On conjecture no. 58 arising from the OEIS

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In [1] certain conjectures arising from the numerical data in the online encyclopedia of integer sequences ([2]) are presented. Problem no. 58 is to prove that for the sequence (a_n) with $a_1 = 1$, $a_2 = 2$ and a_n the smallest number not of the form a_i, $a_{n-1} + a_i$ or $|a_{n-1} - a_i|$ with $i \leq n-1$ we have

$$a_n = 3n + 3 - 2 \cdot 2^{\lfloor \lg(n+2) \rfloor} \qquad \text{for } n > 2.$$

The sequence (a_n) can be found in [2] as *A034702*.
Let n be a natural number. Then n can be written in a unique way as

$$n = 2^k - 2 + r$$

with $r < 2^k$. We define a sequence (b_n) by

$$b_n := 2^k - 3 + 3r \qquad \text{for} \qquad n = 2^k - 2 + r.$$

An alternative description of this sequence is

$$b_1 = 2, \qquad b_n = \begin{cases} \frac{b_{n-1}}{2}, & n = 2^k - 2, \\ b_{n-1} + 3, & \text{otherwise.} \end{cases}$$

Indeed we also have

$$b_n = 3n + 3 - 2 \cdot 2^{\lfloor \lg(n+2) \rfloor} \qquad \text{for } n > 2.$$

We are going to show that $a_n = b_n$ (except for $n = 1, 2$; the values of a_1 and a_2 should be interchanged). This is done in the following three easy lemmata.

Lemma 1 *Let $n = 2^k - 2 + r$, $r < 2^k$. We have*

$$\begin{aligned} \{b_i, i \leq n\} = &\{x | x \equiv b_n \mod 3, x \leq 2^k - 3 + 3r = b_n\} \\ &\cup \{y | y \equiv -b_n \mod 3, y \leq 2 \cdot (2^k - 3)\}. \end{aligned}$$

Proof:
The proof is by induction with respect to n. The case $n = 1$ is trivial. Now let $n > 1$. We distinguish between two cases:
Case 1: If $r > 0$ then $n-1 = 2^k - 2 + (r-1)$, $b_n = b_{n-1} + 3 \equiv b_{n-1} \mod 3$ and

$$\begin{aligned}\{b_i, i \le n\} =& \{b_i, i \le n-1\} \cup \{b_n\} \\ =& \{x | x \equiv b_{n-1} \mod 3, x \le 2^k - 3 + 3(r-1)\} \\ & \cup \{y | y \equiv -b_{n-1} \mod 3, y \le 2 \cdot (2^k - 3)\} \\ & \cup \{2^k - 3 + 3r\} \\ =& \{x | x \equiv b_n \mod 3, x \le 2^k - 3 + 3r\} \\ & \cup \{y | y \equiv -b_n \mod 3, y \le 2 \cdot (2^k - 3)\}.\end{aligned}$$

Case 2: If $r = 0$ then $n - 1 = 2^{k-1} - 2 + (2^{k-1} - 1)$, $b_n = b_{n-1}/2 \equiv -b_{n-1} \mod 3$ and

$$\begin{aligned}\{b_i, i \le n\} =& \{b_i, i \le n-1\} \cup \{b_n\} \\ =& \{x | x \equiv b_{n-1} \mod 3, x \le 2^{k-1} - 3 + 3(2^{k-1} - 1)\} \\ & \cup \{y | y \equiv -b_{n-1} \mod 3, y \le 2 \cdot (2^{k-1} - 3)\} \\ & \cup \{2^k - 3\} \\ =& \{x | x \equiv -b_n \mod 3, x \le 2 \cdot (2^k - 3)\} \\ & \cup \{y | y \equiv b_n \mod 3, y \le 2^k - 6\} \\ & \cup \{2^k - 3\} \\ =& \{x | x \equiv b_n \mod 3, x \le 2^k - 3\} \\ & \cup \{y | y \equiv -b_n \mod 3, y \le 2 \cdot (2^k - 3)\}.\end{aligned}$$

□

Lemma 2 *Let $n = 2^k - 2 + r$, $r < 2^k$, $x < b_n$. Then we have $x = b_i$ or $x = b_{n-1} + b_i$ or $x = |b_{n-1} - b_i|$ for some $i < n$.*

Proof:
We distinguish again between two cases:
Case 1: Let $r = 0$. If $x \equiv 1 \mod 3$ or $x \equiv 2 \mod 3$ then $x = b_i$ for some $i < n$ by the previous lemma. If $x \equiv 0 \mod 3$ then

$$x = 2b_n - \beta = b_{n-1} - \beta$$

with $\beta \equiv 2b_n \equiv -b_n \mod 3$ and $\beta < 2b_n = 2 \cdot (2^k - 3)$, so $\beta = b_i$ for some $i < n$ by the previous lemma.

