general) half-pyramids by Q, the generating function for strict (resp. general) pyramids, counted by their number of dimers (x) and their right width (v) is

$$P(x,v) = \frac{Q(x)}{1-vQ(x)}. \tag{1}$$

In particular, the generating functions $P_s(x,1)$ and $P_t(x,1)$ for directed animals on the square (resp. triangular) lattice are given by:

$$\begin{aligned} P_s(x,1) &= \frac{1}{2}\left(\sqrt{\frac{1+x}{1-3x}}-1\right) \\ P_t(x,1) &= P_s\left(\frac{x}{1-x},1\right) = \frac{1}{2}\left(\frac{1}{\sqrt{1-4x}}-1\right). \end{aligned}$$

Consequently, the number of n-celled directed animals on the square (resp. triangular) lattice is asymptotic to

$$\frac{1}{\sqrt{3\pi}}\,3^n n^{-1/2} \qquad \left(\textit{resp.}\ \frac{1}{2\sqrt{\pi}}\,4^n n^{-1/2}\right).$$

Their average width is asymptotic to

$$6\sqrt{3\pi}\,n^{1/2} \qquad \left(\textit{resp.}\ 16\sqrt{\pi}\,n^{1/2}\right).$$

Proof. These expressions are obtained by factoring pyramids in a canonical way; this factorisation is depicted in Figure 7.

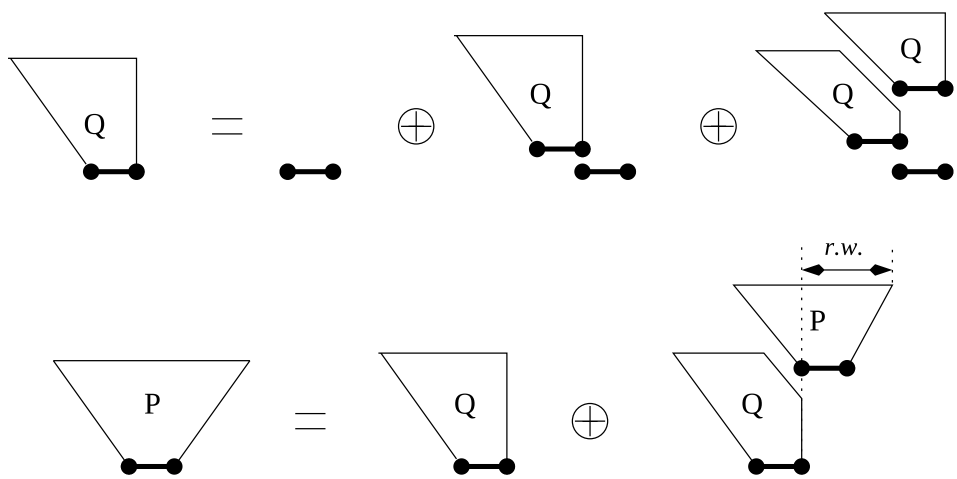

Figure 7: The factorisation of strict half-pyramids and pyramids.

Let us begin with strict half-pyramids. Consider a strict half-pyramid H having several dimers. If there is only one dimer (the minimal one) in the rightmost column, then H is the product of its minimal dimer and a half-pyramid. Otherwise, by pushing upwards the lowest non-minimal dimer of the rightmost column, we factor H into two half-pyramids and a minimal dimer. This gives $Q_s(x) = x + xQ_s(x) + xQ_s(x)^2$. This equation is readily solved, yielding the expression of $Q_s(x)$ given in the proposition. Observe that the expansion operation of Figure 6 does not change the right width and so preserves the property of being a half-pyramid; this gives the announced result for $Q_t(x)$. Alternatively, we could directly factor general half-pyramids to obtain an algebraic equation satisfied by $Q_t(x)$; namely $Q_t(x) = x + 2xQ_t(x) + xQ_t(x)^2$.

A pyramid (be it strict or general) is either a half-pyramid, or the product of a half-pyramid and a pyramid. With the notations of the proposition, this implies that $P(x,v) = Q(x)(1+vP(x,v))$, both for the strict and general pyramid model.

The asymptotic results follow from the general correspondence between the position and nature of singularities of a generating function and the asymptotic behaviour of its coefficients [20]. The average *right* width is obtained by differentiating (1) with respect to v, and the average width is twice the average right width plus one.

■

2.3 Multi-directed animals and stacked directed animals

Encouraged by the success of the mapping between pyramids and directed animals, it is natural to ask whether we can extend this mapping to larger classes of heaps and animals. The map V, that transforms animals into connected heaps, can be shown to be surjective. We describe below a natural class of triangular lattice animals that are in bijection with connected heaps of dimers.

Let H be a connected heap. We proceed by induction on the number of minimal pieces of H. If H is a pyramid, let $\overline{V}(H)$ be the corresponding triangular lattice directed animal. If H has k minimal dimers, with $k > 1$, let us push upwards the $(k-1)$ leftmost minimal dimers: this yields a connected heap H', placed "far above" a remaining pyramid P_k. Replace H' and P_k by their corresponding animals $\overline{V}(H')$ and $\overline{V}(P_k)$, and push back $\overline{V}(H')$ downwards until it connects to $\overline{V}(P_k)$. Let $\overline{V}(H)$ be the resulting animal (Figure 8).

In this way, we recursively define a class of triangular lattice animals that is in one-to-one correspondence with connected heaps of dimers. For obvious reasons, we call these animals *multi-directed animals.*

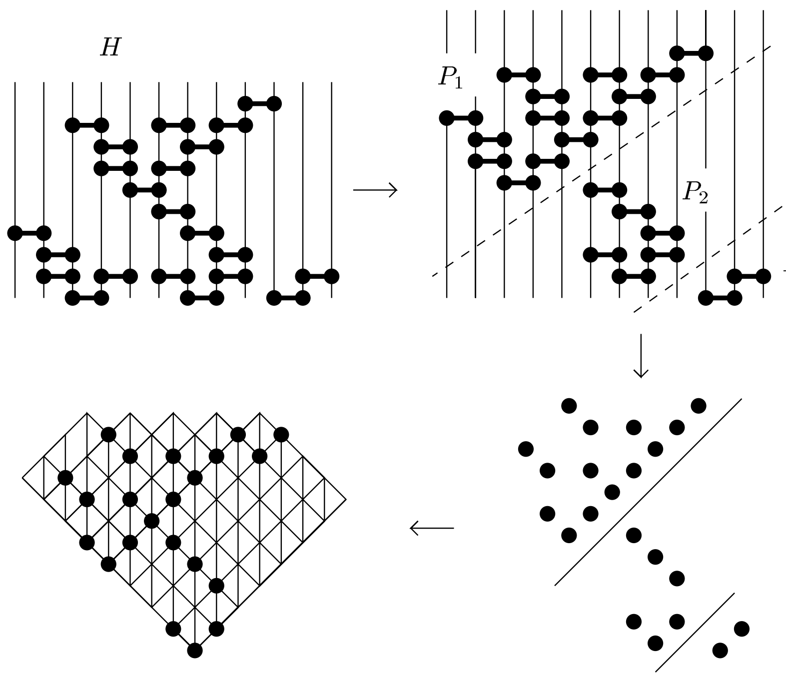

Figure 8: From connected heaps to multi-directed animals: the transformation $\overline{V}$.

The case of square lattice animals is a bit more delicate. The transformation V maps some of them to non-strict heaps (Figure 4). But, clearly, not all connected heaps can be obtained in this way: for instance, the column of height 2 has no preimage among square lattice animals. It seems that the image of square lattice animals under V has no simple description. However all is not lost: if we apply $\overline{V}$ to a *strict* heap H we obtain a *square lattice* animal: hence $\overline{V}$ induces a one-to-one correspondence between strict connected heaps and square lattice multi-directed animals.

We now define a subclass of multi-directed animals that will be easy to enumerate. Take a connected heap H with k minimal pieces. The multi-directed animal $\overline{V}(H)$ is formed by the concatenation of k directed animals. Let us denote by $P_1, P_2, \ldots, P_k$, from left to right, the corresponding pyramidal factors of H. The fact that H is connected means that for $1 < i \leq k$, the projection of P_i onto the horizontal axis intersects the projection of some P_j, for $j < i$. In the example of Figure 8, the projection of P_2 and P_3 intersect the projection of P_1. Let us call *stacked pyramids* the connected heaps such that for $1 < i \leq k$, the horizontal projection of P_i intersects the horizontal projection of P_{i-1}. Let us call *stacked directed animals* the corresponding animals (Figure 9). We define the *right width* of a stacked pyramid to be the right width of its rightmost pyramidal factor.

These objects are easier to enumerate than connected heaps (but less general): the recursive description of stacked pyramids easily translates into algebraic equations for their generating function, which again turns out to be quadratic. In terms of growth constants, this simple model is already larger than all previous exactly solved models.

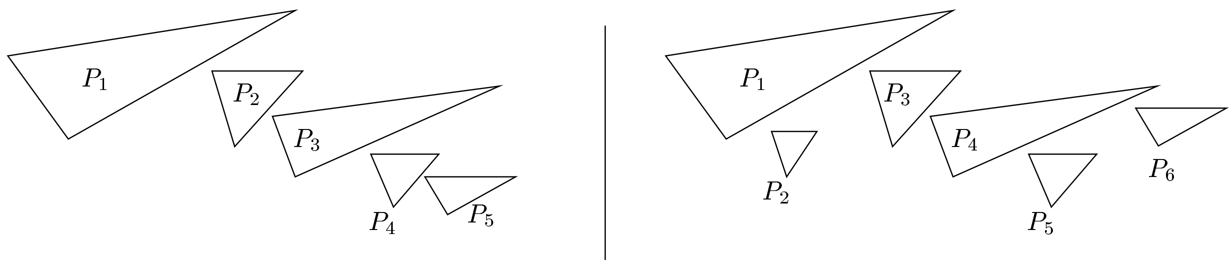

Figure 9: The structure of stacked directed animals (left) and multi-directed animals (right). Each triangle represents a directed animal. In a stacked directed animal each directed animal component, P_i lies below P_{i-1}, whereas in a multi-directed animal P_i lies below P_j for some $j < i$.

Proposition 2 (Stacked directed animals) *Let $Q(x) \equiv Q$ denote the generating function for strict (resp. general) half-pyramids. Let $P(x, v)$ denote the generating function for strict (resp. general) pyramids, where the variable v enumerates the right width. Then the generating function for strict (resp. general) stacked pyramids, counted by their number of dimers (x), right width (v) and number of minimal dimers (t), is*

$$S(x, v, t) = \frac{tP(x, v)}{1 - tP(x, 1)^2} = \frac{tQ(1 - Q)^2}{(1 - vQ)[(1 - Q)^2 - tQ^2]}.$$

In particular, the generating function for strict (resp. general) stacked pyramids is:

$$\begin{aligned} S_s(x) &= \frac{(1 - 2x)(1 - 3x) - (1 - 4x)\sqrt{(1 - 3x)(1 + x)}}{2x(2 - 7x)}, \\ S_t(x) &= S_s\left(\frac{x}{1 - x}\right) = \frac{(1 - 3x)(1 - 4x) - (1 - 5x)\sqrt{1 - 4x}}{2x(2 - 9x)}. \end{aligned}$$

Consequently, the number of n-celled stacked directed animals on the square (resp. triangular) lattice is asymptotic to

$$\frac{3}{28}\, 3.5^n \qquad \left(\textit{resp.}\ \ \frac{1}{12}\, 4.5^n\right).$$

The number of minimal dimers in the corresponding stacked pyramids, which is a lower bound on their width, is asymptotic to

$$\frac{3}{28}\, n \qquad \left(\textit{resp.}\ \ \frac{1}{12}\, n\right).$$

The width is trivially bounded above by n.

Proof. We follow the description of $\overline{V}$ given above. Each stacked pyramid H is:

- either a single pyramid (if it has only one minimal piece),
- or the product of a pyramid $P \equiv P_k$ and a stacked pyramid H' placed above P (see Figure 10). The number of ways the pyramid P can be placed is equal to the right width of H'. The right width of H is then the right width of P.

This shows that

$$S(x,v,t) = tP(x,v)\left(1 + S'_v(x,1,t)\right).$$

We differentiate this equation with respect to v and then set v to 1 in order to compute the derivative $S'_v(x,1,t)$. We finally obtain

$$S(x,v,t) = \frac{tP(x,v)}{1 - tP'_v(x,1)},$$

and we use Proposition 1 to obtain the announced expression of $S(x,v,t)$. The asymptotic results follow as per the proof of Proposition 1.

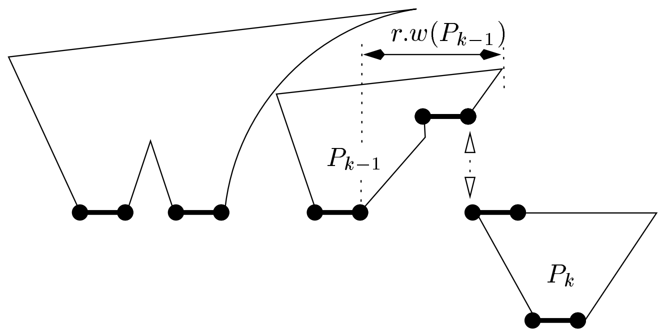

Figure 10: A recursive construction of stacked pyramids.

An alternative construction consists in adding a new pyramid $P \equiv P_1$ to the above left of a stacked pyramid H', rather than to its below right (see Figure 11). The pyramid P can be placed in a number of ways equal to its right width. This yields directly

$$S(x,v,t) = tP(x,v) + tS(x,v,t)P'_v(x,1).$$

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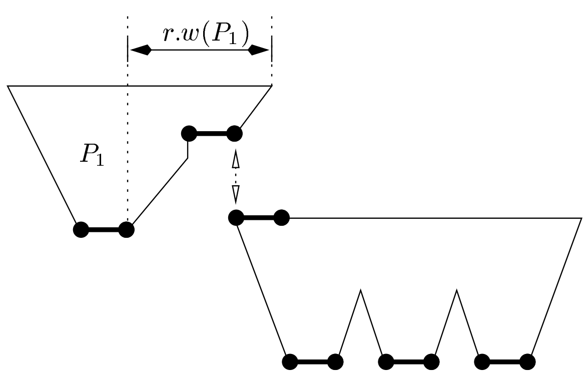

Figure 11: Another recursive construction of stacked pyramids.

For comparison, we state here the result for connected heaps; the proof is given in Section 5.

Proposition 3 (Multi-directed animals) *Let $Q(x) \equiv Q$ denote the generating function for strict (resp. general) half-pyramids. Then the generating function for strict (resp. general) connected heaps of dimers is*

$$C(x) = \frac{Q}{(1-Q)\left[1 - \sum_{k\geq 1} \frac{Q^{k+1}}{1-Q^k(1+Q)}\right]}.$$

This series is not algebraic, nor even D-finite. The number of n-celled multi-directed animals on the square (resp. triangular) lattice is asymptotic to $A\mu^n$ for some positive constant A, with $\mu = 3.58789436...$ (resp. $\mu = 4.58789436...$). Their average width is asymptotic to Bn, for some positive constant B.

Once again, we note that it is sufficient to prove this result for general connected heaps: with obvious notations,

$$C_s(x) = C_t(x/(1+x)) \quad \text{and} \quad Q_s(x) = Q_t(x/(1+x)).$$

Since a connected heap may also be constructed recursively (by adding a new pyramid to the below right of another connected heap) the above result begs a number of questions: Why is the result for connected heaps so much more complicated than the result for stacked pyramids? Why are we unable to apply the method of Proposition 2? Let us explain why the approach of Proposition 2 fails for connected heaps.

Take a pyramid $P \equiv P_k$, and a connected heap H' having $k-1$ minimal dimers. The number of ways we can place P to the below right of H' so as to form a new connected heap H having k minimal dimers is the *modified right width* of H', which we define to be the number of nonempty columns to the right of the rightmost minimal dimer of H' (Figure 12); this is not simply the right width of P_{k-1}. If we are to proceed by the same construction then we have to take the modified right width into account in the enumeration. So the next question is: what is the modified right width of the connected heap H we have obtained? If P has been placed in ith position (the leftmost position corresponding to $i = 1$), then, with obvious notations,

$$m.r.w(H) = r.w(P) + \max(0, m.r.w(H') - i - w(P)),$$

and this rather complicated formula, combined with the fact that the width of pyramids is an inherently non-algebraic parameter (see Section 6), explains why the first construction we used in the proof of Proposition 2 fails for connected heaps.

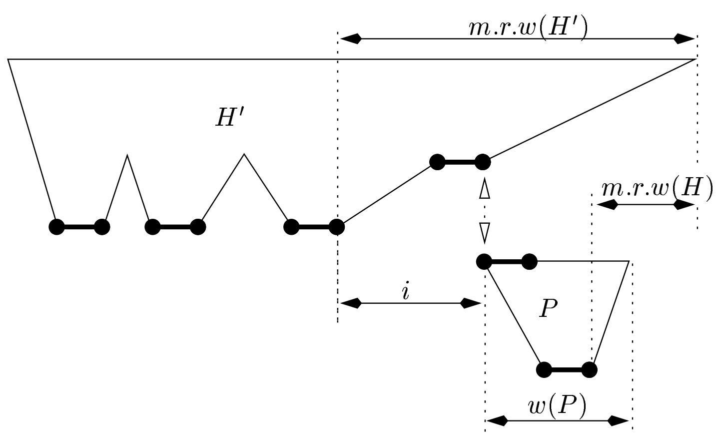

Figure 12: A recursive construction of connected heaps.

In the same proof, we described a second construction of stacked pyramids, that consisted of adding a new pyramid $P \equiv P_1$ to the above left of another stacked pyramid. The principle of this construction also works for connected heaps, but, when we try to use it,

- we actually have to count *all* heaps, because the connectivity is not necessarily preserved when removing the leftmost pyramidal factor from a connected heap (Figure 8);
- we have to take into account a width-like parameter whose transformation during the construction depends simultaneously on the right width and left width of the pyramid P.

For these reasons, it is perhaps not too surprising that in order to prove Proposition 3 we will have to make a slight detour and examine heaps of fixed width.

2.4 Pictures of typical animals

Before moving onto the proofs of our results, we wish to give some sort of qualitative picture of the classes of animals discussed in this paper. Using the recursive constructions of animals described in Sections 2 and 3, it is easy to write, in the spirit of [19], algorithms that generate uniformly at random animals of a fixed size in any of the main three above mentioned classes.

Below we give pictures of "typical" animals from the set of directed animals and the two new classes we have constructed; these give a good qualitative picture of the results we have obtained on the average width of the various animals under consideration. In particular, note how directed animals are very elongated, while the animals in the two new classes appear to have heights and widths roughly equal.

The reader can identify "by hand" the directed animal components in the animals of Figures 14 and 15, and thus observe that none of the multi-directed animals of Figure 15 are stacked directed animals.

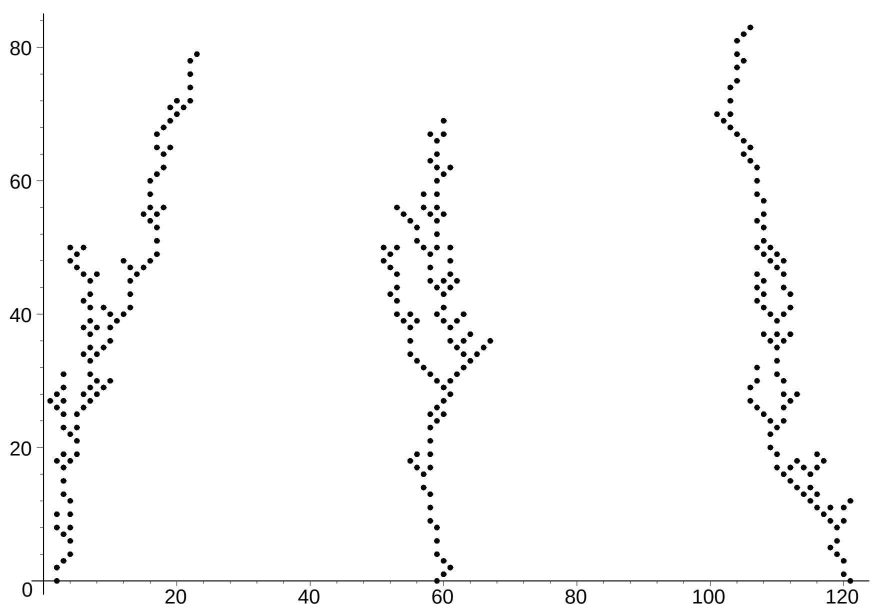

Figure 13: Three random directed animal on the triangular lattice with 100 cells.

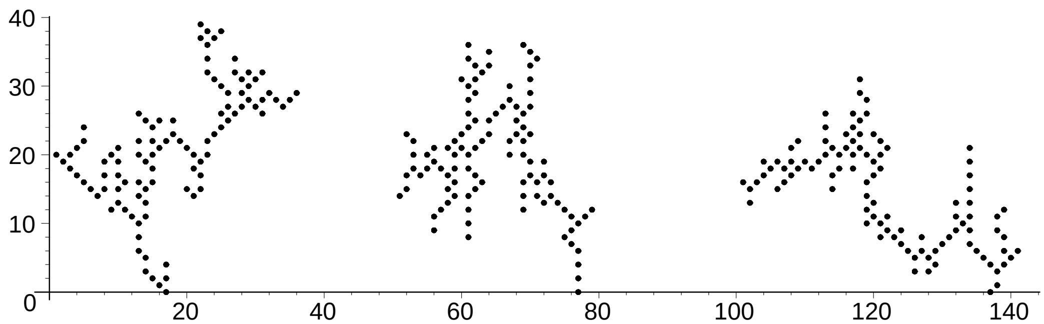

Figure 14: Three random stacked directed animal on the triangular lattice with 100 cells.