Case 2: Let $r > 0$. If $x \equiv -b_n \mod 3$ with $x \leq 2 \cdot (2^k - 3)$ or $x \equiv b_n \mod 3$ then $x = b_i$ for some $i < n$ by the previous lemma.
Now let $x \equiv -b_n \mod 3$ with $x > 2 \cdot (2^k - 3)$. Because of $2 \cdot (2^k - 3) \equiv 2^{k+1} \equiv -2^k \equiv -b_n \mod 3$ we even have $x \geq 2 \cdot (2^k - 3) + 3$. Now let

$$x = b_{n-1} - \beta$$

with $\beta \equiv b_{n-1} - x \equiv b_{n-1} + b_{n-1} \equiv -b_n \mod 3$. We have

$$\beta = b_{n-1} - x = 2^k - 3 + 3(r-1) - x \leq 2^k - 3 + 3(r-1) - 2 \cdot (2^k - 3) - 3 = 2^{k+1} - 3,$$

so $\beta = b_i$ for some $i < n$ by the previous lemma. Note that if $\beta < 0$ then $\beta \in \{-1, -2\}$ and we have $b_n = b_{n-1} + b_2$ resp. $b_n = b_{n-1} + b_1$.
Finally let $x \equiv 0 \mod 3$. If $x > b_{n-1} = b_n - 3$ then $b_n = b_{n-1} + b_2$ or $b_n = b_{n-1} + b_1$. Otherwise we can write

$$x = b_n - 3 - \gamma = b_{n-1} - \gamma$$

with $0 < \gamma \equiv b_n \mod 3$ and $\gamma \leq b_n - 3 - x \leq b_n$, so $\gamma = b_i$ for some $i < n$ by the previous lemma. □

Lemma 3 *Let $n = 2^k - 2 + r$, $r < 2^k$. Then we have $b_n \neq b_i$, $b_n \neq b_{n-1} + b_i$, $b_n \neq |b_{n-1} - b_i|$ for all $i < n$.*

Proof:
Assume first that $b_n = b_i$, i.e.,

$$2^k - 3 + 3r = 2^l - 3 + 3s$$

for some $k \leq l$, $r < 2^k$, $s < 2^l$. We have

$$2^k + 3r = 2^l + 3s < 4 \cdot 2^l = 2^{l+2}$$

from which we conclude that $k = l$ or $k = l + 1$. If $k = l$ then $r = s$ follows. If $k = l + 1$ then $2^{l+1} - 2^l = 2^l = 3(s - r)$, a contradiction. Thus we find that $b_n \neq b_i$ for all $i < n$. For the rest of the proof we distinguish again between two cases:
Case 1: Let $r = 0$. Then we have $b_{n-1} = 2b_n > b_n$. If we assume that $b_n = |b_{n-1} - b_i|$ then we conclude that $b_i = b_n$ which is impossible.
Case 2: Let $r > 0$. Then we have $b_{n-1} = b_n - 3 \equiv b_n \mod 3$. If we assume that $b_n = b_{n-1} + b_i$ or $b_n = |b_{n-1} - b_i|$ then we conclude that $b_i \equiv 0 \mod 3$ which is impossible by the first lemma. □

Corollary 1 *For all $n > 2$ (actually for all $n > 0$ if we adjust the definition of (a_n)) we have*

$$a_n = b_n = 3n + 3 - 2 \cdot 2^{\lfloor \lg(n+2) \rfloor}.$$

Proof:
By the second and the third lemma the number (b_n) is the smallest number not of the form b_i, $b_{n-1} + b_i$ or $|b_{n-1} - b_i|$ with $i \leq n-1$. But this is exactly the definition of a_n. Also the initial values are the same. $\square$

Remark 1 *The sequence a_n produces exactly the natural numbers that are not divisible be 3 (in a strange order).*

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References

[1] Stephan, R., *Prove or disprove 100 conjectures from the OEIS*, preprint (2004), math.CO/0409509.

[2] Sloane, N. J. A., *The On-Line Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences/index.html`.