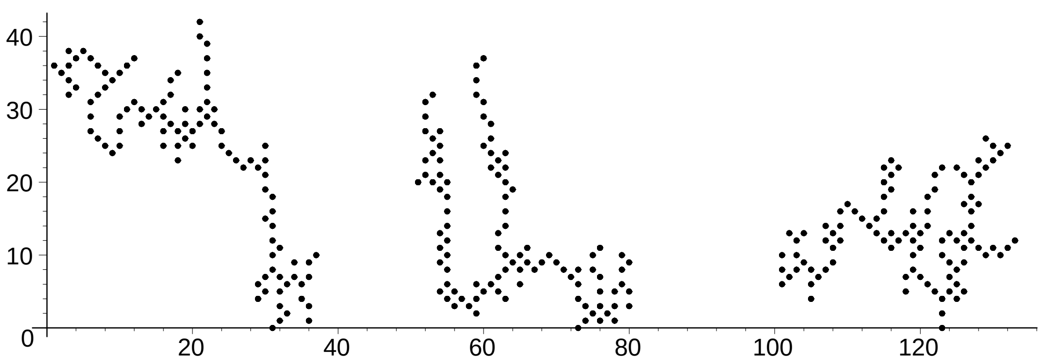

Figure 15: Three random multi-directed animal on the triangular lattice with 100 cells.

3 A column-by-column construction of heaps

In this section we focus on general heaps and triangular lattice animals. The expansion/contraction procedure of Figure 6 can be used to provide analogous results for square lattice animals. We attack the enumeration of heaps via a recursive construction that is even more elementary than the factorisation described in Section 2: it simply consists of building a heap column-by-column. This construction provides functional equations for the corresponding generating functions and allows us to take into account the width of heaps (Section 3.1). This approach does *not* work for stacked pyramids but suggests a new class of connected heaps, which we call *saw-tooth heaps*. In Section 3.2 we solve the functional equations in terms of the Fibonacci polynomials.

3.1 Establishing functional equations

Let $\mathbf{Q}(x,y,u)$ be the generating function for half-pyramids, where u counts the width, y the dimers in the rightmost column, and x the other dimers. Similarly, let $\overline{\mathbf{Q}}(x,y,u)$ be the generating function for half-pyramids, where y now counts dimers of the *leftmost* column. Let $\mathbf{P}(x,y,u,v)$ be the generating function for pyramids, where u counts the left width, v the right width, y the dimers in the rightmost column, and x the other dimers (Figure 16). Finally, let $\mathbf{C}(x,y,u)$ count connected heaps by their width (u), number of dimers in the rightmost column (y), and number of remaining dimers (x). For instance, the expansion of $\mathbf{C}(x,y,u)$ starts with

$$\mathbf{C}(x,y,u) = uy + \left(uy^2 + 2uxy\right) + \left(uy^3 + 3u^2x^2y + 3u^2xy^2 + 4u^3x^2y\right) + \cdots$$

these terms corresponding to connected heaps with at most three dimers.

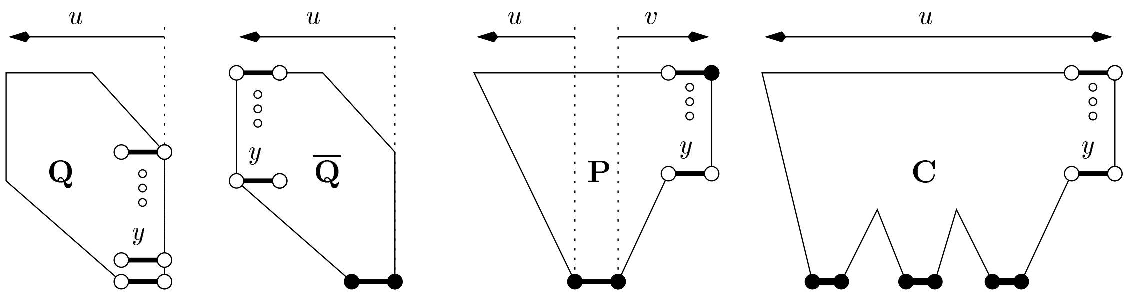

Figure 16: The generating functions $\mathbf{Q}, \overline{\mathbf{Q}}, \mathbf{P}$ and $\mathbf{C}$.

Constructing these classes of heaps column by column yields the following functional equations. For simplicity, we denote $\mathbf{Q}(x,y,u) \equiv \mathbf{Q}(y)$, etc.

Proposition 4 *The heap generating functions defined above are governed by the following functional equations:*

$$\begin{aligned}
\mathbf{Q}(y) &= \frac{uy}{1-y} + \frac{uy}{1-y}\mathbf{Q}\left(\frac{x}{1-y}\right), \\
\overline{\mathbf{Q}}(y) &= \frac{uy}{1-y} + u\overline{\mathbf{Q}}\left(\frac{x}{1-y}\right) - u\overline{\mathbf{Q}}(x), \\
\mathbf{P}(y) &= \frac{1}{u}\mathbf{Q}(y) + v\mathbf{P}\left(\frac{x}{1-y}\right) - v\mathbf{P}(x), \\
\mathbf{C}(y) &= \frac{uy}{1-y} + \frac{u}{1-y}\mathbf{C}\left(\frac{x}{1-y}\right) - u\mathbf{C}(x).
\end{aligned}$$

Proof. Let us begin with the equation governing $\mathbf{Q}$. Take a half-pyramid and delete the dimers in its rightmost column; the remaining dimers (if any) form a new half-pyramid. Conversely, any half-pyramid H of width $m \geq 2$ can be grown from a half-pyramid H' of width $m-1$ by creating a new column to the right of H'. More precisely:

1. we insert a (possibly empty) column of dimers to the above-right of each dimer in the rightmost column of H'. This corresponds to the substitution $y \longmapsto \frac{x}{1-y}$ in the generating function $\mathbf{Q}$;

2. we insert a nonempty column of dimers to the below right of the minimal dimer of H'. The bottom dimer of this column will be the minimal piece of H. This operation is represented by multiplying by $\frac{y}{1-y}$.

This procedure, illustrated by Figure 17, generates all half-pyramids of width at least two. The term $uy/(1-y)$ accounts for half-pyramids of width one.

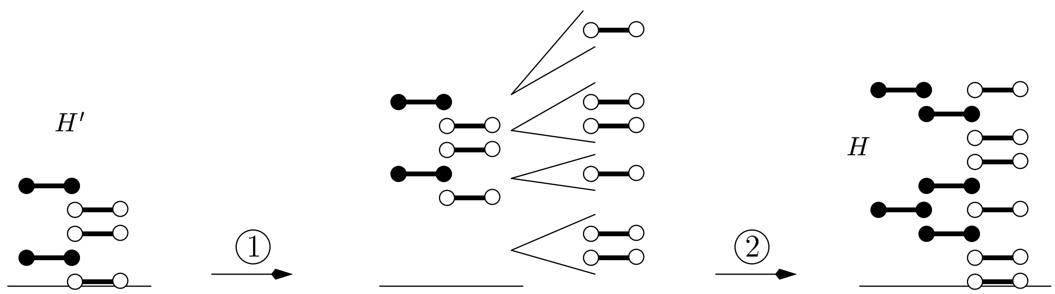

Figure 17: Constructing half-pyramids column by column.

Let us now prove the equation governing $\overline{\mathbf{Q}}$. This time, we construct half-pyramids of width $m \geq 2$ by creating a new column to the *left* of a pyramid H' of width $m-1$. More precisely, we insert a (possibly empty) column of dimers to the above-left of each dimer in the leftmost column of H'. This corresponds to the substitution $y \longmapsto \frac{x}{1-y}$ in the generating function $\overline{\mathbf{Q}}$. Since each of these columns of dimers we have just added can be empty, it is possible that we might not have added any dimers at all. To avoid this unwanted case, we simply subtract $\overline{\mathbf{Q}}(x,x,u)$.

The two other equations are proved in a similar manner.

■

Remarks

1. Saw-tooth heaps. By symmetry, the series $\mathbf{C}(x,y,u)$ also counts connected heaps by their width (u), *leftmost* dimers (y) and remaining dimers (x). Let us compare the functional equations defining the series $\overline{\mathbf{Q}}$ and $\mathbf{C}$. They reflect a recursive construction of half-pyramids (resp. connected heaps) where a new column of dimers is created at each step to the left of the object. The difference between the two equations comes from the fact that, in the construction of connected heaps, we can create a new minimal dimer by adding a new column of dimers to the below left of the lowest leftmost dimer, whereas this is forbidden when we construct half-pyramids. If we bound the height of this new column of dimers by h, we obtain a family of heaps that interpolates between half-pyramids ($h=0$) and connected heaps ($h=\infty$). In particular, for $h=1$, we obtain a class of heaps whose generating function $\mathbf{N}(x,y,u) \equiv \mathbf{N}(y)$ satisfies:

$$\mathbf{N}(y) = \frac{uy}{1-y} + u(1+y)\mathbf{N}\left(\frac{x}{1-y}\right) - u\mathbf{N}(x). \tag{2}$$

The lower edge of these heaps is constrained so that (when drawn from left to right) it may only grow upwards diagonally, but is able to grow straight down, something like the edge of a saw-tooth. Because of this similarity, we call these heaps *saw-tooth heaps* (Figure 18).

2. Connected heaps and Klarner's animals. Let $c_k(n;a_k)$ denote the number of connected heaps of width k having n dimers, a_k of which lie in the rightmost column. The equation defining $\mathbf{C}(x,y,u)$, given in Proposition 4, is equivalent to the following recursion:

$$c_k(n;a_k) = \begin{cases} 1 & \text{if } k=1 \text{ and } a_k = n, \\ \displaystyle\sum_{a_{k-1}=1}^{n-a_k} \binom{a_{k-1}+a_k}{a_{k-1}} c_{k-1}(n-a_k;a_{k-1}) & \text{if } k>1 \text{ and } a_k < n, \\ 0 & \text{otherwise.} \end{cases}$$

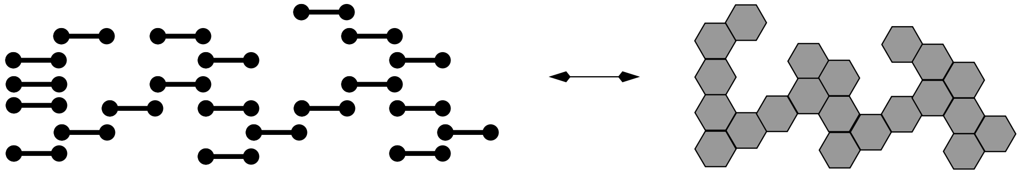

Figure 18: A saw-tooth heap and the corresponding honeycomb lattice polyomino.

Equivalently, the number of connected heaps of width k having a_i dimers in the ith column, for $1 \le i \le k$, is

$$\binom{a_1+a_2}{a_1}\binom{a_2+a_3}{a_2}\cdots\binom{a_{k-1}+a_k}{a_{k-1}}. \tag{3}$$

In [29, Section 4], Klarner describes a class of *square lattice* animals counted by a sequence

$$b(n) = \sum f(a_1,a_2)f(a_2,a_3)\cdots f(a_{k-1},a_k), \tag{4}$$

where the sum extends over all compositions $(a_1, a_2, \ldots, a_k)$ of n in positive parts, and the generating function for the numbers $f(a,b)$ is given by

$$\sum_{a,b\ge 0} f(a,b)x^a y^b = \frac{1-xy}{1-x-y+x^2y^2}.$$

Then, he mentions the existence of an analogous class of animals *on the triangular lattice*, which are enumerated by (4) with $f(a,b) = \binom{a+b}{a}$. The description of these two families is a little ambiguous, and while studying them we realized that (i) the triangular lattice model was equivalent to (general) connected heaps of dimers and (ii) this model could be solved "more exactly" than the treatment in Klarner's paper where he only gives Eq. (3) and the lower bound of 4 on the growth constant. These realisations were the starting point of this paper. Let us underline that the class of *square lattice* animals described by Klarner is *not* equivalent to strict connected heaps: its growth constant is approximately 3.72 (instead of 3.58). Unfortunately, we have so far not been able to solve the associated functional equation.

3.2 Solving functional equations

The first idea that comes to mind to solve the equations of Proposition 4 – iteration – turns out to be the right one. Let us, for instance, start iterating the equation ruling $\mathbf{Q}$:

$$\begin{aligned}
\mathbf{Q}(y) &= \frac{uy}{1-y}\left[1+\mathbf{Q}\left(\frac{x}{1-y}\right)\right],\\
&= \frac{uy}{1-y} + \frac{u^2xy}{(1-y)(1-x-y)}\left[1+\mathbf{Q}\left(\frac{x(1-y)}{1-x-y}\right)\right],\\
&= \frac{uy}{1-y} + \frac{u^2xy}{(1-y)(1-x-y)} + \frac{u^3x^2y}{(1-x-y)(1-2x-y(1-x))}\left[1+\mathbf{Q}\left(\frac{x(1-x-y)}{1-2x-y(1-x)}\right)\right],\\
&= \frac{uy}{1-y} + \frac{u^2xy}{(1-y)(1-x-y)} + \frac{u^3x^2y}{(1-x-y)(1-2x-y(1-x))}\\
&\quad + \frac{u^4x^3y}{(1-2x-y(1-x))(1-3x+x^2-y(1-2x))}\left[1+\mathbf{Q}\left(\frac{x(1-2x-y(1-x))}{1-3x+x^2-y(1-2x)}\right)\right].
\end{aligned}$$

After a few iterations, it becomes clear that the sequence of polynomials involved here, which starts with $1, 1-x, 1-2x, 1-3x+x^2$, is the sequence of *Fibonacci polynomials* F_n, related to the Tchebytcheff polynomials of the second kind, and defined by $F_0 = F_1 = 1$, and for $n \ge 2$,

$$F_n = F_{n-1} - xF_{n-2}. \tag{5}$$

This is not surprising, since these polynomials arise in the enumeration of Dyck paths of bounded height [32, Section 6], and a general bijection between walks on a graph and heaps of cycles [41, Section 6] puts Dyck walks of height m in one-to-one correspondence with half-pyramids of width m. Another way of understanding the role of these polynomials is based on general heap enumeration results and on the fact that F_n counts *trivial* heaps of dimers on a segment of n vertices. This approach will be detailed in the next section. To state our results concisely, it is convenient to introduce the bivariate polynomials $\tilde{F}_n(x,y) \equiv \tilde{F}_n$ defined for $n \geq 2$ by

$$\tilde{F}_n = F_{n-1} - yF_{n-2}, \tag{6}$$

with $\tilde{F}_0 = \tilde{F}_1 = 1$. By convention, $F_i = \tilde{F}_i = 0$ if $i < 0$, so that (5) and (6) hold for $n \geq 1$. Observe that $\tilde{F}_n(x,x) = F_n$.

Proposition 5 *The heap generating functions* $\mathbf{Q}, \overline{\mathbf{Q}}, \mathbf{P}, \mathbf{C}$ *and* $\mathbf{N}$ *defined above are given by:*

$$\begin{aligned}
\mathbf{Q}(x,y,u) &= \sum_{n\geq 1} \frac{yx^{n-1}u^n}{\tilde{F}_n \tilde{F}_{n+1}}, \\
\overline{\mathbf{Q}}(x,y,u) &= \sum_{n\geq 1} \frac{yx^{n-1}u^n}{F_n \tilde{F}_{n+1}}, \\
\mathbf{P}(x,y,u,v) &= \sum_{m\geq 0}\sum_{n\geq 0} yx^n u^m v^n \left[\frac{F_m F_{m+1}}{F_{m+n+1}\tilde{F}_{m+n+2}} - \frac{F_{m-1}F_m}{F_{m+n}\tilde{F}_{m+n+1}}\right] \\
&= \sum_{m\geq 0} \frac{u^m x^m y}{\tilde{F}_{m+1}\tilde{F}_{m+2}} + \sum_{m\geq 0}\sum_{n\geq 1} x^{m+1}u^m v^n \left[\frac{\tilde{F}_n \tilde{F}_{n+1}}{\tilde{F}_{m+n+1}\tilde{F}_{m+n+2}} - \frac{F_{n-1}F_n}{F_{m+n}F_{m+n+1}}\right], \\
\mathbf{N}(x,y,u) &= \sum_{n\geq 1} \frac{x^{n-1}yu^n}{\tilde{F}_{n+1}} \prod_{i=1}^{n-1}\left(1 + \frac{F_i}{F_{i+1}}\right), \\
\mathbf{C}(x,y,u) &= \frac{\displaystyle\sum_{n\geq 0} \frac{u^n}{\tilde{F}_{n+1}}}{\displaystyle\sum_{n\geq 0} \frac{u^n}{F_n}} - 1.
\end{aligned}$$

Observe that the last expression defines $\mathbf{C}(x,y,u)$ *as a power series in* u *with rational coefficients in* x *and* y*, but that it is not well-defined when we attempt to set* $u = 1$*.*

Proof. The easiest, but least illuminating way of proving this proposition is to check that the series defined by the above expressions satisfy the functional equations of Proposition 4 and Eq. (2). This verification is straightforward, all the identities resulting from the definition of the polynomials F_n and $\tilde{F}_n$, and from the fact that for $n \geq 1$,

$$\tilde{F}_n\left(x, \frac{x}{1-y}\right) = \frac{\tilde{F}_{n+1}}{1-y}.$$

To prove that the first expression of $\mathbf{P}(x,y,u,v)$ satisfies the third equation of Proposition 4, we also need the identity

$$F_m\tilde{F}_{m+1} - F_{m-1}\tilde{F}_{m+2} = F_m^2 - F_{m-1}F_{m+1} = x^m, \qquad m \geq 0, \tag{7}$$

which is easily proved by induction on m.

■

This is, perhaps, a little dissatisfying and it is more interesting to describe how one can *discover* these identities. Clearly, the first two can be conjectured by inspection: we iterate the corresponding equation a few times to obtain the coefficient of u^n, for small values of n, and the pattern soon becomes clear. The same approach also works for $\mathbf{N}$. Iterating the equation defining $\mathbf{C}$, however, does not suggest at once any simple pattern for the numerator:

$$\mathbf{C}(y) = \frac{uy}{1-y} + \frac{u^2xy(2-x-y)}{(1-x)(1-y)(1-x-y)} + \frac{u^3x^2y(4-9x+4x^2-6y+10xy+2y^2-2xy^2-2x^2y)}{(1-x)(1-2x)(1-y)(1-x-y)(1-2x-y+xy)} + O(u^4).$$

Let us explain how to obtain the expression of $\mathbf{C}$. It is more convenient to start with the series $\overline{\mathbf{C}}(x,y,u) \equiv \overline{\mathbf{C}}(y) = 1 + \mathbf{C}(x,y,u)$, which satisfies

$$\overline{\mathbf{C}}(y) = 1 + \frac{u}{1-y}\overline{\mathbf{C}}\left(\frac{x}{1-y}\right) - u\overline{\mathbf{C}}(x). \tag{8}$$

Let us introduce two operators Φ and Ψ that act on power series of $\mathbb{Q}[[x,y,u]]$:

$$\Phi(\mathbf{S}(x,y,u)) = \frac{u}{1-y}\mathbf{S}\left(x, \frac{x}{1-y}, u\right) \quad \text{and} \quad \Psi(\mathbf{S}(x,y,u)) = -u\mathbf{S}(x,x,u).$$

Observe that, if $\mathbf{S}$ does not depend on y, then

$$\Phi(\mathbf{S}) = \mathbf{S}\Phi(1) \quad \text{and} \quad \Psi(\mathbf{S}) = \mathbf{S}\Psi(1). \tag{9}$$

Using these operators, Eq. (8) can be rewritten as

$$\overline{\mathbf{C}}(y) = 1 + (\Phi + \Psi)\overline{\mathbf{C}}(y).$$

Iterating this equation provides

$$\overline{\mathbf{C}}(y) = \sum_{n\geq 0}(\Phi+\Psi)^n(1) = \frac{1}{1-\Phi-\Psi}(1). \tag{10}$$

Equivalently, we can consider $\overline{\mathbf{C}}(y)$ to be the sum of every word in the language $\{\Phi,\Psi\}^*$ acting on 1,

$$\overline{\mathbf{C}}(y) = \sum_{w\in\{\Phi,\Psi\}^*} w(1).$$

The language $\{\Phi,\Psi\}^*$ can be written as the disjoint union of Φ^* and $\{\Phi,\Psi\}^*\Psi\Phi^*$. It is easy to prove by induction on $n \geq 0$ that

$$\Phi^n(1) = \frac{u^n}{\tilde{F}_{n+1}} \quad \text{and} \quad \Psi\Phi^n(1) = -\frac{u^{n+1}}{F_{n+1}}. \tag{11}$$

Observe that $\Psi\Phi^n(1)$ is independent of y. Applying these facts gives,

$$\begin{aligned}
\overline{\mathbf{C}}(y) &= \sum_{w\in\Phi^*} w(1) + \sum_{w\in\{\Phi,\Psi\}^*\Psi\Phi^*} w(1) \\
&= \sum_{n\geq 0}\Phi^n(1) + \frac{1}{1-\Phi-\Psi}\left(\sum_{n\geq 0}\Psi\Phi^n(1)\right) \\
&= \sum_{n\geq 0}\frac{u^n}{\tilde{F}_{n+1}} - \frac{1}{1-\Phi-\Psi}\left(\sum_{n\geq 0}\frac{u^{n+1}}{F_{n+1}}\right) && \text{(by (11))} \\
&= \sum_{n\geq 0}\frac{u^n}{\tilde{F}_{n+1}} - \left(\sum_{n\geq 0}\frac{u^{n+1}}{F_{n+1}}\right)\frac{1}{1-\Phi-\Psi}(1) && \text{(by (9))} \\
&= \sum_{n\geq 0}\frac{u^n}{\tilde{F}_{n+1}} - \sum_{n\geq 0}\frac{u^{n+1}}{F_{n+1}}\,\overline{\mathbf{C}}(y) && \text{(by (10)).}
\end{aligned}$$

This provides the expression of $\mathbf{C} = \overline{\mathbf{C}} - 1$ given in Proposition 5.

We can proceed similarly for solving the functional equation that defines the pyramid generating function $\mathbf{P}(x,y,u,v) \equiv \mathbf{P}(y)$. This equation can be written

$$\mathbf{P}(y) = \sum_{m\geq 0}\frac{u^m x^m y}{\tilde{F}_{m+1}\tilde{F}_{m+2}} + (\Phi+\Psi)\mathbf{P}(y)$$

where the operators Φ and Ψ are now defined by

$$\Phi(\mathbf{S}(x,y,u,v)) = v\mathbf{S}\left(x, \frac{x}{1-y}, u, v\right) \quad \text{and} \quad \Psi(\mathbf{S}(x,y,u,v)) = -v\mathbf{S}(x,x,u,v).$$

Clearly, $\mathbf{P}(y) = \sum_{m\geq 0} u^m \mathbf{P}_m(y)$ where

$$\mathbf{P}_m(y) = \frac{x^m y}{\tilde{F}_{m+1}\tilde{F}_{m+2}} + (\Phi+\Psi)\mathbf{P}_m(y). \tag{12}$$

It is easy to prove by induction on n that for $n \geq 1$ and $m \geq 0$,

$$\Phi^n\left(\frac{x^m y}{\tilde{F}_{m+1}\tilde{F}_{m+2}}\right) = \frac{v^n x^{m+1}\tilde{F}_n\tilde{F}_{n+1}}{\tilde{F}_{m+n+1}\tilde{F}_{m+n+2}}$$

and for $n \geq 0$ and $m \geq 0$,

$$\Psi\Phi^n\left(\frac{x^m y}{\tilde{F}_{m+1}\tilde{F}_{m+2}}\right) = -\frac{v^{n+1}x^{m+1}F_nF_{n+1}}{F_{m+n+1}F_{m+n+2}}.$$

Moreover, $\Phi(1) = v$ and $\Psi(1) = -v$, so that

$$\frac{1}{1-\Phi-\Psi}(1) = 1.$$

The iteration of Eq. (12) provides

$$\begin{aligned}
\mathbf{P}_m(y) &= \frac{1}{1-\Phi-\Psi}\left(\frac{x^m y}{\tilde{F}_{m+1}\tilde{F}_{m+2}}\right) \\
&= \sum_{n\geq 0}\Phi^n\left(\frac{x^m y}{\tilde{F}_{m+1}\tilde{F}_{m+2}}\right) + \frac{1}{1-\Phi-\Psi}\left(\sum_{n\geq 0}\Psi\Phi^n\left(\frac{x^m y}{\tilde{F}_{m+1}\tilde{F}_{m+2}}\right)\right) \\
&= \frac{x^m y}{\tilde{F}_{m+1}\tilde{F}_{m+2}} + \sum_{n\geq 1}\frac{v^n x^{m+1}\tilde{F}_n\tilde{F}_{n+1}}{\tilde{F}_{m+n+1}\tilde{F}_{m+n+2}} - \sum_{n\geq 0}\frac{v^{n+1}x^{m+1}F_nF_{n+1}}{F_{m+n+1}F_{m+n+2}}\frac{1}{1-\Phi-\Psi}(1) \\
&= \frac{x^m y}{\tilde{F}_{m+1}\tilde{F}_{m+2}} + \sum_{n\geq 1}\frac{v^n x^{m+1}\tilde{F}_n\tilde{F}_{n+1}}{\tilde{F}_{m+n+1}\tilde{F}_{m+n+2}} - \sum_{n\geq 1}\frac{v^n x^{m+1}F_{n-1}F_n}{F_{m+n}F_{m+n+1}}, \\
&= \frac{x^m y}{\tilde{F}_{m+1}\tilde{F}_{m+2}} + \sum_{n\geq 1} v^n x^{m+1}\left[\frac{\tilde{F}_n\tilde{F}_{n+1}}{\tilde{F}_{m+n+1}\tilde{F}_{m+n+2}} - \frac{F_{n-1}F_n}{F_{m+n}F_{m+n+1}}\right].
\end{aligned}$$

This yields the second expression of $\mathbf{P}(x,y,u,v)$ given in Proposition 5.

4 Applications of Viennot's inversion lemma

In the previous section we obtained closed form expressions for five heap generating functions (Proposition 5). These expressions were obtained by solving the functional equations of Proposition 4. In this section we put a new light on four of these five expressions: a simple combinatorial lemma, which is sometimes called the *inversion lemma* and actually applies to any kind of heaps, provides another explanation for them. We have not yet been able to re-explain the generating function for saw-tooth heaps in this way.

So far, we have implicitly defined heaps and animals up to a translation. This will no longer be the case in this section, and we will consider the axis on which the minimal dimers lie to be graded. In this context it is convenient to consider heaps as words of a partially commutative monoid, as explained in [41]. Let $A \subset \mathbb{Z}$, and let A^* be the free monoid on A, that is, the set of words $a_1 \cdots a_k$ on the alphabet A, equipped with the concatenation product: $a_1 \cdots a_k \circ b_1 \cdots b_\ell = a_1 \cdots a_k b_1 \cdots b_\ell$. We consider the congruence on A^* defined by the following partial commutations: for i, j in $\mathbb{Z}$,

$$ij \equiv ji \Longleftrightarrow |i - j| > 1.$$

The elements of the quotient monoid $A^*/\equiv$ can be represented graphically by heaps of dimers above a graded axis: each letter i corresponds to a dimer whose projection on the horizontal axis is centred at i.

Let $x_1, \ldots, x_m$ be m formal variables, and let us endow the letters of A with a weight

$$w : A \longrightarrow \mathbb{R}[x_1, \ldots, x_m].$$

The weight of a word is defined to be the product of the weights of its letters. We assume that the series $\sum_{u \in A^*} w(u)$ is a well-defined formal power series in the x_i. We denote by $|u|$ the number of letters (dimers) of a word u.

The results concerning pyramids and half-pyramids stated in Proposition 5 are consequences of a simple inversion lemma, due to Viennot [41], which actually holds in any partially commutative monoid. We say that a heap is *trivial* if all its dimers are minimal. Given $B \subseteq A$, we denote by $\mathcal{T}(B)$ the set of trivial heaps whose letters (or dimers) belong to B.

Lemma 6 *Let $B \subset A$. The generating function for heaps of $A^*/\equiv$ having all their minimal dimers in B is*

$$\sum_{\substack{H \in A^*/\equiv \\ \min(H) \subset B}} w(H) = \frac{\sum_{T \in \mathcal{T}(A \setminus B)} (-1)^{|T|} w(T)}{\sum_{T \in \mathcal{T}(A)} (-1)^{|T|} w(T)}.$$

4.1 Half-pyramids and pyramids

We first review briefly the derivation of the first three results of Proposition 5 based on the above lemma. To recover the first one, we note that any half-pyramid of width at most n has its dimers in the alphabet $[1-n, 0]$, with its minimal dimer centred at 0, and so we take $A = [1-n, 0]$ and $B = \{0\}$. We choose $w(0) = y$ and $w(i) = x$ for $i < 0$. Using the recurrence relation satisfied by the Fibonacci polynomials, it is easy to check by induction on n that for $n \geq 0$,

$$\sum_{T \in \mathcal{T}(A)} (-1)^{|T|} w(T) = \tilde{F}_{n+1}$$

and that

$$\sum_{T \in \mathcal{T}(A \setminus B)} (-1)^{|T|} w(T) = F_n.$$

By Lemma 6, the generating function for half-pyramids of minimal dimer 0 and width *at most* n is $F_n/\tilde{F}_{n+1}$. This series counts, among others, the empty heap. The generating function for half-pyramids of minimal dimer 0 and width *exactly* n is then

$$\frac{F_n}{\tilde{F}_{n+1}} - \frac{F_{n-1}}{\tilde{F}_n} = \frac{F_n \tilde{F}_n - F_{n-1} \tilde{F}_{n+1}}{\tilde{F}_n \tilde{F}_{n+1}} = \frac{x^{n-1} y}{\tilde{F}_n \tilde{F}_{n+1}},$$

where we have made use of (7).

For the second result of Proposition 5, we change the weights, such that $w(1-n) = y$ and $w(i) = x$ for $i > 1-n$. Then the generating function for half-pyramids of minimal dimer 0 and width *at most* n is $\tilde{F}_n/\tilde{F}_{n+1}$, and the generating function for half-pyramids of minimal dimer 0 and width *exactly* n is

$$\frac{\tilde{F}_n}{\tilde{F}_{n+1}} - \frac{F_{n-1}}{F_n} = \frac{x^{n-1} y}{F_n \tilde{F}_{n+1}}.$$

Finally, to evaluate the generating function of pyramids of left width m and right width n, we take $A = [-m, n]$ and $B = \{0\}$. We choose $w(n) = y$ and $w(i) = x$ for $i < n$. Then the generating function for pyramids of minimal dimer 0, left width at most m and right width at most n is

$$\frac{F_{m+1}\tilde{F}_{n+1}}{\tilde{F}_{m+n+2}},$$

and the generating function for pyramids of minimal dimer 0, left width exactly m and right width exactly n is

$$\frac{F_{m+1}\tilde{F}_{n+1}}{\tilde{F}_{m+n+2}} - \frac{F_m\tilde{F}_{n+1}}{\tilde{F}_{m+n+1}} - \frac{F_{m+1}F_n}{F_{m+n+1}} + \frac{F_mF_n}{F_{m+n}}.$$

If we separate the first two terms of this expression from the last two terms then simplification gives the second expression of $\mathbf{P}(x, y, u, v)$ given in Proposition 5, thanks to the following identity, valid for $m \geq 0$:

$$F_{m+1}\tilde{F}_{m+n+1} - F_m\tilde{F}_{m+n+2} = \begin{cases} yx^m & \text{if } n = 0, \\ x^{m+1}\tilde{F}_n & \text{otherwise.} \end{cases}$$

Similarly, if we separate the first and third terms from the second and fourth terms then we obtain the first expression of $\mathbf{P}(x, y, u, v)$ given in Proposition 5, thanks to the following identity, valid for $m \geq -1, n \geq 0$:

$$\tilde{F}_{n+1}F_{m+n+1} - F_n\tilde{F}_{m+n+2} = x^n y F_m.$$

4.2 Connected heaps

We now give a combinatorial explanation of the connected heap result. Let $\overline{\mathbf{C}}(x, y, u) = 1 + \mathbf{C}(x, y, u)$ denote the generating function for (possibly empty) connected heaps, considered up to a translation. Proposition 5 states that

$$\sum_{n\geq 0} \frac{u^n}{\tilde{F}_{n+1}} = \overline{\mathbf{C}}(x, y, u) \sum_{m\geq 0} \frac{u^m}{F_m}. \tag{13}$$

If we scan any general heap from left to right, we find that it is made up of several connected heaps, with some non-zero number of empty columns between them. The above identity, roughly speaking, comes from the fact that any general heap can be split into its rightmost connected component and a remaining smaller heap. More precisely, let us write

$$\overline{\mathbf{C}}(x, y, u) = \sum_{k\geq 0} \overline{\mathbf{C}}_k(x, y)u^k,$$

where $\overline{\mathbf{C}}_k(x, y)$ counts connected heaps of dimers of width k (the variable y counts the dimers in the rightmost column, and the variable x counts the remaining dimers). Extracting from (13) the coefficient of u^n yields the identity

$$\frac{1}{\tilde{F}_{n+1}(x, y)} = \sum_{m\geq 0} \frac{1}{F_m(x)} \cdot \overline{\mathbf{C}}_{n-m}(x, y),$$

which we are going to explain combinatorially.

Take $A = [0, n-1]$, $w(n-1) = y$ and $w(i) = x$ for $i < n-1$. Then $1/\tilde{F}_{n+1}$ is the generating function for all heaps on the alphabet A. Let H be such a heap. Let C be the connected component of H that contains all the letters $n-1$ occurring in H (see Figure 19). If there is no such letter in H, then C is empty. Let $n - m$ be the width of C, with $0 \leq m \leq n$. Having deleted C from H, we are left with a heap H' on the alphabet $[0, m-2]$. By the inversion lemma, such heaps are counted by $1/F_m$. The result follows.

5 The generating functions for connected heaps and saw-tooth heaps

The expressions given in Proposition 5 for the five heap generating functions raise two questions. Firstly, can one recover from the first three of them the simple algebraic results of Proposition 1? Secondly, can

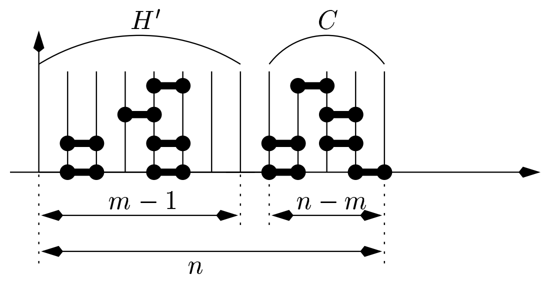

Figure 19: The combinatorics of the connected heaps generating function.

one go further with the expressions obtained for connected heaps and saw-tooth heaps? In particular, we aim at proving Proposition 3, but the expression of the series $\mathbf{C}(x, y, u)$ we have obtained so far is not even well-defined when $u = 1$.

This section brings answers to these questions. The key tool is to express the Fibonacci polynomials in terms of the generating function for Catalan numbers:

$$Q = \frac{1 - 2x - \sqrt{1 - 4x}}{2x} = \sum_{n \geq 1} \frac{1}{n+1} \binom{2n}{n} x^n, \tag{14}$$

which is also the generating function for half-pyramids. The recursion defining the Fibonacci polynomials is equivalent to

$$\sum_{n \geq 0} F_n u^n = \frac{1}{1 - u + xu^2}.$$

Expanding this rational function in partial fractions of u and taking the coefficient of u^n yields, for $n \geq 0$,

$$F_n = \frac{1 - Q^{n+1}}{(1 - Q)(1 + Q)^n}. \tag{15}$$

Moreover, x and Q are related by

$$x = \frac{Q}{(1 + Q)^2}. \tag{16}$$

Using these identities we can recover from Proposition 5 the results of Proposition 1 about half-pyramids and pyramids. For instance, the generating function for half-pyramids is

$$\begin{aligned}
\mathbf{Q}(x, x, 1) &= \sum_{n \geq 1} \frac{x^n}{F_n F_{n+1}} \\
&= (1 - Q^2) \sum_{n \geq 1} \frac{Q^n (1 - Q)}{(1 - Q^{n+1})(1 - Q^{n+2})} \\
&= (1 - Q^2) \sum_{n \geq 1} \left(\frac{Q^n}{1 - Q^{n+1}} - \frac{Q^{n+1}}{1 - Q^{n+2}} \right) \\
&= (1 - Q^2) \frac{Q}{1 - Q^2} = Q, \qquad (17)
\end{aligned}$$

as stated in Proposition 1.

More generally, all the series of Proposition 5 can be expressed in terms of u, v, y and Q, using Eqs. (15) and (16). In particular, we can now complete the enumeration of connected heaps.

Proposition 7 (Multi-directed animals) *Let $Q(x) \equiv Q$ denote the generating function for half-pyramids. Then the generating function for connected heaps of dimers is*

$$\mathbf{C}(x,x,1) = \frac{Q}{(1-Q)\left[1-\sum_{k\geq 1}\frac{Q^{k+1}}{1-Q^k(1+Q)}\right]},$$

as stated in Proposition 3. It is not D-finite. The number of connected heaps having n dimers (or multi-directed animals having n cells) is asymptotic to $A\mu^n$ for some positive constant A, with $\mu = 4.58789436...$ The average width of these heaps is asymptotic to Bn for some positive constant B.

Proof. Let us start from the following expression of $\mathbf{C}(x,x,u)$ derived from Proposition 5 (remembering that $\tilde{F}_n(x,x) = F_n$):

$$\begin{aligned}\mathbf{C}(x,x,u) &= \frac{1}{u} - 1 - \left[\sum_{n\geq 0}\frac{u^{n+1}}{F_n}\right]^{-1} \\ &= \frac{1}{u} - 1 - \frac{1}{1-Q}\left[\sum_{n\geq 1}\frac{u^n(1+Q)^{n-1}}{1-Q^n}\right]^{-1}. \end{aligned} \quad (18)$$

We cannot replace u by 1 directly; let us isolate the part that becomes singular at $u=1$:

$$\begin{aligned}\mathbf{C}(x,x,u) &= \frac{1}{u} - 1 - \frac{1}{1-Q}\left[\sum_{n\geq 1}u^n(1+Q)^{n-1}\left(1+\frac{Q^n}{1-Q^n}\right)\right]^{-1} \\ &= \frac{1}{u} - 1 - \frac{1-u(1+Q)}{u(1-Q)}\left[1+(1-u(1+Q))\sum_{n\geq 1}\frac{u^{n-1}Q^n(1+Q)^{n-1}}{1-Q^n}\right]^{-1}.\end{aligned}$$

Hence, at $u=1$,

$$\mathbf{C}(x,x,1) = \frac{Q}{(1-Q)\left[1-\sum_{n\geq 1}\frac{Q^{n+1}(1+Q)^{n-1}}{1-Q^n}\right]}.$$

Finally, to recover the expression of Proposition 7, we observe that

$$\begin{aligned}\sum_{n\geq 1}\frac{Q^{n+1}(1+Q)^{n-1}}{1-Q^n} &= \sum_{n\geq 1}Q^{n+1}(1+Q)^{n-1}\sum_{k\geq 0}Q^{nk} \quad (19)\\ &= \frac{Q}{1+Q}\sum_{k\geq 0}\sum_{n\geq 1}\left(Q^{k+1}(1+Q)\right)^n \\ &= \sum_{k\geq 1}\frac{Q^{k+1}}{1-Q^k(1+Q)}. \quad (20)\end{aligned}$$

Let us now prove that $\mathbf{C}(x,x,1)$ is not D-finite. Let $f(q) = q/[(1-q)g(q)]$, where

$$g(q) = 1 - \sum_{k\geq 1}\frac{q^{k+1}}{1-q^k(1+q)}.$$

Hence $\mathbf{C}(x,x,1) = f(Q(x))$, where $Q(x)$ is the algebraic function of x given by (14). Conversely,

$$f(q) = \mathbf{C}(q/(1+q)^2, q/(1+q)^2, 1).$$

These relations imply that $\mathbf{C}(x,x,1)$ is D-finite if and only if $f(q)$ is D-finite (substituting an algebraic series in a D-finite one preserves holonomy, see [38, Theorem 2.7]). We will show that this is not the case.

The series $g(q)$ is meromorphic on $\{q : |q| < 1\}$, with simple poles at all values of q satisfying

$$q^k(1+q) = 1, \quad |q| < 1.$$

These values can be seen to accumulate on $\mathcal{P} = \{e^{i\theta}, -2\pi/3 \le \theta \le 2\pi/3\}$ as $k \to \infty$ (Figure 20). Hence $f(q)$ is meromorphic on $\{q : |q| < 1\}$, and has an accumulation of zeroes on $\mathcal{P}$. Consequently, any point of $\mathcal{P}$ is a singularity of f (an analytic continuation of f would be zero on a segment of the unit circle, which is impossible for a non-trivial holomorphic function). As a D-finite function has only finitely many singularities, $f(q)$ cannot be D-finite.

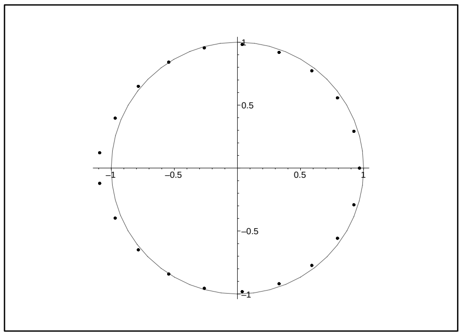

Figure 20: The 21 zeroes of $q^{20}(1+q) - 1$.

Let us write $\mathbf{C}(x,x,1) = \sum_n c_n x^n$, so that c_n is the number of connected heaps having n dimers. In order to determine the asymptotic behaviour of c_n, we need to study the *dominant* singularities of $\mathbf{C}(x,x,1) = f(Q(x))$, that is, the singularities of smallest modulus. The singularities of $\mathbf{C}(x,x,1)$ are to be found among the singularities of $Q(x)$ and the values of x such that $Q(x)$ is a singularity of f. The series $Q(x)$ has a unique singularity at $x = 1/4$. What about f?

The series $f(q)$ is meromorphic on $|q| < 1$. Its singularities in this domain are isolated poles corresponding to the zeroes of g. The series $1 - g(q) = \sum_{k\ge1} \frac{q^{k+1}}{1-q^k(1+q)}$ has positive coefficients, is holomorphic on $|q| < (\sqrt{5}-1)/2$ and tends to infinity as q tends to $(\sqrt{5}-1)/2$. This implies that $g(q)$ has a unique zero of minimal modulus, denoted q_c, which can be estimated numerically: $q_c = 0.4727898832...$ Moreover, this zero is simple: $g'(q_c) < 0$. This zero is the unique dominant singularity of f. It is a simple pole.

Now $Q(x)$ has positive coefficients, $Q(0) = 0$ and $Q(1/4) = 1$. Therefore there exists $x_c \in [0, 1/4]$ such that $Q(x_c) = q_c$. This implies that $\mathbf{C}(x,x,1) = f(Q(x))$ has a unique dominant singularity, $x_c = q_c/(1+q_c)^2 = 0.2179649141...$, which is a simple isolated pole. It follows that $c_n \sim A x_c^{-n} = A(4.58789436...)^n$ as $n \to \infty$.

Finally, let us study the average width of connected heaps. We differentiate the expression of $\mathbf{C}(x,x,u)$ given in Eq. (18) with respect to u. We apply to this series the same treatment that led to the expression of $\mathbf{C}(x,x,1)$. We obtain:

$$\begin{aligned} \mathbf{C}'_u(x,x,1) &= \frac{1 + \sum_{n\ge1} \frac{nQ^{n+2}(1+Q)^{n-1}}{1-Q^n}}{(1-Q)\left[1 - \sum_{n\ge1} \frac{Q^{n+1}(1+Q)^{n-1}}{1-Q^n}\right]^2} - 1 \\ &= \frac{1 + \sum_{k\ge1} \frac{Q^{k+2}}{(1-Q^k(1+Q))^2}}{(1-Q)\left[1 - \sum_{k\ge1} \frac{Q^{k+1}}{1-Q^k(1+Q)}\right]^2} - 1. \end{aligned} \tag{21}$$

We have seen that $\mathbf{C}(x,x,1)$ has a unique dominant singularity, which is a simple pole, at $x_c = q_c/(1+q_c)^2$, where q_c is the smallest positive solution of $1 = \sum_{k\ge1} \frac{q^{k+1}}{1-q^k(1+q)}$. Similarly, we derive from (21) that

$\mathbf{C}'_u(x,x,1)$ has a unique dominant singularity, which is a double pole, at x_c. Hence the coefficient of x^n in $\mathbf{C}'_u(x,x,1)$ is asymptotic to Bnx_c^{-n}. The result follows.

■

Let us now turn our attention to saw-tooth animals.

Proposition 8 (Saw-tooth animals) *Let $Q(x) \equiv Q$ denote the generating function for half-pyramids. Then the generating function $\mathbf{N}(x,x,1)$ for saw-tooth heaps is*

$$\mathbf{N}(x,x,1) = (1-Q)\sum_{n\geq 1} \frac{Q^n}{(1+Q)^{n-1}(1-Q^{n+2})} \prod_{i=1}^{n-1} \left(1 + \frac{(1+Q)(1-Q^{i+1})}{1-Q^{i+2}}\right).$$

This series is not D-finite. The number of n-celled saw-tooth animals is asymptotic to $A\mu^n$ for some positive constant A, with $\mu = \sqrt{5}+2 = 4.236\ldots$. The average width of these animals is asymptotic to Bn for some positive constant B.

Proof. We obtain the expression of $\mathbf{N}(x,x,1)$ from Proposition 5 by replacing x and the Fibonacci polynomials F_i by their expressions in terms of Q (see Eqns. (15) and (16)). This leads us to introduce the following power series in the formal variable q:

$$\begin{aligned} f(q) &= (1-q)\sum_{n\geq 1} \frac{q^n}{(1+q)^{n-1}(1-q^{n+2})} \prod_{i=1}^{n-1} \left[1 + \frac{(1+q)(1-q^{i+1})}{1-q^{i+2}}\right], \\ &= (1-q)\sum_{n\geq 1} \frac{q^n(2+q)^{n-1}}{(1+q)^{n-1}(1-q^{n+2})} \prod_{i=1}^{n-1} \left[1 - \frac{(1-q^2)q^{i+1}}{(2+q)(1-q^{i+2})}\right]. \end{aligned}$$

Again, $\mathbf{N}(x,x,1)$ is D-finite if and only $f(q)$ is D-finite. We are going to prove that this is not the case, as $f(q)$ is meromorphic inside the unit circle, with infinitely many poles.

We first want to show that the product in the expression of $f(q)$ converges inside the unit circle and so we can exclude it from our analysis of the singularities. The equation $(1-q^2)q^{i+1} = (2+q)(1-q^{i+2})$ can be rewritten

$$q^i = \frac{1+2q^{-1}}{1+2q}.$$

It is not difficult to see that $|1+2q^{-1}| \geq |1+2q|$ if and only if $|q| \leq 1$. This implies that for $i \geq 0$, all roots of the above equation lie on the unit circle. Moreover, for $|q|<1$, the series $\sum_i q^{i+1}/(1-q^{i+2})$ is convergent. Consequently, as n goes to infinity,

$$\lim_{n\to\infty} \prod_{i=1}^{n-1} \left[1 - \frac{(1-q^2)q^{i+1}}{(2+q)(1-q^{i+2})}\right] = \prod_{i\geq 1} \left[1 - \frac{(1-q^2)q^{i+1}}{(2+q)(1-q^{i+2})}\right] \equiv W(q),$$

and the function $W(q)$ is an holomorphic function with no zero inside the unit circle. Hence, for $|q|<1$, we can write

$$f(q) = (1-q)W(q)\sum_{n\geq 1} \frac{q^n(2+q)^{n-1}}{(1+q)^{n-1}(1-q^{n+2})} \prod_{i\geq 1} \left[1 - \frac{(1-q^2)q^{n+i}}{(2+q)(1-q^{n+i+1})}\right]^{-1}.$$

In order to determine the poles of $f(q)$ in the domain $|q|<1$, we shall now perform a resummation that is similar to the transformation of (19) into (20). We have:

$$f(q) = (1-q)W(q)\sum_{n\geq 1} \frac{q^n(2+q)^{n-1}}{(1+q)^{n-1}} g(q^n, q), \tag{22}$$

where

$$g(z,q) = \frac{1}{1-zq^2} \prod_{i\geq 1} \left[1 - \frac{(1-q^2)zq^i}{(2+q)(1-zq^{i+1})}\right]^{-1}$$

is an holomorphic function of z for $|z| < 1$. Let us expand $g(z,q)$ in z:

$$g(z,q) = \sum_{k \geq 0} z^k g_k(q),$$

where $g_k(q)$ is a rational function of q, of denominator $(2+q)^k$, that remains positive on the interval $(0,1)$. In the expression (22) of $f(q)$, we replace $g(q^n, q)$ by its expansion, exchange the summations on n and k, perform the summation on $n$and end up with

$$f(q) = (1-q)W(q) \sum_{k \geq 0} \frac{g_k(q) q^{k+1}}{1 - \frac{q^{k+1}(2+q)}{1+q}}.$$

We can now work out the analytic structure of f. The above expression shows that $f(q)$ is meromorphic inside the unit circle and has only simple poles, that are to be found among the roots of $q^{k+1}(2+q) = 1+q$, $k \geq 0$. For $k \geq 0$, the solution of smallest modulus of the equation $q^{k+1}(2+q) = 1+q$, denoted q_k, belongs to $[0,1]$, and increases to 1 as k goes to infinity. As $g_k(q) > 0$ for $q \in (0,1)$, we see that each q_k is indeed a simple pole of f. Hence f has infinitely many singularities, and cannot be D-finite.

The radius of f is given by its smallest pole, $q_0 = (\sqrt{5}-1)/2$. As the only singularity of $Q(x)$ is at $x = 1/4$, this implies that $\mathbf{N}(x,x,1)$ has a unique dominant singularity, $x_c = q_0/(1+q_0)^2 = \sqrt{5}-2$, which is a simple pole. Hence the number of n-celled saw-tooth animals is asymptotic to $A(\sqrt{5}+2)^n$.

The study of the average width of saw-tooth heaps is extremely close to what we have already done. With the notations used above, we have

$$\begin{aligned}
\mathbf{N}'_u(x,x,1) &= (1-Q) \sum_{n \geq 1} \frac{nQ^n (2+Q)^{n-1}}{(1+Q)^{n-1}(1-Q^{n+2})} \prod_{i=1}^{n-1} \left[1 - \frac{(1-Q^2)Q^{i+1}}{(2+Q)(1-Q^{i+2})} \right] \\
&= (1-Q)W(Q) \sum_{n \geq 1} \frac{nQ^n(2+Q)^{n-1}}{(1+Q)^{n-1}} g(Q^n, Q), \\
&= (1-Q)W(Q) \sum_{k \geq 0} \frac{g_k(Q) Q^{k+1}}{\left(1 - \frac{Q^{k+1}(2+Q)}{1+Q}\right)^2}.
\end{aligned}$$

From this, we conclude that $\mathbf{N}'_u(x,x,1)$ has a unique dominant singularity, which is a double pole at $x_c = \sqrt{5}-2$. The result follows.

■

6 Half-pyramids and pyramids: what parameters are algebraic?

Using the results obtained in Section 3 on the enumeration of heaps of given width we can show that certain series are transcendental, and even non-D-finite. For instance, counting pyramids (or directed animals) by their width and number of dimers yields a non-D-finite series. Recall the definition of the series $\mathbf{Q}(x,y,u)$, $\overline{\mathbf{Q}}(x,y,u)$ and $\mathbf{P}(x,y,u,v)$, illustrated in Figure 16. Our results are summarized in Table 2, in which the operator D_u denotes the differentiation with respect to the variable u. These results are stated and proved for general heaps (triangular lattice animals), but the contraction/expansion operation implies that they also hold for square lattice animals.

To prove the first two results of the table, we use the factorisation of (general) half-pyramids and pyramids described in Section 2. For half-pyramids, we have to add to the three cases of Figure 7 a fourth case, where there is a dimer directly above the minimal one. We obtain:

$$\mathbf{Q}(x,y,1) = y + y\mathbf{Q}(x,x,1) + y\mathbf{Q}(x,x,1)\mathbf{Q}(x,y,1) + y\mathbf{Q}(x,y,1) = y(1+\mathbf{Q}(x,x,1))(1+\mathbf{Q}(x,y,1)).$$

Then, Figure 7 gives the following equation for pyramids:

$$\mathbf{P}(x,y,1,v) = \mathbf{Q}(x,y,1) + v\mathbf{Q}(x,x,1)\mathbf{P}(x,y,1,v).$$

Type of heaps	Parameters	Series	Nature
Half-pyramids	area + dimers in the rightmost column	$\mathbf{Q}(x,y,1)$	algebraic
Pyramids	+ right width	$\mathbf{P}(x,y,1,v)$	algebraic
Half-pyramids	area + dimers in the leftmost column	$\overline{\mathbf{Q}}(x,y,1)$	non DF
Half-pyramids with a marked column	area	$(D_u\mathbf{Q})(x,x,1)$	non DF
Pyramids with a marked column	area	$D_u(u\mathbf{P}(x,x,u,u))\vert_{u=1}$	algebraic
Half-pyramids	area + width	$\mathbf{Q}(x,x,u)$	non DF
Pyramids	area + width	$u\mathbf{P}(x,x,u,u)$	non DF

Table 2: The nature of various generating functions for pyramids and half-pyramids.

From these two equations, we obtain

$$\mathbf{Q}(x,x,1) = Q, \quad \mathbf{Q}(x,y,1) = \frac{y(1+Q)}{1-y(1+Q)} \quad \text{and } P(x,y,1,v) = \frac{y(1+Q)}{[1-y(1+Q)][1-vQ]}.$$

We could also derive these results from the expressions of Proposition 5, as we have done at the beginning of Section 5 for $\mathbf{Q}(x,x,1)$, but this would be more tedious and less combinatorial.

Let us now focus on the transcendence results. To prove the first one, it suffices to prove that the generating function for half-pyramids having only one dimer in their leftmost column is not D-finite. From Proposition 5, this series is

$$[y]\overline{\mathbf{Q}}(x,y,1) = \sum_{n\geq 1} \frac{x^{n-1}}{F_n^2}.$$

We replace x and F_n by their expressions in terms of Q and obtain

$$(D_y\overline{\mathbf{Q}})(x,0,1) = (1-Q^2)^2 \sum_{n\geq 1} \frac{Q^{n-1}}{(1-Q^{n+1})^2}.$$

Similarly, after a few reductions comparable to (17), we find:

$$(D_u\mathbf{Q})(x,x,1) = (1-Q^2) \sum_{n\geq 1} \frac{Q^n}{1-Q^{n+1}}.$$

In order to prove that these series are not D-finite, we have to prove that the divisor generating functions

$$\sum_{n\geq 1} \frac{q^n}{1-q^n} = \sum_{m\geq 1} q^m \sum_{d|m} 1 \qquad \text{and} \qquad \sum_{n\geq 1} \frac{q^n}{(1-q^n)^2} = \sum_{m\geq 1} q^m \sum_{d|m} d$$

are not D-finite. These series have integer coefficients, and have radius of convergence 1. By a theorem of Pólya-Carlson (a series with integer coefficients that converges inside the unit disk is either rational or has the unit circle as a natural boundary [13]), these series are either rational, or not D-finite. But these two series are actually known to be irrational (this can be proven [2] by adapting an argument of [3]). Hence they are not D-finite. As $(D_u\mathbf{Q})(x,x,1)$ is not D-finite, the series $\mathbf{Q}(x,x,u)$ cannot be D-finite either.

There remains to prove the two results of Table 2 concerning the width of pyramids. The generating function of pyramids according to their number of dimers and width is $\overline{\mathbf{P}}(x,u) := u\mathbf{P}(x,x,u,u)$. The series $D_u\overline{\mathbf{P}}(x,1)$ counts pyramids where a column is marked, and *is algebraic*. This can be derived from the expression of $\mathbf{P}(x, y, u, v)$ given in Proposition 5, but is more easily obtained via the factorisation of pyramids described in Figure 21. This factorisation gives:

$$\begin{aligned} D_u \overline{\mathbf{P}}(x,1) &= \overline{\mathbf{P}}(x,1) + 2\overline{\mathbf{P}}(x,1)^2 \\ &= \frac{Q}{1-Q} + 2\frac{Q^2}{(1-Q)^2} \\ &= \frac{Q(1+Q)}{(1-Q)^2}. \end{aligned}$$

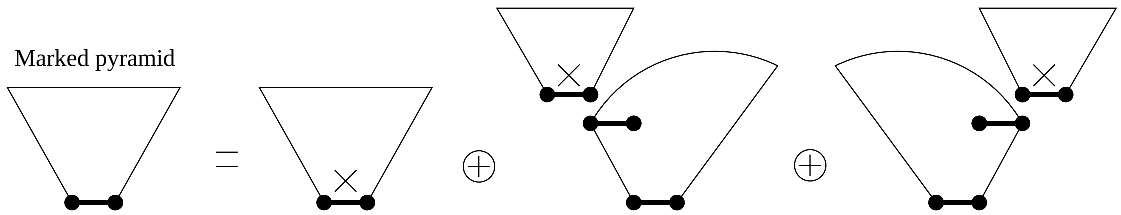

Figure 21: The generating function for pyramids with a marked column is algebraic.

From the first expression of $\mathbf{P}(x, y, u, v)$ given in Proposition 5, we find that the generating function $\mathbf{P}_N$ for pyramids of width N is

$$\mathbf{P}_N = [u^N]\overline{\mathbf{P}}(x,u) = \sum_{m+n=N-1} \frac{Q^{n+1}(1-Q^{m+1})}{1-Q^{N+1}} \left[\frac{1-Q^{m+2}}{1-Q^{N+2}} - \frac{1-Q^m}{1-Q^N} \right].$$

This sum can be easily evaluated:

$$\begin{aligned} \mathbf{P}_N &= \frac{(1-Q^2)Q^N}{(1-Q^N)(1-Q^{N+1})(1-Q^{N+2})} \left[N(1+Q^{N+1}) - 2Q\frac{1-Q^N}{1-Q} \right] \\ &= \frac{Q^N(1+Q)}{1-Q} \left[\frac{N}{1-Q^N} - 2\frac{Q(N+1)}{1-Q^{N+1}} + \frac{Q^2(N+2)}{1-Q^{N+2}} \right]. \end{aligned}$$

The series $\overline{\mathbf{P}}(x, u)$ can be seen as a formal power series in u with rational coefficients in x. If it were D-finite, so would be the following series:

$$\mathbf{S}(q,u) := \overline{\mathbf{P}}\left(\frac{q}{(1+q)^2}, u \right) = \sum_{N \geq 1} \mathbf{S}_N(q) u^N$$

with

$$\mathbf{S}_N(q) = \frac{q^N(1+q)}{1-q} \left[\frac{N}{1-q^N} - 2\frac{q(N+1)}{1-q^{N+1}} + \frac{q^2(N+2)}{1-q^{N+2}} \right].$$

For $N \geq 3$, any Nth primitive root of 1 is a pole of $\mathbf{S}_N(q)$. This property, combined with the following lemma, shows that $\mathbf{S}(q, u)$, and hence $\overline{\mathbf{P}}(x, u)$, cannot be D-finite.

■

Lemma 9 *Let $S(q,u) = \sum_n S_n(q)u^n$ be a formal power series in u with coefficients in $\mathbb{C}(q)$. Assume that $S(q,u)$ is D-finite in u. For $n \geq 0$, let $\mathcal{P}_n$ be the set of poles of $S_n(q)$, and let $\mathcal{P} = \cup_n \mathcal{P}_n$. Then $\mathcal{P}$ has only a finite number of limit points.*

Proof. By assumption, the sequence $S_n(q)$ satisfies a linear recurrence relation with coefficients in $\mathbb{C}[q, n]$ (obtained by extracting the coefficient of u^n in the differential equation satisfied by $S(q, u)$):

$$a_0(q,n)S_n(q) = a_1(q,n)S_{n-1}(q) + \cdots + a_k(q,n)S_{n-k}(q),$$

for $n \geq n_0$, with $a_i(q,n) \in \mathbb{C}[q,n]$, and $a_0(q,n) \not\equiv 0$. Consequently, $S_n(q)$ can be written as a rational function of denominator

$$D_n(q) = I(q) \prod_{m=1}^{n} a_0(q,m),$$

where $I(q)$ is a polynomial accounting for the denominators of the $S_m(q)$, $m < n_0$. Let $\ell \in \mathbb{C}$ be a limit point of the set $\mathcal{P}$: there exists a sequence (q_i, n_i) such that $q_i \to \ell$, $n_i \to \infty$ and $a_0(q_i, n_i) = 0$ for all i. Let the degree of $a_0(q,n)$ in n be d, and let us write

$$a_0(q,n) = \sum_{k=0}^{d} b_k(q) n^k.$$

Then

$$0 = \frac{a_0(q_i, n_i)}{n_i^d} = \sum_{k=0}^{d} b_k(q_i) n_i^{k-d},$$

and this converges to $b_d(\ell)$ as i goes to infinity. Hence all limit points of $\mathcal{P}$ are roots of the polynomial $b_d(q)$. This concludes the proof.

■

7 Summary of the results, and tables

We have enumerated exactly two new families of square lattice animals: stacked directed animals and multi-directed animals. Both include directed animals. Each of these families is in one-to-one correspondence with a natural set of (strict) heaps of dimers. The corresponding growth constants are 3.5 and 3.58... respectively, improving on the growth constants of directed animals and column-convex animals (3 and 3.20... respectively).

By removing the condition that heaps should be strict, we have enumerated exactly two analogous families of triangular lattice animals. The growth constants are 4.5 and 4.58... respectively, which should be compared to the growth constant 4 of directed animals on the triangular lattice. On the triangular lattice, we have also enumerated a third new class of animals, called saw-tooth animals, which have a growth constant of $2 + \sqrt{5} = 4.2...$. They have no simple square lattice counterpart and are perhaps more "artificial" that the other two classes.

We have used two kinds of recursive constructions for heaps of dimers: the first one concatenates two heaps, and is well-suited to the enumeration of directed animals and stacked directed animals. The second one adds a new column to a heap, is well-suited to the enumeration of directed animals and multi-directed animals.

We have highlighted the analytic structure of the generating functions and the nature of their singularities. The generating function for stacked directed animals is algebraic, while the generating function for multi-directed animals is not D-finite. Both have a simple pole as their dominant singularity. We have also proved that certain series, such as the generating function for directed animals counted by their area and width, are not D-finite.

Our results are summarised in Table 3 below. The data in the last line of this table are estimates: see [14] for the square lattice growth constant, [26] for the average width, and [24, 43] for the triangular lattice growth constant. The number of n-celled animals in each family, for small values of n, is given in Tables 4 (square lattice) and 5 (triangular lattice).

Let us conclude by mentioning a parameter, defined on directed animals, that seems to resist the two constructions we have used in this paper but still yields an algebraic generating function. Let A be a directed animal on the square lattice. We say that a cell c of A is *only supported on the right* if the south-east neighbour of c belongs to A, but not its south-west neighbour. Similarly, in a triangular lattice directed animal A, we say that a cell c is *only supported on the right* if, in addition to the above two conditions, the south neighbour of c does not belong to A. For instance, the first animal of Figure 5 has 3 cells only supported on the right, while the second animal has 2 such cells.

Model	Growth constant	Average width	Nature of the GF
Column convex polyominoes	3.20... □ 3.86... △	n	rational (Temperley [39]) rational (Klarner [29])
Directed animals (pyramids)	3 □ 4 △	$\sqrt{n}$ (Gouyou-Viennot [23])	quadratic (Dhar [15])
Stacked directed animals (stacked pyramids)	3.5 □ 4.5 △	$O(n)$	quadratic (Proposition 2)
Saw-tooth animals (saw-tooth heaps)	 4.23... △	n	not D-finite (Proposition 7)
Multi-directed animals (connected heaps)	3.58... □ 4.58... △	n	not D-finite (Proposition 8)
All animals	4.06... □ 5.18... △	$n^{0.64...}$	?

Table 3: Comparison of the various models.

Using a formal connection between one-dimensional gas models and directed animals on the square and triangular lattices it has been proved that the generating function of directed animals, counted by their area and number of cells only supported on the right, is algebraic [10]. So far, this result has resisted heap-based approaches. Moreover, its seems that the column-by-column construction of Section 3 is not well-suited either. We hope that a more transparent and combinatorial proof will be found for this apparently simple result. The construction recently given by Shapiro [37] is elegant and simple, but it does not seem well-adapted to this parameter either, at least at first sight.

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cells	*directed*	*column − convex*	*stacked dir.*	*multi − directed*	*all*
1	1	1	1	1	1
2	2	2	2	2	2
3	5	6	6	6	6
4	13	19	19	19	19
5	35	61	63	63	63
6	96	196	213	214	216
7	267	629	729	738	760
8	750	2017	2513	2571	2725
9	2123	6466	8703	9020	9910
10	6046	20727	30232	31806	36446
11	17303	66441	105236	112572	135268
12	49721	212980	366849	399548	505861
13	143365	682721	1280131	1421145	1903890
14	414584	2188509	4470354	5063254	7204874
15	1201917	7015418	15619386	18062902	27394666
16	3492117	22488411	54595869	64505148	104592937
17	10165779	72088165	190891131	230547424	400795844
18	29643870	231083620	667590414	824547052	1540820542
19	86574831	740754589	2335121082	2950565215	5940738676
20	253188111	2374540265	8168950665	10562978104	22964779660

Table 4: Square lattice data

cells	*directed*	*column − convex*	*stacked directed*	*multi − directed*	*all*
1	1	1	1	1	1
2	3	3	3	3	3
3	10	11	11	11	11
4	35	42	44	44	44
5	126	162	184	184	186
6	462	626	789	790	814
7	1716	2419	3435	3450	3652
8	6435	9346	15100	15242	16689
9	24310	36106	66806	67895	77359
10	92378	139483	296870	304267	362671
11	352716	538841	1323318	1369761	1716033
12	1352078	2081612	5911972	6188002	8182213
13	5200300	8041537	26455294	28031111	39267086
14	20058300	31065506	118528793	127253141	189492795
15	77558760	120010109	531540891	578694237	918837374
16	300540195	463614741	2385375732	2635356807	4474080844
17	1166803110	1791004361	10710619014	12015117401	21866153748
18	4537567650	6918884013	48112492938	54831125131	107217298977
19	17672631900	26728553546	216195753066	250418753498	527266673134
20	68923264410	103255896932	971744791032	1144434017309	2599804551168

Table 5: Triangular lattice data (even though directed animals have a larger connective constant than column-convex animals, there are more n-celled column-convex animals than n-celled directed animals up until $n = 42$).

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